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Phys- ics

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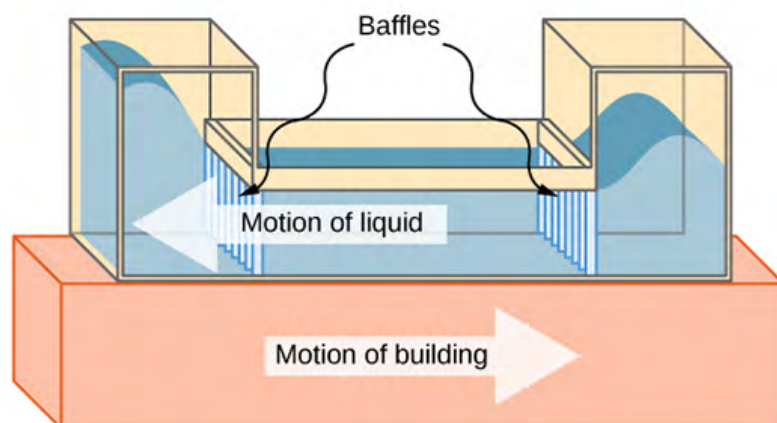
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CHAPTER 15

Oscillations



(a)



(b)

Figure 15.1 (a) The Comcast Building in Philadelphia, Pennsylvania, looming high above the skyline, is approximately 305 meters (1000 feet) tall. At this height, the top floors can oscillate back and forth due to seismic activity and fluctuating winds. (b) Shown above is a schematic drawing of a tuned, liquid-column mass damper, installed at the top of the Comcast, consisting of a 300,000-gallon reservoir of water to reduce oscillations.

Chapter Outline

[15.1 Simple Harmonic Motion](#)

[15.2 Energy in Simple Harmonic Motion](#)

[15.3 Comparing Simple Harmonic Motion and Circular Motion](#)

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INTRODUCTION We begin the study of oscillations with simple systems of pendulums and springs. Although these systems may seem quite basic, the concepts involved have many real-life applications. For example, the Comcast Building in Philadelphia, Pennsylvania, stands approximately 305 meters (1000 feet) tall. As

buildings are built taller, they can act as inverted, physical pendulums, with the top floors oscillating due to seismic activity and fluctuating winds. In the Comcast Building, a tuned-mass damper is used to reduce the oscillations. Installed at the top of the building is a tuned, liquid-column mass damper, consisting of a 300,000-gallon reservoir of water. This U-shaped tank allows the water to oscillate freely at a frequency that matches the natural frequency of the building. Damping is provided by tuning the turbulence levels in the moving water using baffles.

15.1 Simple Harmonic Motion

Learning Objectives

By the end of this section, you will be able to:

- Define the terms period and frequency
- List the characteristics of simple harmonic motion
- Explain the concept of phase shift
- Write the equations of motion for the system of a mass and spring undergoing simple harmonic motion
- Describe the motion of a mass oscillating on a vertical spring

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time ([Figure 15.2](#)). The string vibrates around an equilibrium position, and one oscillation is completed when the string starts from the initial position, travels to one of the extreme positions, then to the other extreme position, and returns to its initial position. We define **periodic motion** to be any motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by a child swinging on a swing. In this section, we study the basic characteristics of oscillations and their mathematical description.



Figure 15.2 When a guitar string is plucked, the string oscillates up and down in periodic motion. The vibrating string causes the surrounding air molecules to oscillate, producing sound waves. (credit: Yutaka Tsutano)

Period and Frequency in Oscillations

In the absence of friction, the time to complete one oscillation remains constant and is called the **period (T)**. Its units are usually seconds, but may be any convenient unit of time. The word ‘period’ refers to the time for some event whether repetitive or not, but in this chapter, we shall deal primarily in periodic motion, which is by definition repetitive.

A concept closely related to period is the frequency of an event. **Frequency (f)** is defined to be the number of events per unit time. For periodic motion, frequency is the number of oscillations per unit time. The relationship between frequency and period is

$$f = \frac{1}{T}.$$

15.1

The SI unit for frequency is the *hertz* (Hz) and is defined as one *cycle per second*:

$$1 \text{ Hz} = 1 \frac{\text{cycle}}{\text{s}} \quad \text{or} \quad 1 \text{ Hz} = \frac{1}{\text{s}} = 1 \text{ s}^{-1}.$$

A cycle is one complete **oscillation**.



EXAMPLE 15.1

Determining the Frequency of Medical Ultrasound

Ultrasound machines are used by medical professionals to make images for examining internal organs of the body. An ultrasound machine emits high-frequency sound waves, which reflect off the organs, and a computer receives the waves, using them to create a picture. We can use the formulas presented in this module to determine the frequency, based on what we know about oscillations. Consider a medical imaging device that produces ultrasound by oscillating with a period of $0.400 \mu\text{s}$. What is the frequency of this oscillation?

Strategy

The period (T) is given and we are asked to find frequency (f).

Solution

Substitute $0.400 \mu\text{s}$ for T in $f = \frac{1}{T}$:

$$f = \frac{1}{T} = \frac{1}{0.400 \times 10^{-6} \text{ s}}.$$

Solve to find

$$f = 2.50 \times 10^6 \text{ Hz}.$$

Significance

This frequency of sound is much higher than the highest frequency that humans can hear (the range of human hearing is 20 Hz to 20,000 Hz); therefore, it is called ultrasound. Appropriate oscillations at this frequency generate ultrasound used for noninvasive medical diagnoses, such as observations of a fetus in the womb.

Characteristics of Simple Harmonic Motion

A very common type of periodic motion is called **simple harmonic motion (SHM)**. A system that oscillates with SHM is called a **simple harmonic oscillator**.

Simple Harmonic Motion

In simple harmonic motion, the acceleration of the system, and therefore the net force, is proportional to the displacement and acts in the opposite direction of the displacement.

A good example of SHM is an object with mass m attached to a spring on a frictionless surface, as shown in [Figure 15.3](#). The object oscillates around the equilibrium position, and the net force on the object is equal to the force provided by the spring. This force obeys Hooke's law $F_s = -kx$, as discussed in a previous chapter.

If the net force can be described by Hooke's law and there is no *damping* (slowing down due to friction or other nonconservative forces), then a simple harmonic oscillator oscillates with equal displacement on either side of the equilibrium position, as shown for an object on a spring in [Figure 15.3](#). The maximum displacement from equilibrium is called the **amplitude (A)**. The units for amplitude and displacement are the same but depend on the type of oscillation. For the object on the spring, the units of amplitude and displacement are meters.

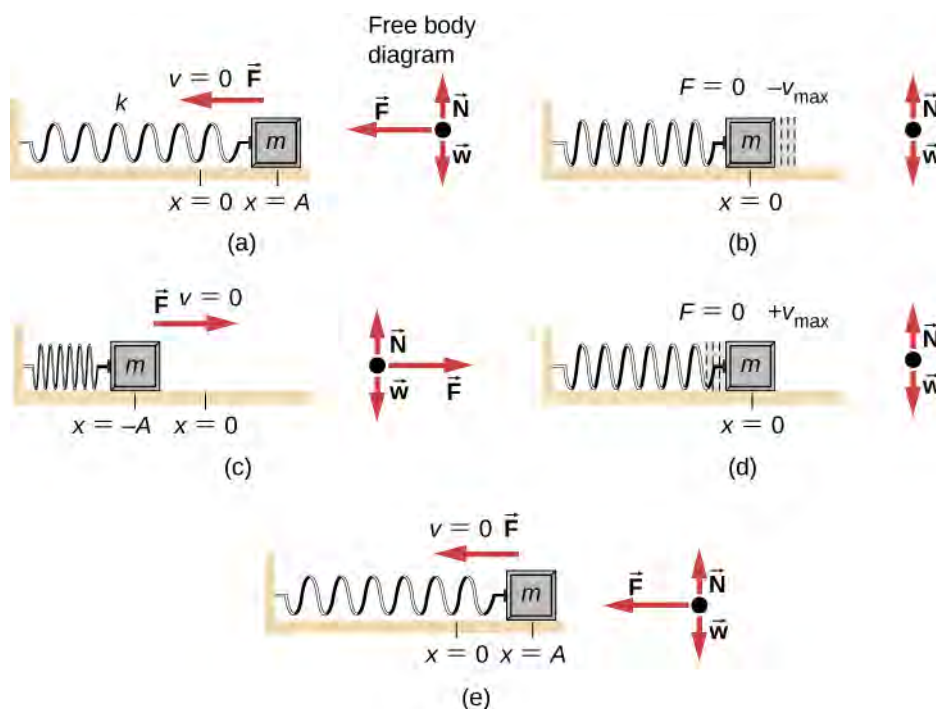


Figure 15.3 An object attached to a spring sliding on a frictionless surface is an uncomplicated simple harmonic oscillator. In the above set of figures, a mass is attached to a spring and placed on a frictionless table. The other end of the spring is attached to the wall. The position of the mass, when the spring is neither stretched nor compressed, is marked as $x = 0$ and is the equilibrium position. (a) The mass is displaced to a position $x = A$ and released from rest. (b) The mass accelerates as it moves in the negative x -direction, reaching a maximum negative velocity at $x = 0$. (c) The mass continues to move in the negative x -direction, slowing until it comes to a stop at $x = -A$. (d) The mass now begins to accelerate in the positive x -direction, reaching a positive maximum velocity at $x = 0$. (e) The mass then continues to move in the positive direction until it stops at $x = A$. The mass continues in SHM that has an amplitude A and a period T . The object's maximum speed occurs as it passes through equilibrium. The stiffer the spring is, the smaller the period T . The greater the mass of the object is, the greater the period T .

What is so significant about SHM? For one thing, the period T and frequency f of a simple harmonic oscillator are independent of amplitude. The string of a guitar, for example, oscillates with the same frequency whether plucked gently or hard.

Two important factors do affect the period of a simple harmonic oscillator. The period is related to how stiff the system is. A very stiff object has a large **force constant** (k), which causes the system to have a smaller period. For example, you can adjust a diving board's stiffness—the stiffer it is, the faster it vibrates, and the shorter its period. Period also depends on the mass of the oscillating system. The more massive the system is, the longer the period. For example, a heavy person on a diving board bounces up and down more slowly than a light one. In fact, the mass m and the force constant k are the *only* factors that affect the period and frequency of SHM. To derive an equation for the period and the frequency, we must first define and analyze the equations of motion. Note that the force constant is sometimes referred to as the *spring constant*.

Equations of SHM

Consider a block attached to a spring on a frictionless table ([Figure 15.4](#)). The **equilibrium position** (the position where the spring is neither stretched nor compressed) is marked as $x = 0$. At the equilibrium position, the net force is zero.

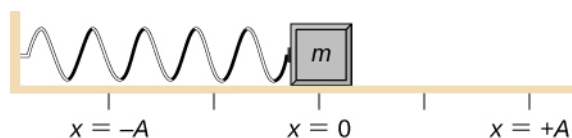


Figure 15.4 A block is attached to a spring and placed on a frictionless table. The equilibrium position, where the spring is neither

extended nor compressed, is marked as $x = 0$.

Work is done on the block to pull it out to a position of $x = +A$, and it is then released from rest. The maximum x -position (A) is called the amplitude of the motion. The block begins to oscillate in SHM between $x = +A$ and $x = -A$, where A is the amplitude of the motion and T is the period of the oscillation. The period is the time for one oscillation. Figure 15.5 shows the motion of the block as it completes one and a half oscillations after release. Figure 15.6 shows a plot of the position of the block versus time. When the position is plotted versus time, it is clear that the data can be modeled by a cosine function with an amplitude A and a period T . The cosine function $\cos\theta$ repeats every multiple of 2π , whereas the motion of the block repeats every period T . However, the function $\cos\left(\frac{2\pi}{T}t\right)$ repeats every integer multiple of the period. The maximum of the cosine function is one, so it is necessary to multiply the cosine function by the amplitude A .

$$x(t) = A \cos\left(\frac{2\pi}{T}t\right) = A \cos(\omega t).$$

15.2

Recall from the chapter on rotation that the angular frequency equals $\omega = \frac{d\theta}{dt}$. In this case, the period is constant, so the angular frequency is defined as 2π divided by the period, $\omega = \frac{2\pi}{T}$.

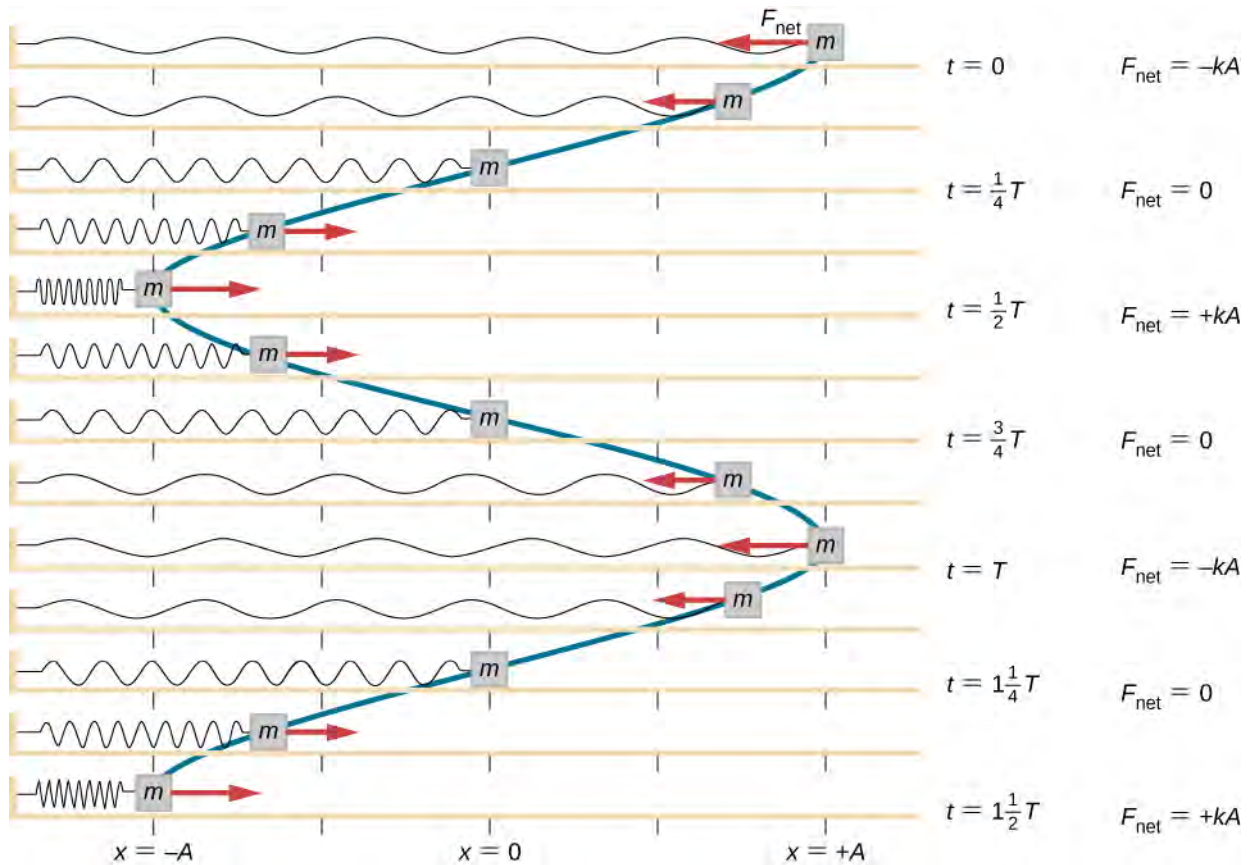


Figure 15.5 A block is attached to one end of a spring and placed on a frictionless table. The other end of the spring is anchored to the wall. The equilibrium position, where the net force equals zero, is marked as $x = 0$ m. Work is done on the block, pulling it out to $x = +A$, and the block is released from rest. The block oscillates between $x = +A$ and $x = -A$. The force is also shown as a vector.

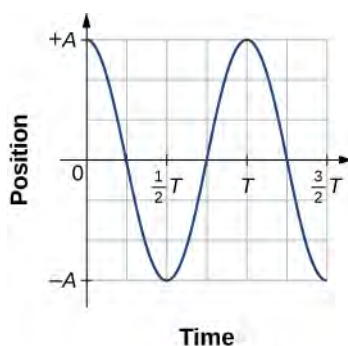


Figure 15.6 A graph of the position of the block shown in [Figure 15.5](#) as a function of time. The position can be modeled as a periodic function, such as a cosine or sine function.

The equation for the position as a function of time $x(t) = A \cos(\omega t)$ is good for modeling data, where the position of the block at the initial time $t = 0.00$ s is at the amplitude A and the initial velocity is zero. Often when taking experimental data, the position of the mass at the initial time $t = 0.00$ s is not equal to the amplitude and the initial velocity is not zero. Consider 10 seconds of data collected by a student in lab, shown in [Figure 15.7](#).

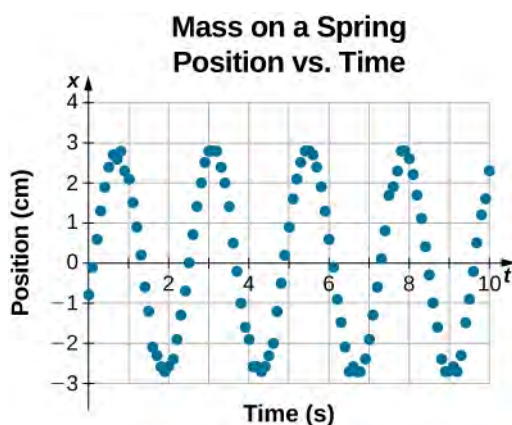


Figure 15.7 Data collected by a student in lab indicate the position of a block attached to a spring, measured with a sonic range finder. The data are collected starting at time $t = 0.00$ s, but the initial position is near position $x \approx -0.80$ cm $\neq 3.00$ cm, so the initial position does not equal the amplitude $x_0 = +A$. The velocity is the time derivative of the position, which is the slope at a point on the graph of position versus time. The velocity is not $v = 0.00$ m/s at time $t = 0.00$ s, as evident by the slope of the graph of position versus time, which is not zero at the initial time.

The data in [Figure 15.7](#) can still be modeled with a periodic function, like a cosine function, but the function is shifted to the right. This shift is known as a **phase shift** and is usually represented by the Greek letter phi (ϕ). The equation of the position as a function of time for a block on a spring becomes

$$x(t) = A \cos(\omega t + \phi).$$

This is the generalized equation for SHM where t is the time measured in seconds, ω is the angular frequency with units of inverse seconds, A is the amplitude measured in meters or centimeters, and ϕ is the phase shift measured in radians ([Figure 15.8](#)). It should be noted that because sine and cosine functions differ only by a phase shift, this motion could be modeled using either the cosine or sine function.

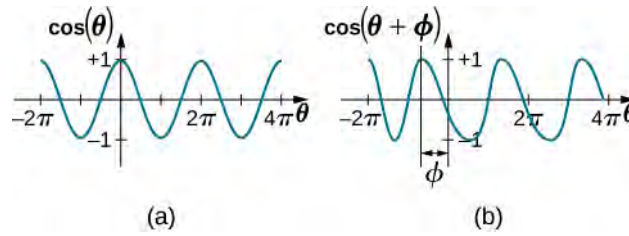


Figure 15.8 (a) A cosine function. (b) A cosine function shifted to the left by an angle ϕ . The angle ϕ is known as the phase shift of the function.

The velocity of the mass on a spring, oscillating in SHM, can be found by taking the derivative of the position equation:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(A \cos(\omega t + \phi)) = -A\omega \sin(\omega t + \phi) = -v_{\max} \sin(\omega t + \phi).$$

Because the sine function oscillates between -1 and $+1$, the maximum velocity is the amplitude times the angular frequency, $v_{\max} = A\omega$. The maximum velocity occurs at the equilibrium position ($x = 0$) when the mass is moving toward $x = +A$. The maximum velocity in the negative direction is attained at the equilibrium position ($x = 0$) when the mass is moving toward $x = -A$ and is equal to $-v_{\max}$.

The acceleration of the mass on the spring can be found by taking the time derivative of the velocity:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(-A\omega \sin(\omega t + \phi)) = -A\omega^2 \cos(\omega t + \phi) = -a_{\max} \cos(\omega t + \phi).$$

The maximum acceleration is $a_{\max} = A\omega^2$. The maximum acceleration occurs at the position ($x = -A$), and the acceleration at the position ($x = -A$) and is equal to $-a_{\max}$.

Summary of Equations of Motion for SHM

In summary, the oscillatory motion of a block on a spring can be modeled with the following equations of motion:

$x(t) = A \cos(\omega t + \phi)$	15.3
$v(t) = -v_{\max} \sin(\omega t + \phi)$	15.4
$a(t) = -a_{\max} \cos(\omega t + \phi)$	15.5
$x_{\max} = A$	15.6
$v_{\max} = A\omega$	15.7
$a_{\max} = A\omega^2$	15.8

Here, A is the amplitude of the motion, T is the period, ϕ is the phase shift, and $\omega = \frac{2\pi}{T} = 2\pi f$ is the angular frequency of the motion of the block.



EXAMPLE 15.2

Determining the Equations of Motion for a Block and a Spring

A 2.00-kg block is placed on a frictionless surface. A spring with a force constant of $k = 32.00$ N/m is attached to the block, and the opposite end of the spring is attached to the wall. The spring can be compressed or extended. The equilibrium position is marked as $x = 0.00$ m.

Work is done on the block, pulling it out to $x = +0.02$ m. The block is released from rest and oscillates between $x = +0.02$ m and $x = -0.02$ m. The period of the motion is 1.57 s. Determine the equations of motion.

Strategy

We first find the angular frequency. The phase shift is zero, $\phi = 0.00$ rad, because the block is released from rest at $x = A = +0.02$ m. Once the angular frequency is found, we can determine the maximum velocity and maximum acceleration.

Solution

The angular frequency can be found and used to find the maximum velocity and maximum acceleration:

$$\begin{aligned}\omega &= \frac{2\pi}{1.57 \text{ s}} = 4.00 \text{ s}^{-1}; \\ v_{\text{max}} &= A\omega = 0.02 \text{ m} (4.00 \text{ s}^{-1}) = 0.08 \text{ m/s}; \\ a_{\text{max}} &= A\omega^2 = 0.02 \text{ m} (4.00 \text{ s}^{-1})^2 = 0.32 \text{ m/s}^2.\end{aligned}$$

All that is left is to fill in the equations of motion:

$$\begin{aligned}x(t) &= A \cos(\omega t + \phi) = (0.02 \text{ m}) \cos(4.00 \text{ s}^{-1} t); \\ v(t) &= -v_{\text{max}} \sin(\omega t + \phi) = (-0.08 \text{ m/s}) \sin(4.00 \text{ s}^{-1} t); \\ a(t) &= -a_{\text{max}} \cos(\omega t + \phi) = (-0.32 \text{ m/s}^2) \cos(4.00 \text{ s}^{-1} t).\end{aligned}$$

Significance

The position, velocity, and acceleration can be found for any time. It is important to remember that when using these equations, your calculator must be in radians mode.

The Period and Frequency of a Mass on a Spring

One interesting characteristic of the SHM of an object attached to a spring is that the angular frequency, and therefore the period and frequency of the motion, depend on only the mass and the force constant, and not on other factors such as the amplitude of the motion. We can use the equations of motion and Newton's second law ($\vec{F}_{\text{net}} = m\vec{a}$) to find equations for the angular frequency, frequency, and period.

Consider the block on a spring on a frictionless surface. There are three forces on the mass: the weight, the normal force, and the force due to the spring. The only two forces that act perpendicular to the surface are the weight and the normal force, which have equal magnitudes and opposite directions, and thus sum to zero. The only force that acts parallel to the surface is the force due to the spring, so the net force must be equal to the force of the spring:

$$F_x = -kx;$$

$$ma = -kx;$$

$$\begin{aligned}m \frac{d^2x}{dt^2} &= -kx; \\ \frac{d^2x}{dt^2} &= -\frac{k}{m}x.\end{aligned}$$

Substituting the equations of motion for x and a gives us

$$-A\omega^2 \cos(\omega t + \phi) = -\frac{k}{m} A \cos(\omega t + \phi).$$

Cancelling out like terms and solving for the angular frequency yields

$$\omega = \sqrt{\frac{k}{m}}. \quad 15.9$$

The angular frequency depends only on the force constant and the mass, and not the amplitude. The angular frequency is defined as $\omega = 2\pi/T$, which yields an equation for the period of the motion:

$$T = 2\pi\sqrt{\frac{m}{k}}. \quad 15.10$$

The period also depends only on the mass and the force constant. The greater the mass, the longer the period. The stiffer the spring, the shorter the period. The frequency is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \quad 15.11$$

Vertical Motion and a Horizontal Spring

When a spring is hung vertically and a block is attached and set in motion, the block oscillates in SHM. In this case, there is no normal force, and the net effect of the force of gravity is to change the equilibrium position. Consider [Figure 15.9](#). Two forces act on the block: the weight and the force of the spring. The weight is constant and the force of the spring changes as the length of the spring changes.

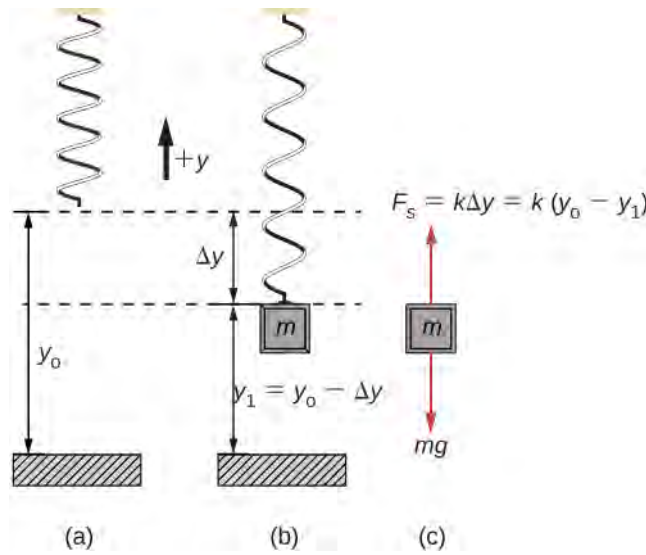


Figure 15.9 A spring is hung from the ceiling. When a block is attached, the block is at the equilibrium position where the weight of the block is equal to the force of the spring. (a) The spring is hung from the ceiling and the equilibrium position is marked as y_0 . (b) A mass is attached to the spring and a new equilibrium position is reached ($y_1 = y_0 - \Delta y$) when the force provided by the spring equals the weight of the mass. (c) The free-body diagram of the mass shows the two forces acting on the mass: the weight and the force of the spring.

When the block reaches the equilibrium position, as seen in [Figure 15.9](#), the force of the spring equals the weight of the block, $F_{\text{net}} = F_s - mg = 0$, where

$$-k(-\Delta y) = mg.$$

From the figure, the change in the position is $\Delta y = y_0 - y_1$ and since $-k(-\Delta y) = mg$, we have

$$k(y_0 - y_1) - mg = 0.$$

If the block is displaced and released, it will oscillate around the new equilibrium position. As shown in [Figure 15.10](#), if the position of the block is recorded as a function of time, the recording is a periodic function.

If the block is displaced to a position y , the net force becomes $F_{\text{net}} = k(y - y_0) - mg = 0$. But we found that at the equilibrium position, $mg = k\Delta y = ky_0 - ky_1$. Substituting for the weight in the equation yields

$$F_{\text{net}} = ky - ky_0 - (ky_0 - ky_1) = -k(y - y_1).$$

Recall that y_1 is just the equilibrium position and any position can be set to be the point $y = 0.00\text{ m}$. So let's set y_1 to $y = 0.00\text{ m}$. The net force then becomes

$$F_{\text{net}} = -ky;$$

$$m \frac{d^2 y}{dt^2} = -ky.$$

This is just what we found previously for a horizontally sliding mass on a spring. The constant force of gravity only served to shift the equilibrium location of the mass. Therefore, the solution should be the same form as for a block on a horizontal spring, $y(t) = A \cos(\omega t + \phi)$. The equations for the velocity and the acceleration also have the same form as for the horizontal case. Note that the inclusion of the phase shift means that the motion can actually be modeled using either a cosine or a sine function, since these two functions only differ by a phase shift.

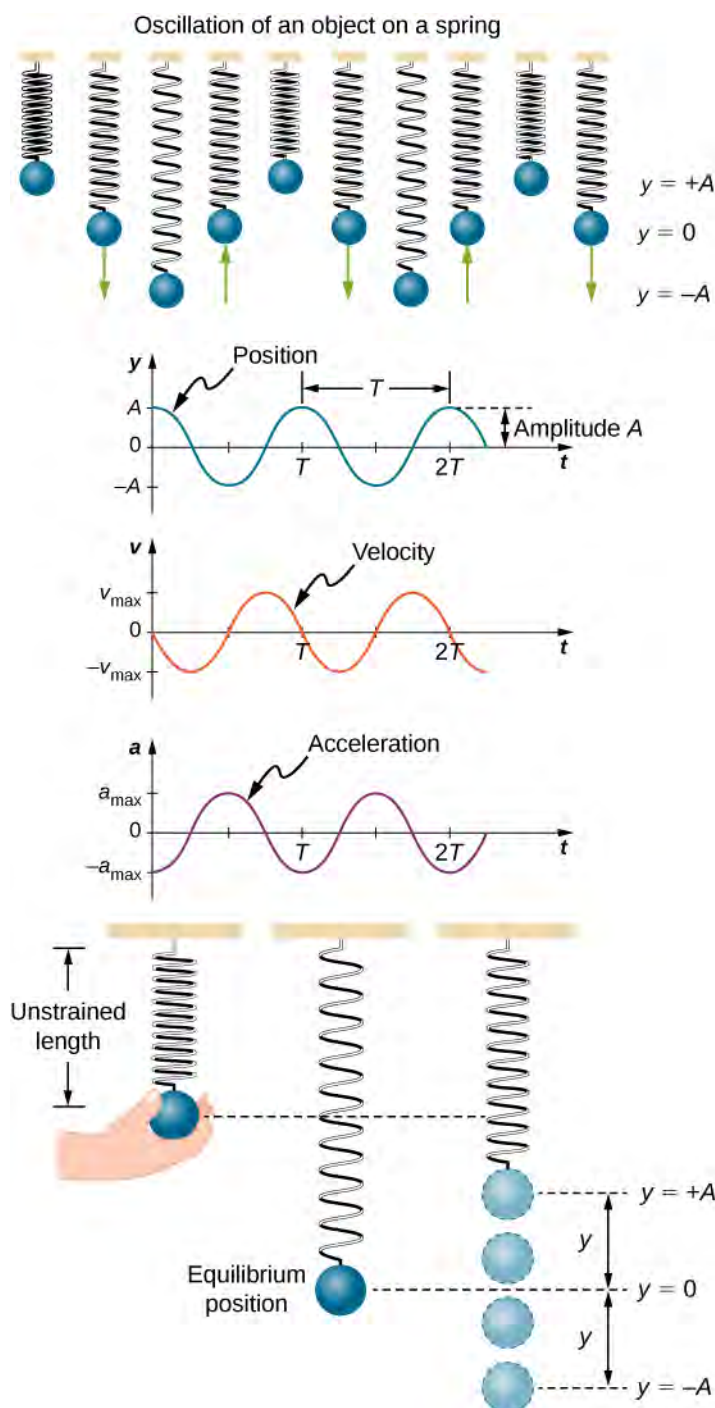


Figure 15.10 Graphs of $y(t)$, $v(t)$, and $a(t)$ versus t for the motion of an object on a vertical spring. The net force on the object can be described by Hooke's law, so the object undergoes SHM. Note that the initial position has the vertical displacement at its maximum value A ; v is initially zero and then negative as the object moves down; the initial acceleration is negative, back toward the equilibrium position and becomes zero at that point.

15.2 Energy in Simple Harmonic Motion

Learning Objectives

By the end of this section, you will be able to:

- Describe the energy conservation of the system of a mass and a spring
- Explain the concepts of stable and unstable equilibrium points

To produce a deformation in an object, we must do work. That is, whether you pluck a guitar string or compress a car's shock absorber, a force must be exerted through a distance. If the only result is deformation, and no work goes into thermal, sound, or kinetic energy, then all the work is initially stored in the deformed object as some form of potential energy.

Consider the example of a block attached to a spring on a frictionless table, oscillating in SHM. The force of the spring is a conservative force (which you studied in the chapter on potential energy and conservation of energy), and we can define a potential energy for it. This potential energy is the energy stored in the spring when the spring is extended or compressed. In this case, the block oscillates in one dimension with the force of the spring acting parallel to the motion:

$$W = \int_{x_i}^{x_f} F_x dx = \int_{x_i}^{x_f} -kx dx = \left[-\frac{1}{2}kx^2 \right]_{x_i}^{x_f} = - \left[\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \right] = - [U_f - U_i] = -\Delta U.$$

When considering the energy stored in a spring, the equilibrium position, marked as $x_i = 0.00$ m, is the position at which the energy stored in the spring is equal to zero. When the spring is stretched or compressed a distance x , the potential energy stored in the spring is

$$U = \frac{1}{2}kx^2.$$

Energy and the Simple Harmonic Oscillator

To study the energy of a simple harmonic oscillator, we need to consider all the forms of energy. Consider the example of a block attached to a spring, placed on a frictionless surface, oscillating in SHM. The potential energy stored in the deformation of the spring is

$$U = \frac{1}{2}kx^2.$$

In a simple harmonic oscillator, the energy oscillates between kinetic energy of the mass $K = \frac{1}{2}mv^2$ and potential energy $U = \frac{1}{2}kx^2$ stored in the spring. In the SHM of the mass and spring system, there are no dissipative forces, so the total energy is the sum of the potential energy and kinetic energy. In this section, we consider the conservation of energy of the system. The concepts examined are valid for all simple harmonic oscillators, including those where the gravitational force plays a role.

Consider [Figure 15.11](#), which shows an oscillating block attached to a spring. In the case of undamped SHM, the energy oscillates back and forth between kinetic and potential, going completely from one form of energy to the other as the system oscillates. So for the simple example of an object on a frictionless surface attached to a spring, the motion starts with all of the energy stored in the spring as **elastic potential energy**. As the object starts to move, the elastic potential energy is converted into kinetic energy, becoming entirely kinetic energy at the equilibrium position. The energy is then converted back into elastic potential energy by the spring as it is stretched or compressed. The velocity becomes zero when the kinetic energy is completely converted, and this cycle then repeats. Understanding the conservation of energy in these cycles will provide extra insight here and in later applications of SHM, such as alternating circuits.

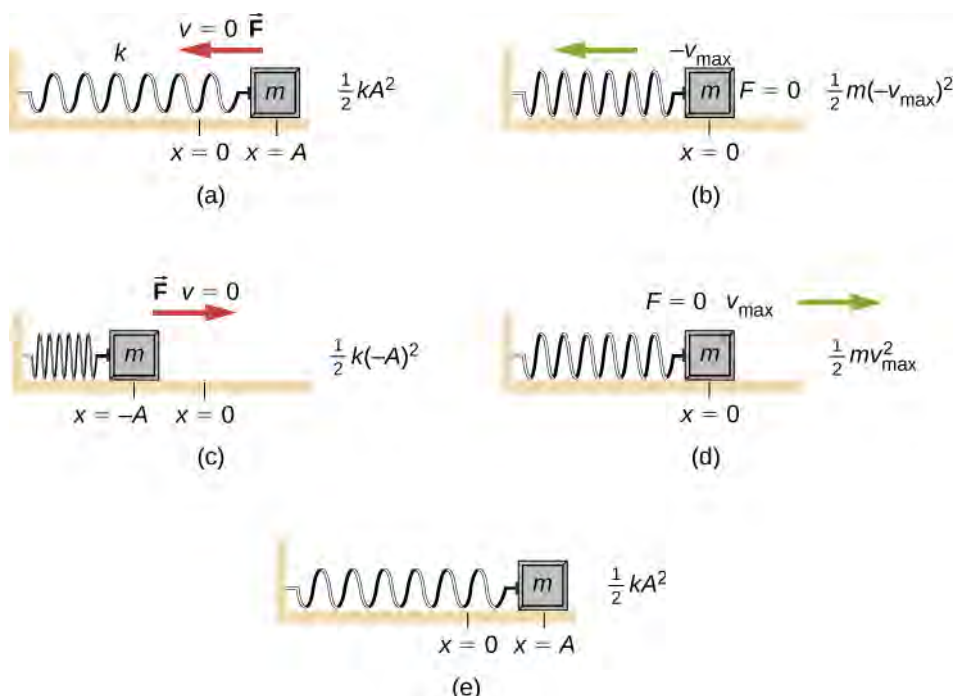


Figure 15.11 The transformation of energy in SHM for an object attached to a spring on a frictionless surface. (a) When the mass is at the position $x = +A$, all the energy is stored as potential energy in the spring $U = \frac{1}{2}kA^2$. The kinetic energy is equal to zero because the velocity of the mass is zero. (b) As the mass moves toward $x = -A$, the mass crosses the position $x = 0$. At this point, the spring is neither extended nor compressed, so the potential energy stored in the spring is zero. At $x = 0$, the total energy is all kinetic energy where $K = \frac{1}{2}m(-v_{\max})^2$. (c) The mass continues to move until it reaches $x = -A$ where the mass stops and starts moving toward $x = +A$. At the position $x = -A$, the total energy is stored as potential energy in the compressed $U = \frac{1}{2}k(-A)^2$ and the kinetic energy is zero. (d) As the mass passes through the position $x = 0$, the kinetic energy is $K = \frac{1}{2}mv_{\max}^2$ and the potential energy stored in the spring is zero. (e) The mass returns to the position $x = +A$, where $K = 0$ and $U = \frac{1}{2}kA^2$.

Consider [Figure 15.11](#), which shows the energy at specific points on the periodic motion. While staying constant, the energy oscillates between the kinetic energy of the block and the potential energy stored in the spring:

$$E_{\text{Total}} = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2.$$

The motion of the block on a spring in SHM is defined by the position $x(t) = A\cos(\omega t + \phi)$ with a velocity of $v(t) = -A\omega\sin(\omega t + \phi)$. Using these equations, the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ and $\omega = \sqrt{\frac{k}{m}}$, we can find the total energy of the system:

$$\begin{aligned} E_{\text{Total}} &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}mA^2\left(\frac{k}{m}\right)\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}kA^2\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)) \\ &= \frac{1}{2}kA^2. \end{aligned}$$

The total energy of the system of a block and a spring is equal to the sum of the potential energy stored in the spring plus the kinetic energy of the block and is proportional to the square of the amplitude $E_{\text{Total}} = (1/2)kA^2$. The total energy of the system is constant.

A closer look at the energy of the system shows that the kinetic energy oscillates like a sine-squared function, while the potential energy oscillates like a cosine-squared function. However, the total energy for the system is

constant and is proportional to the amplitude squared. [Figure 15.12](#) shows a plot of the potential, kinetic, and total energies of the block and spring system as a function of time. Also plotted are the position and velocity as a function of time. Before time $t = 0.0$ s, the block is attached to the spring and placed at the equilibrium position. Work is done on the block by applying an external force, pulling it out to a position of $x = +A$. The system now has potential energy stored in the spring. At time $t = 0.00$ s, the position of the block is equal to the amplitude, the potential energy stored in the spring is equal to $U = \frac{1}{2}kA^2$, and the force on the block is maximum and points in the negative x -direction ($F_S = -kA$). The velocity and kinetic energy of the block are zero at time $t = 0.00$ s. At time $t = 0.00$ s, the block is released from rest.

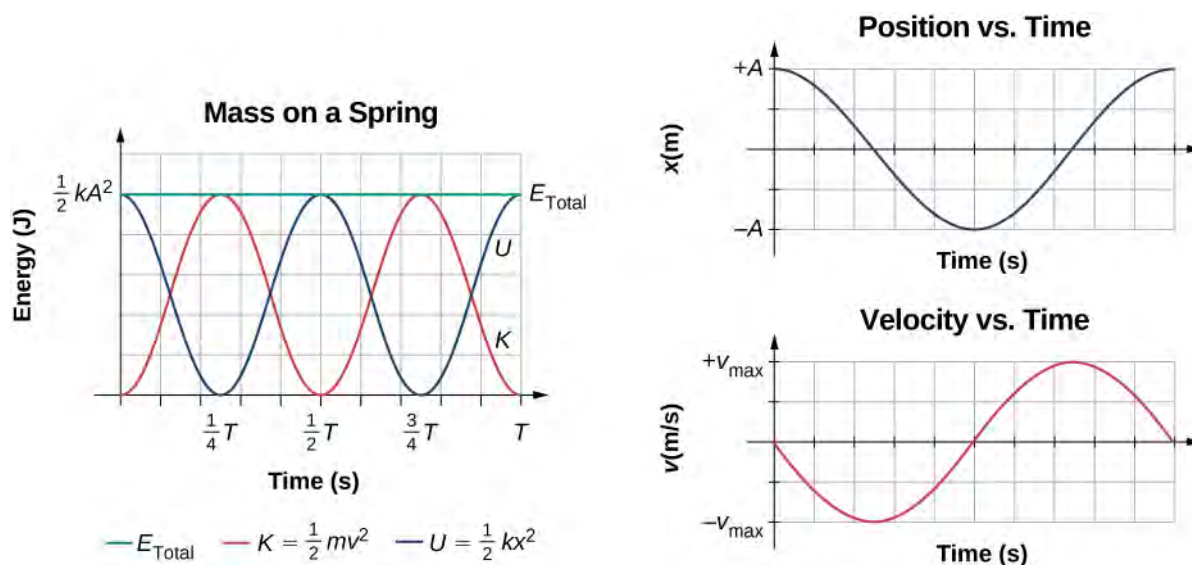


Figure 15.12 Graph of the kinetic energy, potential energy, and total energy of a block oscillating on a spring in SHM. Also shown are the graphs of position versus time and velocity versus time. The total energy remains constant, but the energy oscillates between kinetic energy and potential energy. When the kinetic energy is maximum, the potential energy is zero. This occurs when the velocity is maximum and the mass is at the equilibrium position. The potential energy is maximum when the speed is zero. The total energy is the sum of the kinetic energy plus the potential energy and it is constant.

Oscillations About an Equilibrium Position

We have just considered the energy of SHM as a function of time. Another interesting view of the simple harmonic oscillator is to consider the energy as a function of position. [Figure 15.13](#) shows a graph of the energy versus position of a system undergoing SHM.

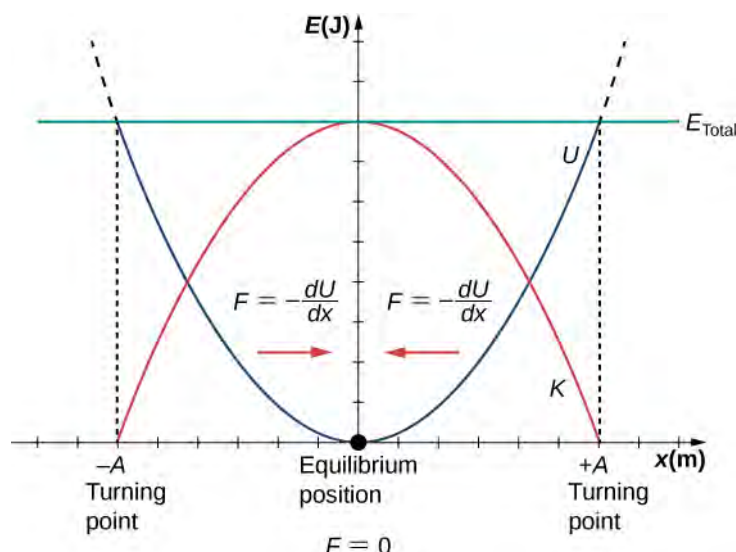


Figure 15.13 A graph of the kinetic energy (red), potential energy (blue), and total energy (green) of a simple harmonic oscillator. The force is equal to $F = -\frac{dU}{dx}$. The equilibrium position is shown as a black dot and is the point where the force is equal to zero. The force is positive when $x < 0$, negative when $x > 0$, and equal to zero when $x = 0$.

The potential energy curve in [Figure 15.13](#) resembles a bowl. When a marble is placed in a bowl, it settles to the equilibrium position at the lowest point of the bowl ($x = 0$). This happens because a **restoring force** points toward the equilibrium point. This equilibrium point is sometimes referred to as a **fixed point**. When the marble is disturbed to a different position ($x = +A$), the marble oscillates around the equilibrium position. Looking back at the graph of potential energy, the force can be found by looking at the slope of the potential energy graph ($F = -\frac{dU}{dx}$). Since the force on either side of the fixed point points back toward the equilibrium point, the equilibrium point is called a **stable equilibrium point**. The points $x = A$ and $x = -A$ are called the turning points. (See [Potential Energy and Conservation of Energy](#).)

Stability is an important concept. If an equilibrium point is stable, a slight disturbance of an object that is initially at the stable equilibrium point will cause the object to oscillate around that point. The stable equilibrium point occurs because the force on either side is directed toward it. For an unstable equilibrium point, if the object is disturbed slightly, it does not return to the equilibrium point.

Consider the marble in the bowl example. If the bowl is right-side up, the marble, if disturbed slightly, will oscillate around the stable equilibrium point. If the bowl is turned upside down, the marble can be balanced on the top, at the equilibrium point where the net force is zero. However, if the marble is disturbed slightly, it will not return to the equilibrium point, but will instead roll off the bowl. The reason is that the force on either side of the equilibrium point is directed away from that point. This point is an unstable equilibrium point.

[Figure 15.14](#) shows three conditions. The first is a stable equilibrium point (a), the second is an unstable equilibrium point (b), and the last is also an unstable equilibrium point (c), because the force on only one side points toward the equilibrium point.

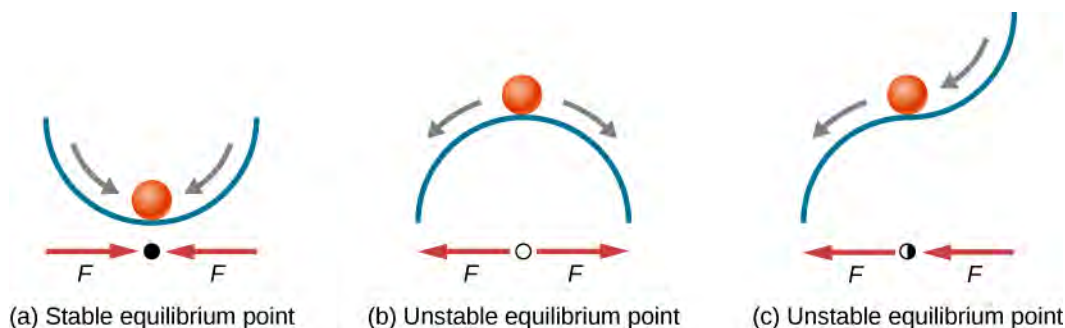


Figure 15.14 Examples of equilibrium points. (a) Stable equilibrium point; (b) unstable equilibrium point; (c) unstable equilibrium point

(sometimes referred to as a half-stable equilibrium point).

The process of determining whether an equilibrium point is stable or unstable can be formalized. Consider the potential energy curves shown in [Figure 15.15](#). The force can be found by analyzing the slope of the graph. The force is $F = -\frac{dU}{dx}$. In (a), the fixed point is at $x = 0.00$ m. When $x < 0.00$ m, the force is positive. When $x > 0.00$ m, the force is negative. This is a stable point. In (b), the fixed point is at $x = 0.00$ m. When $x < 0.00$ m, the force is negative. When $x > 0.00$ m, the force is also negative. This is an unstable point.

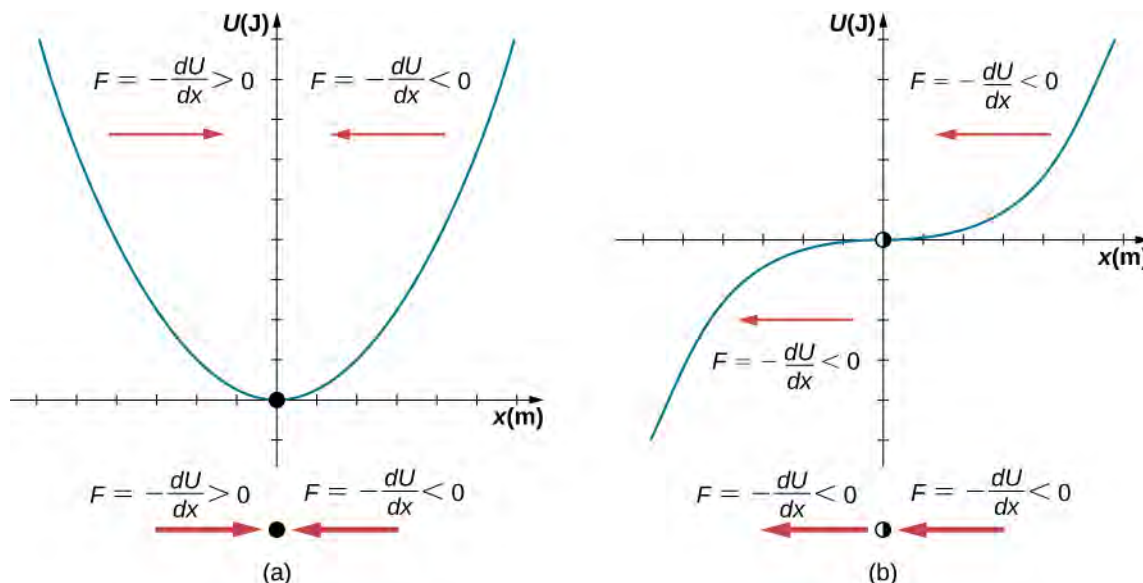


Figure 15.15 Two examples of a potential energy function. The force at a position is equal to the negative of the slope of the graph at that position. (a) A potential energy function with a stable equilibrium point. (b) A potential energy function with an unstable equilibrium point. This point is sometimes called half-stable because the force on one side points toward the fixed point.

A practical application of the concept of stable equilibrium points is the force between two neutral atoms in a molecule. If two molecules are in close proximity, separated by a few atomic diameters, they can experience an attractive force. If the molecules move close enough so that the electron shells of the other electrons overlap, the force between the molecules becomes repulsive. The attractive force between the two atoms may cause the atoms to form a molecule. The force between the two molecules is not a linear force and cannot be modeled simply as two masses separated by a spring, but the atoms of the molecule can oscillate around an equilibrium position when displaced a small amount from the equilibrium position. The atoms oscillate due the attractive force and repulsive force between the two atoms.

Consider one example of the interaction between two atoms known as the van Der Waals interaction. It is beyond the scope of this chapter to discuss in depth the interactions of the two atoms, but the oscillations of the atoms can be examined by considering one example of a model of the potential energy of the system. One suggestion to model the potential energy of this molecule is with the Lennard-Jones 6-12 potential:

$$U(x) = 4\epsilon \left[\left(\frac{\sigma}{x} \right)^{12} - \left(\frac{\sigma}{x} \right)^6 \right].$$

A graph of this function is shown in [Figure 15.16](#). The two parameters ϵ and σ are found experimentally.

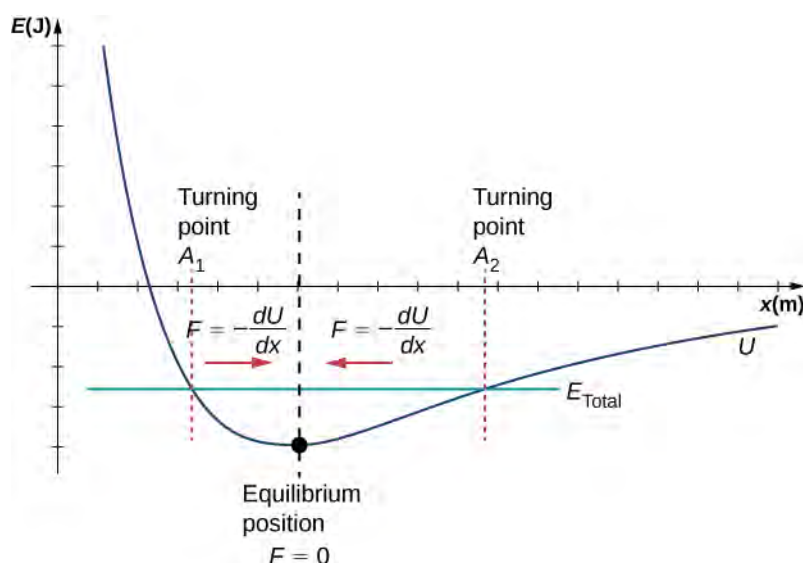


Figure 15.16 The Lennard-Jones potential energy function for a system of two neutral atoms. If the energy is below some maximum energy, the system oscillates near the equilibrium position between the two turning points.

From the graph, you can see that there is a potential energy well, which has some similarities to the potential energy well of the potential energy function of the simple harmonic oscillator discussed in [Figure 15.13](#). The Lennard-Jones potential has a stable equilibrium point where the potential energy is minimum and the force on either side of the equilibrium point points toward equilibrium point. Note that unlike the simple harmonic oscillator, the potential well of the Lennard-Jones potential is not symmetric. This is due to the fact that the force between the atoms is not a Hooke's law force and is not linear. The atoms can still oscillate around the equilibrium position $x_{\min}$ because when $x < x_{\min}$, the force is positive; when $x > x_{\min}$, the force is negative. Notice that as x approaches zero, the slope is quite steep and negative, which means that the force is large and positive. This suggests that it takes a large force to try to push the atoms close together. As x becomes increasingly large, the slope becomes less steep and the force is smaller and negative. This suggests that if given a large enough energy, the atoms can be separated.

If you are interested in this interaction, find the force between the molecules by taking the derivative of the potential energy function. You will see immediately that the force does not resemble a Hooke's law force ($F = -kx$), but if you are familiar with the binomial theorem:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots,$$

the force can be approximated by a Hooke's law force.

Velocity and Energy Conservation

Getting back to the system of a block and a spring in [Figure 15.11](#), once the block is released from rest, it begins to move in the negative direction toward the equilibrium position. The potential energy decreases and the magnitude of the velocity and the kinetic energy increase. At time $t = T/4$, the block reaches the equilibrium position $x = 0.00$ m, where the force on the block and the potential energy are zero. At the equilibrium position, the block reaches a negative velocity with a magnitude equal to the maximum velocity $v = -A\omega$. The kinetic energy is maximum and equal to $K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2 = \frac{1}{2}kA^2$. At this point, the force on the block is zero, but momentum carries the block, and it continues in the negative direction toward $x = -A$. As the block continues to move, the force on it acts in the positive direction and the magnitude of the velocity and kinetic energy decrease. The potential energy increases as the spring compresses. At time $t = T/2$, the block reaches $x = -A$. Here the velocity and kinetic energy are equal to zero. The force on the block is $F = +kA$ and the potential energy stored in the spring is $U = \frac{1}{2}kA^2$. During the oscillations, the total energy is constant and equal to the sum of the potential energy and the kinetic energy of the system,

$$E_{\text{Total}} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2. \quad 15.12$$

The equation for the energy associated with SHM can be solved to find the magnitude of the velocity at any position:

$$|v| = \sqrt{\frac{k}{m}(A^2 - x^2)}. \quad 15.13$$

The energy in a simple harmonic oscillator is proportional to the square of the amplitude. When considering many forms of oscillations, you will find the energy proportional to the amplitude squared.

✓ CHECK YOUR UNDERSTANDING 15.1

Why would it hurt more if you snapped your hand with a ruler than with a loose spring, even if the displacement of each system is equal?

✓ CHECK YOUR UNDERSTANDING 15.2

Identify one way you could decrease the maximum velocity of a simple harmonic oscillator.

15.3 Comparing Simple Harmonic Motion and Circular Motion

Learning Objectives

By the end of this section, you will be able to:

- Describe how the sine and cosine functions relate to the concepts of circular motion
- Describe the connection between simple harmonic motion and circular motion

An easy way to model SHM is by considering uniform circular motion. [Figure 15.17](#) shows one way of using this method. A peg (a cylinder of wood) is attached to a vertical disk, rotating with a constant angular frequency. [Figure 15.18](#) shows a side view of the disk and peg. If a lamp is placed above the disk and peg, the peg produces a shadow. Let the disk have a radius of $r = A$ and define the position of the shadow that coincides with the center line of the disk to be $x = 0.00 \text{ m}$. As the disk rotates at a constant rate, the shadow oscillates between $x = +A$ and $x = -A$. Now imagine a block on a spring beneath the floor as shown in [Figure 15.18](#).

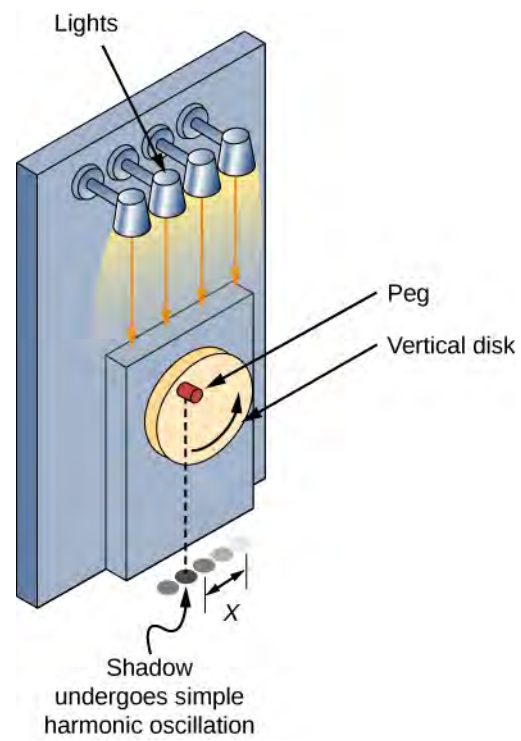


Figure 15.17 SHM can be modeled as rotational motion by looking at the shadow of a peg on a wheel rotating at a constant angular frequency.

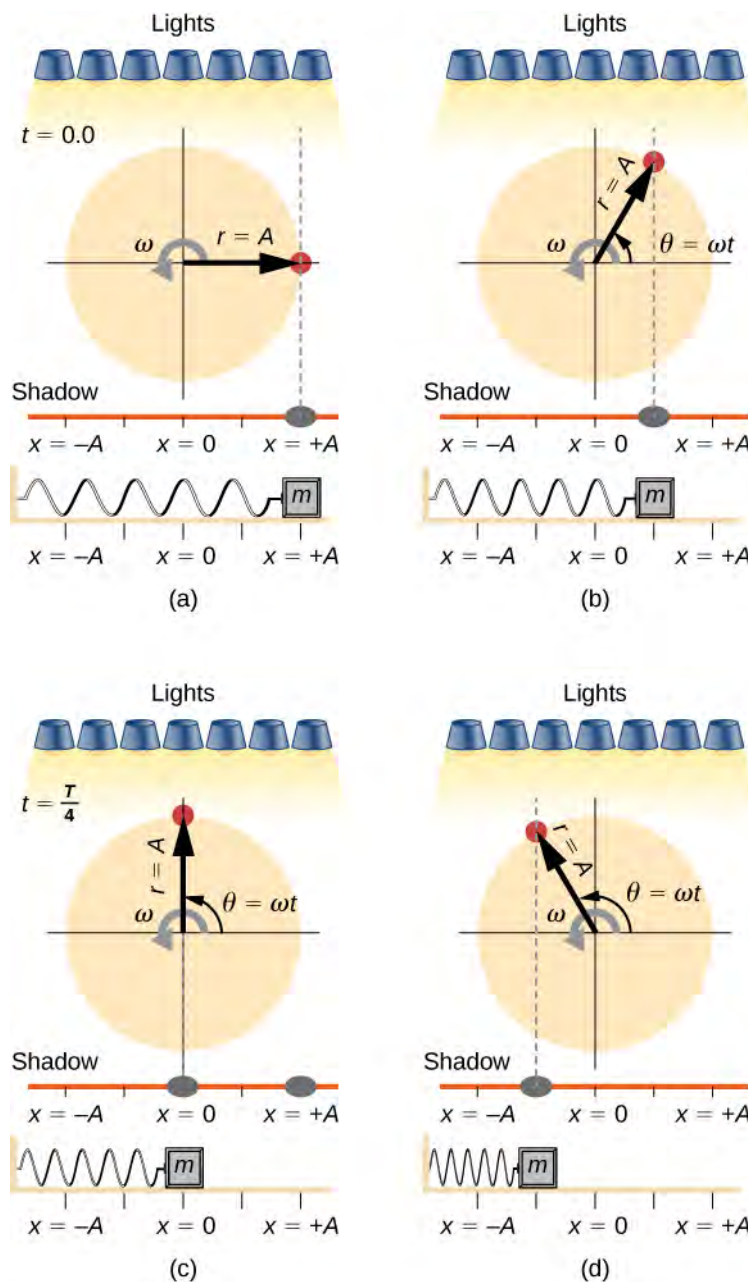


Figure 15.18 Light shines down on the disk so that the peg makes a shadow. If the disk rotates at just the right angular frequency, the shadow follows the motion of the block on a spring. If there is no energy dissipated due to nonconservative forces, the block and the shadow will oscillate back and forth in unison. In this figure, four snapshots are taken at four different times. (a) The wheel starts at $\theta = 0^\circ$ and the shadow of the peg is at $x = +A$, representing the mass at position $x = +A$. (b) As the disk rotates through an angle $\theta = \omega t$, the shadow of the peg is between $x = +A$ and $x = 0$. (c) The disk continues to rotate until $\theta = 90^\circ$, at which the shadow follows the mass to $x = 0$. (d) The disk continues to rotate, the shadow follows the position of the mass.

If the disk turns at the proper angular frequency, the shadow follows along with the block. The position of the shadow can be modeled with the equation

$$x(t) = A\cos(\omega t).$$

15.14

Recall that the block attached to the spring does not move at a constant velocity. How often does the wheel have to turn to have the peg's shadow always on the block? The disk must turn at a constant angular frequency equal to 2π times the frequency of oscillation ($\omega = 2\pi f$).

Figure 15.19 shows the basic relationship between uniform circular motion and SHM. The peg lies at the tip of the radius, a distance A from the center of the disk. The x -axis is defined by a line drawn parallel to the ground, cutting the disk in half. The y -axis (not shown) is defined by a line perpendicular to the ground, cutting the disk into a left half and a right half. The center of the disk is the point $(x = 0, y = 0)$. The projection of the position of the peg onto the fixed x -axis gives the position of the shadow, which undergoes SHM analogous to the system of the block and spring. At the time shown in the figure, the projection has position x and moves to the left with velocity v . The tangential velocity of the peg around the circle equals $v_{\max}$ of the block on the spring. The x -component of the velocity is equal to the velocity of the block on the spring.

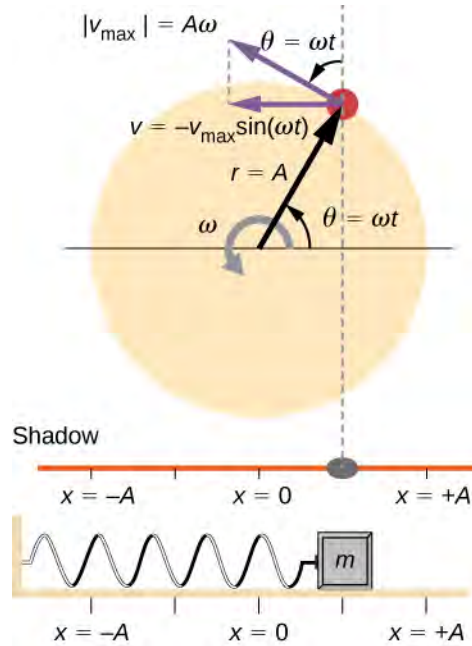


Figure 15.19 A peg moving on a circular path with a constant angular velocity ω is undergoing uniform circular motion. Its projection on the x -axis undergoes SHM. Also shown is the velocity of the peg around the circle, $v_{\max}$, and its projection, which is v . Note that these velocities form a similar triangle to the displacement triangle.

We can use Figure 15.19 to analyze the velocity of the shadow as the disk rotates. The peg moves in a circle with a speed of $v_{\max} = A\omega$. The shadow moves with a velocity equal to the component of the peg's velocity that is parallel to the surface where the shadow is being produced:

$$v = -v_{\max} \sin(\omega t). \quad 15.15$$

It follows that the acceleration is

$$a = -a_{\max} \cos(\omega t). \quad 15.16$$

✓ CHECK YOUR UNDERSTANDING 15.3

Identify an object that undergoes uniform circular motion. Describe how you could trace the SHM of this object.

15.4 Pendulums

Learning Objectives

By the end of this section, you will be able to:

- State the forces that act on a simple pendulum
- Determine the angular frequency, frequency, and period of a simple pendulum in terms of the length of the pendulum and the acceleration due to gravity
- Define the period for a physical pendulum
- Define the period for a torsional pendulum

Pendulums are in common usage. Grandfather clocks use a pendulum to keep time and a pendulum can be used to measure the acceleration due to gravity. For small displacements, a pendulum is a simple harmonic oscillator.

The Simple Pendulum

A **simple pendulum** is defined to have a point mass, also known as the pendulum bob, which is suspended from a string of length L with negligible mass (Figure 15.20). Here, the only forces acting on the bob are the force of gravity (i.e., the weight of the bob) and tension from the string. The mass of the string is assumed to be negligible as compared to the mass of the bob.

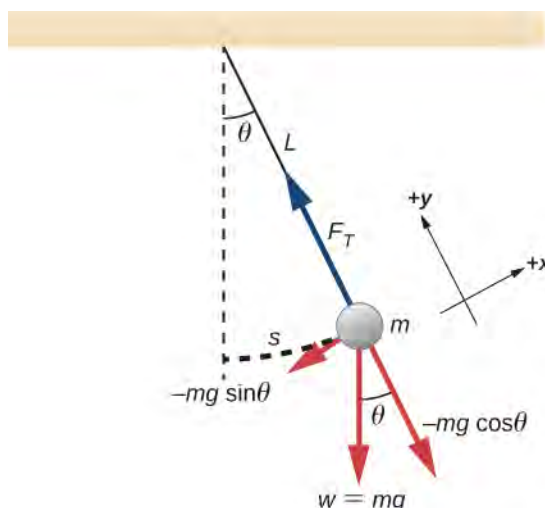


Figure 15.20 A simple pendulum has a small-diameter bob and a string that has a very small mass but is strong enough not to stretch appreciably. The linear displacement from equilibrium is s , the length of the arc. Also shown are the forces on the bob, which result in a net force of $-mg \sin \theta$ toward the equilibrium position—that is, a restoring force.

Consider the torque on the pendulum. The force providing the restoring torque is the component of the weight of the pendulum bob that acts along the arc length. The torque is the length of the string L times the component of the net force that is perpendicular to the radius of the arc. The minus sign indicates the torque acts in the opposite direction of the angular displacement:

$$\begin{aligned}
 \tau &= -L(mg \sin \theta); \\
 I\alpha &= -L(mg \sin \theta); \\
 I \frac{d^2\theta}{dt^2} &= -L(mg \sin \theta); \\
 mL^2 \frac{d^2\theta}{dt^2} &= -L(mg \sin \theta); \\
 \frac{d^2\theta}{dt^2} &= -\frac{g}{L} \sin \theta.
 \end{aligned}$$

The solution to this differential equation involves advanced calculus, and is beyond the scope of this text. But note that for small angles (less than 15 degrees), $\sin \theta$ and θ differ by less than 1%, so we can use the small

angle approximation $\sin \theta \approx \theta$. The angle θ describes the position of the pendulum. Using the small angle approximation gives an approximate solution for small angles,

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{L} \theta. \quad 15.17$$

Because this equation has the same form as the equation for SHM, the solution is easy to find. The angular frequency is

$$\omega = \sqrt{\frac{g}{L}} \quad 15.18$$

and the period is

$$T = 2\pi \sqrt{\frac{L}{g}}. \quad 15.19$$

The period of a simple pendulum depends on its length and the acceleration due to gravity. The period is completely independent of other factors, such as mass and the maximum displacement. As with simple harmonic oscillators, the period T for a pendulum is nearly independent of amplitude, especially if θ is less than about 15° . Even simple pendulum clocks can be finely adjusted and remain accurate.

Note the dependence of T on g . If the length of a pendulum is precisely known, it can actually be used to measure the acceleration due to gravity, as in the following example.



EXAMPLE 15.3

Measuring Acceleration due to Gravity by the Period of a Pendulum

What is the acceleration due to gravity in a region where a simple pendulum having a length 75.000 cm has a period of 1.7357 s?

Strategy

We are asked to find g given the period T and the length L of a pendulum. We can solve $T = 2\pi \sqrt{\frac{L}{g}}$ for g , assuming only that the angle of deflection is less than 15° .

Solution

1. Square $T = 2\pi \sqrt{\frac{L}{g}}$ and solve for g :

$$g = 4\pi^2 \frac{L}{T^2}.$$

2. Substitute known values into the new equation:

$$g = 4\pi^2 \frac{0.75000 \text{ m}}{(1.7357 \text{ s})^2}.$$

3. Calculate to find g :

$$g = 9.8281 \text{ m/s}^2.$$

Significance

This method for determining g can be very accurate, which is why length and period are given to five digits in this example. For the precision of the approximation $\sin \theta \approx \theta$ to be better than the precision of the pendulum length and period, the maximum displacement angle should be kept below about 0.5° .

✓ CHECK YOUR UNDERSTANDING 15.4

An engineer builds two simple pendulums. Both are suspended from small wires secured to the ceiling of a room. Each pendulum hovers 2 cm above the floor. Pendulum 1 has a bob with a mass of 10 kg. Pendulum 2 has a bob with a mass of 100 kg. Describe how the motion of the pendulums will differ if the bobs are both displaced by 12° .

Physical Pendulum

Any object can oscillate like a pendulum. Consider a coffee mug hanging on a hook in the pantry. If the mug gets knocked, it oscillates back and forth like a pendulum until the oscillations die out. We have described a simple pendulum as a point mass and a string. A **physical pendulum** is any object whose oscillations are similar to those of the simple pendulum, but cannot be modeled as a point mass on a string, and the mass distribution must be included into the equation of motion.

As for the simple pendulum, the restoring force of the physical pendulum is the force of gravity. With the simple pendulum, the force of gravity acts on the center of the pendulum bob. In the case of the physical pendulum, the force of gravity acts on the center of mass (CM) of an object. The object oscillates about a point O . Consider an object of a generic shape as shown in [Figure 15.21](#).

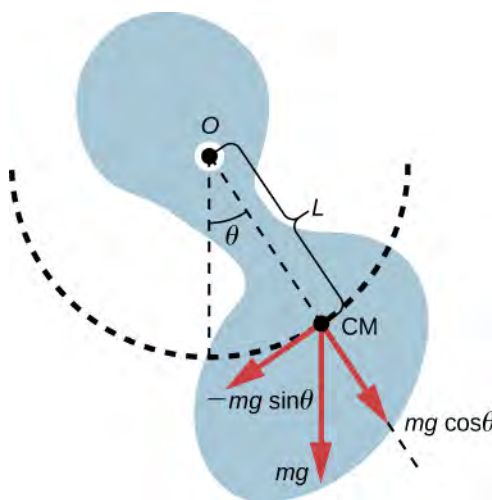


Figure 15.21 A physical pendulum is any object that oscillates as a pendulum, but cannot be modeled as a point mass on a string. The force of gravity acts on the center of mass (CM) and provides the restoring force that causes the object to oscillate. The minus sign on the component of the weight that provides the restoring force is present because the force acts in the opposite direction of the increasing angle θ .

When a physical pendulum is hanging from a point but is free to rotate, it rotates because of the torque applied at the CM, produced by the component of the object's weight that acts tangent to the motion of the CM. Taking the counterclockwise direction to be positive, the component of the gravitational force that acts tangent to the motion is $-mg \sin \theta$. The minus sign is the result of the restoring force acting in the opposite direction of the increasing angle. Recall that the torque is equal to $\vec{\tau} = \vec{r} \times \vec{F}$. The magnitude of the torque is equal to the length of the radius arm times the tangential component of the force applied, $|\tau| = rF \sin \theta$. Here, the length L of the radius arm is the distance between the point of rotation and the CM. To analyze the motion, start with the net torque. Like the simple pendulum, consider only small angles so that $\sin \theta \approx \theta$. Recall from [Fixed-Axis Rotation](#) on rotation that the net torque is equal to the moment of inertia $I = \int r^2 dm$ times the angular acceleration α , where $\alpha = \frac{d^2\theta}{dt^2}$:

$$I\alpha = \tau_{\text{net}} = L(-mg) \sin \theta.$$

Using the small angle approximation and rearranging:

$$\begin{aligned}
 I\alpha &= -L(mg)\theta; \\
 I\frac{d^2\theta}{dt^2} &= -L(mg)\theta; \\
 \frac{d^2\theta}{dt^2} &= -\left(\frac{mgL}{I}\right)\theta.
 \end{aligned}$$

Once again, the equation says that the second time derivative of the position (in this case, the angle) equals minus a constant $\left(-\frac{mgL}{I}\right)$ times the position. The solution is

$$\theta(t) = \Theta \cos(\omega t + \phi),$$

where Θ is the maximum angular displacement. The angular frequency is

$$\omega = \sqrt{\frac{mgL}{I}}. \quad 15.20$$

The period is therefore

$$T = 2\pi\sqrt{\frac{I}{mgL}}. \quad 15.21$$

Note that for a simple pendulum, the moment of inertia is $I = \int r^2 dm = mL^2$ and the period reduces to

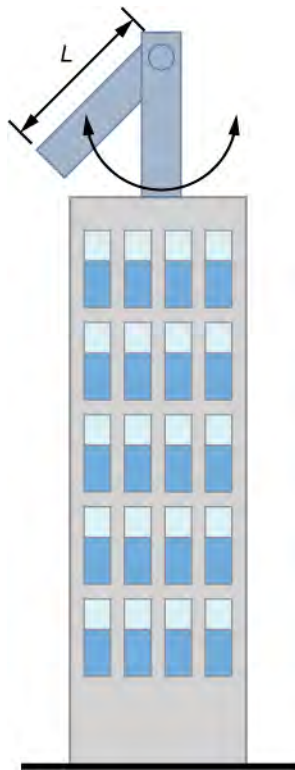
$$T = 2\pi\sqrt{\frac{L}{g}}.$$



EXAMPLE 15.4

Reducing the Swaying of a Skyscraper

In extreme conditions, skyscrapers can sway up to two meters with a frequency of up to 20.00 Hz due to high winds or seismic activity. Several companies have developed physical pendulums that are placed on the top of the skyscrapers. As the skyscraper sways to the right, the pendulum swings to the left, reducing the sway. Assuming the oscillations have a frequency of 0.50 Hz, design a pendulum that consists of a long beam, of constant density, with a mass of 100 metric tons and a pivot point at one end of the beam. What should be the length of the beam?



Strategy

We are asked to find the length of the physical pendulum with a known mass. We first need to find the moment of inertia of the beam. We can then use the equation for the period of a physical pendulum to find the length.

Solution

1. Find the moment of inertia for the CM:
2. Use the parallel axis theorem to find the moment of inertia about the point of rotation:

$$I = I_{\text{CM}} + \frac{L^2}{4} M = \frac{1}{12} ML^2 + \frac{1}{4} ML^2 = \frac{1}{3} ML^2.$$

3. The period of a physical pendulum has a period of $T = 2\pi\sqrt{\frac{I}{mgL}}$. Use the moment of inertia to solve for the length L :

$$T = 2\pi\sqrt{\frac{I}{MgL}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}};$$

$$L = \frac{3T^2g}{8\pi^2} = 1.49 \text{ m}.$$

4. This length L is from the center of mass to the axis of rotation, which is half the length of the pendulum. Therefore the length H of the pendulum is:

$$H = 2L = 5.96 \text{ m}$$

Significance

There are many ways to reduce the oscillations, including modifying the shape of the skyscrapers, using multiple physical pendulums, and using tuned-mass dampers.

Torsional Pendulum

A **torsional pendulum** consists of a rigid body suspended by a light wire or spring (Figure 15.22). When the body is twisted some small maximum angle (Θ) and released from rest, the body oscillates between ($\theta = +\Theta$) and ($\theta = -\Theta$). The restoring torque is supplied by the shearing of the string or wire.

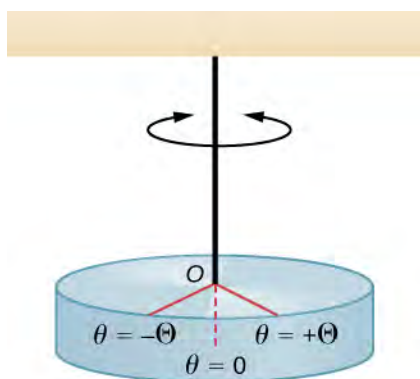


Figure 15.22 A torsional pendulum consists of a rigid body suspended by a string or wire. The rigid body oscillates between $\theta = +\Theta$ and $\theta = -\Theta$.

The restoring torque can be modeled as being proportional to the angle:

$$\tau = -\kappa\theta.$$

The variable kappa (κ) is known as the torsion constant of the wire or string. The minus sign shows that the restoring torque acts in the opposite direction to increasing angular displacement. The net torque is equal to the moment of inertia times the angular acceleration:

$$I \frac{d^2\theta}{dt^2} = -\kappa\theta;$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta.$$

This equation says that the second time derivative of the position (in this case, the angle) equals a negative constant times the position. This looks very similar to the equation of motion for the SHM $\frac{d^2x}{dt^2} = -\frac{k}{m}x$, where the period was found to be $T = 2\pi\sqrt{\frac{m}{k}}$. Therefore, the period of the torsional pendulum can be found using

$$T = 2\pi\sqrt{\frac{I}{\kappa}}. \quad 15.22$$

The units for the torsion constant are $[\kappa] = \text{N}\cdot\text{m} = \left(\text{kg} \frac{\text{m}}{\text{s}^2}\right) \text{m} = \text{kg} \frac{\text{m}^2}{\text{s}^2}$ and the units for the moment of inertia are $[I] = \text{kg}\cdot\text{m}^2$, which show that the unit for the period is the second.



EXAMPLE 15.5

Measuring the Torsion Constant of a String

A rod has a length of $l = 0.30 \text{ m}$ and a mass of 4.00 kg . A string is attached to the CM of the rod and the system is hung from the ceiling (Figure 15.23). The rod is displaced 10 degrees from the equilibrium position and released from rest. The rod oscillates with a period of 0.5 s . What is the torsion constant κ ?

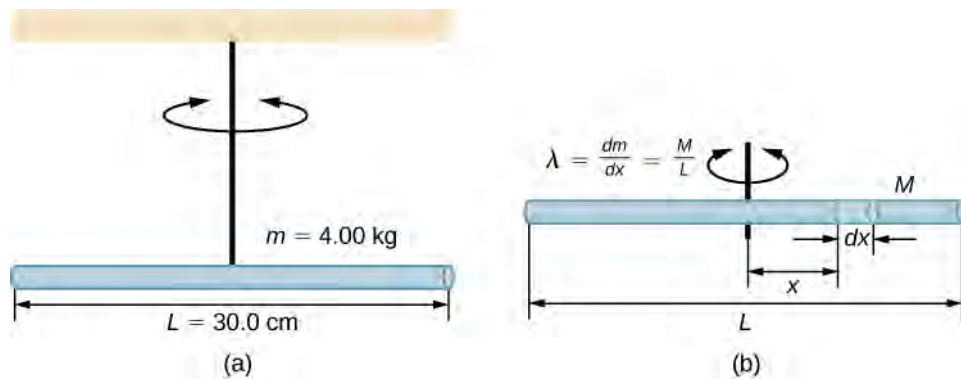


Figure 15.23 (a) A rod suspended by a string from the ceiling. (b) Finding the rod's moment of inertia.

Strategy

We are asked to find the torsion constant of the string. We first need to find the moment of inertia.

Solution

1. Find the moment of inertia for the CM:

$$I_{\text{CM}} = \int x^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda dx = \lambda \left[\frac{x^3}{3} \right]_{-L/2}^{+L/2} = \lambda \frac{2L^3}{24} = \left(\frac{M}{L} \right) \frac{2L^3}{24} = \frac{1}{12} ML^2.$$

2. Calculate the torsion constant using the equation for the period:

$$\begin{aligned} T &= 2\pi \sqrt{\frac{I}{\kappa}}; \\ \kappa &= I \left(\frac{2\pi}{T} \right)^2 = \left(\frac{1}{12} ML^2 \right) \left(\frac{2\pi}{T} \right)^2; \\ &= \left(\frac{1}{12} (4.00 \text{ kg}) (0.30 \text{ m})^2 \right) \left(\frac{2\pi}{0.50 \text{ s}} \right)^2 = 4.73 \text{ N} \cdot \text{m}. \end{aligned}$$

Significance

Like the force constant of the system of a block and a spring, the larger the torsion constant, the shorter the period.

15.5 Damped Oscillations

Learning Objectives

By the end of this section, you will be able to:

- Describe the motion of damped harmonic motion
- Write the equations of motion for damped harmonic oscillations
- Describe the motion of driven, or forced, damped harmonic motion
- Write the equations of motion for forced, damped harmonic motion

In the real world, oscillations seldom follow true SHM. Friction of some sort usually acts to dampen the motion so it dies away, or needs more force to continue. In this section, we examine some examples of damped harmonic motion and see how to modify the equations of motion to describe this more general case.

A guitar string stops oscillating a few seconds after being plucked. To keep swinging on a playground swing, you must keep pushing (Figure 15.24). Although we can often make friction and other nonconservative forces small or negligible, completely undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.



Figure 15.24 To counteract dampening forces, you need to keep pumping a swing. (credit: Bob Mical)

[Figure 15.25](#) shows a mass m attached to a spring with a force constant k . The mass is raised to a position A_0 , the initial amplitude, and then released. The mass oscillates around the equilibrium position in a fluid with viscosity but the amplitude decreases for each oscillation. For a system that has a small amount of damping, the period and frequency are constant and are nearly the same as for SHM, but the amplitude gradually decreases as shown. This occurs because the non-conservative damping force removes energy from the system, usually in the form of thermal energy.

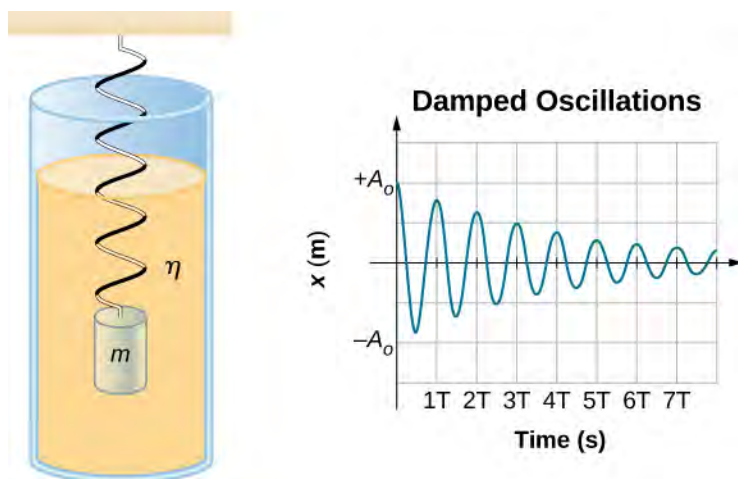


Figure 15.25 For a mass on a spring oscillating in a viscous fluid, the period remains constant, but the amplitudes of the oscillations decrease due to the damping caused by the fluid.

Consider the forces acting on the mass. Note that the only contribution of the weight is to change the equilibrium position, as discussed earlier in the chapter. Therefore, the net force is equal to the force of the spring and the damping force (F_D). If the magnitude of the velocity is small, meaning the mass oscillates slowly, the damping force is proportional to the velocity and acts against the direction of motion ($F_D = -bv$). The net force on the mass is therefore

$$ma = -bv - kx.$$

Writing this as a differential equation in x , we obtain

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0. \quad 15.23$$

To determine the solution to this equation, consider the plot of position versus time shown in [Figure 15.26](#). The curve resembles a cosine curve oscillating in the envelope of an exponential function $A_0 e^{-\alpha t}$ where $\alpha = \frac{b}{2m}$. The solution is

$$x(t) = A_0 e^{-\frac{b}{2m}t} \cos(\omega t + \phi). \quad 15.24$$

It is left as an exercise to prove that this is, in fact, the solution. To prove that it is the right solution, take the first and second derivatives with respect to time and substitute them into [Equation 15.23](#). It is found that [Equation 15.24](#) is the solution if

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

Recall that the angular frequency of a mass undergoing SHM is equal to the square root of the force constant divided by the mass. This is often referred to as the **natural angular frequency**, which is represented as

$$\omega_0 = \sqrt{\frac{k}{m}}. \quad 15.25$$

The angular frequency for damped harmonic motion becomes

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}. \quad 15.26$$

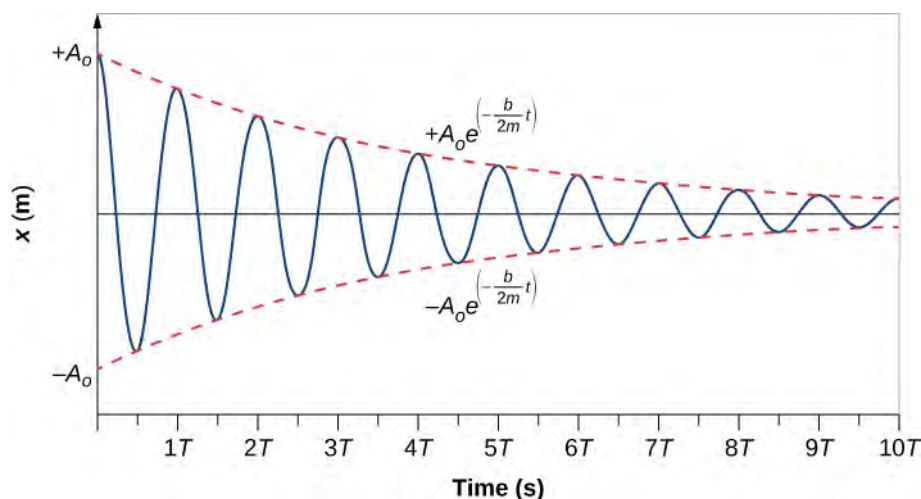


Figure 15.26 Position versus time for the mass oscillating on a spring in a viscous fluid. Notice that the curve appears to be a cosine function inside an exponential envelope.

Recall that when we began this description of damped harmonic motion, we stated that the damping must be small. Two questions come to mind. Why must the damping be small? And how small is small? If you gradually *increase* the amount of damping in a system, the period and frequency begin to be affected, because damping opposes and hence slows the back and forth motion. (The net force is smaller in both directions.) If there is very large damping, the system does not even oscillate—it slowly moves toward equilibrium. The angular frequency is equal to

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

As b increases, $\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ becomes smaller and eventually reaches zero when $b = \sqrt{4mk}$. If b becomes any larger, $\frac{k}{m} - \left(\frac{b}{2m}\right)^2$ becomes a negative number and $\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$ is a complex number.

[Figure 15.27](#) shows the displacement of a harmonic oscillator for different amounts of damping. When the

damping constant is small, $b < \sqrt{4mk}$, the system oscillates while the amplitude of the motion decays exponentially. This system is said to be **underdamped**, as in curve (a). Many systems are underdamped, and oscillate while the amplitude decreases exponentially, such as the mass oscillating on a spring. The damping may be quite small, but eventually the mass comes to rest. If the damping constant is $b = \sqrt{4mk}$, the system is said to be **critically damped**, as in curve (b). An example of a critically damped system is the shock absorbers in a car. It is advantageous to have the oscillations decay as fast as possible. Here, the system does not oscillate, but asymptotically approaches the equilibrium condition as quickly as possible. Curve (c) in [Figure 15.27](#) represents an **overdamped** system where $b > \sqrt{4mk}$. An overdamped system will approach equilibrium over a longer period of time.

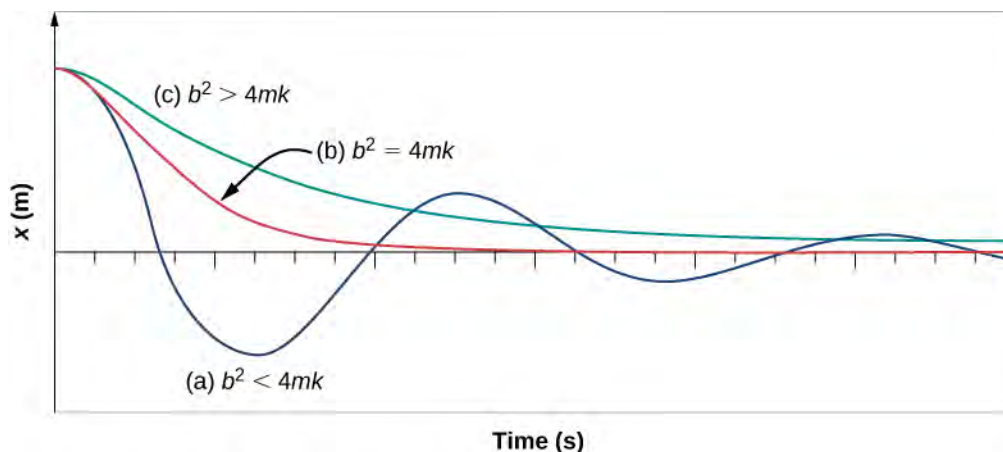


Figure 15.27 The position versus time for three systems consisting of a mass and a spring in a viscous fluid. (a) If the damping is small ($b < \sqrt{4mk}$), the mass oscillates, slowly losing amplitude as the energy is dissipated by the non-conservative force(s). The limiting case is (b) where the damping is ($b = \sqrt{4mk}$). (c) If the damping is very large ($b > \sqrt{4mk}$), the mass does not oscillate when displaced, but attempts to return to the equilibrium position.

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position.

✓ CHECK YOUR UNDERSTANDING 15.5

Why are completely undamped harmonic oscillators so rare?

15.6 Forced Oscillations

Learning Objectives

By the end of this section, you will be able to:

- Define forced oscillations
- List the equations of motion associated with forced oscillations
- Explain the concept of resonance and its impact on the amplitude of an oscillator
- List the characteristics of a system oscillating in resonance

Sit in front of a piano sometime and sing a loud brief note at it with the dampers off its strings ([Figure 15.28](#)). It will sing the same note back at you—the strings, having the same frequencies as your voice, are resonating in response to the forces from the sound waves that you sent to them. This is a good example of the fact that objects—in this case, piano strings—can be forced to oscillate, and oscillate most easily at their natural frequency. In this section, we briefly explore applying a periodic driving force acting on a simple harmonic oscillator. The driving force puts energy into the system at a certain frequency, not necessarily the same as the natural frequency of the system. Recall that the natural frequency is the frequency at which a system would

oscillate if there were no driving and no damping force.

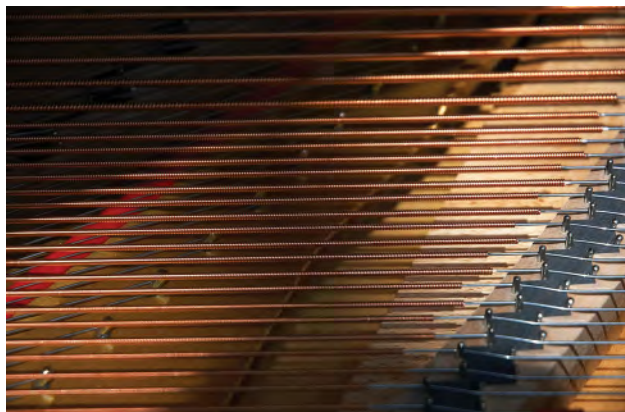


Figure 15.28 You can cause the strings in a piano to vibrate simply by producing sound waves from your voice. (credit: Matt Billings)

Most of us have played with toys involving an object supported on an elastic band, something like the paddle ball suspended from a finger in [Figure 15.29](#). Imagine the finger in the figure is your finger. At first, you hold your finger steady, and the ball bounces up and down with a small amount of damping. If you move your finger up and down slowly, the ball follows along without bouncing much on its own. As you increase the frequency at which you move your finger up and down, the ball responds by oscillating with increasing amplitude. When you drive the ball at its natural frequency, the ball's oscillations increase in amplitude with each oscillation for as long as you drive it. The phenomenon of driving a system with a frequency equal to its natural frequency is called **resonance**. A system being driven at its natural frequency is said to *resonate*. As the driving frequency gets progressively higher than the resonant or natural frequency, the amplitude of the oscillations becomes smaller until the oscillations nearly disappear, and your finger simply moves up and down with little effect on the ball.

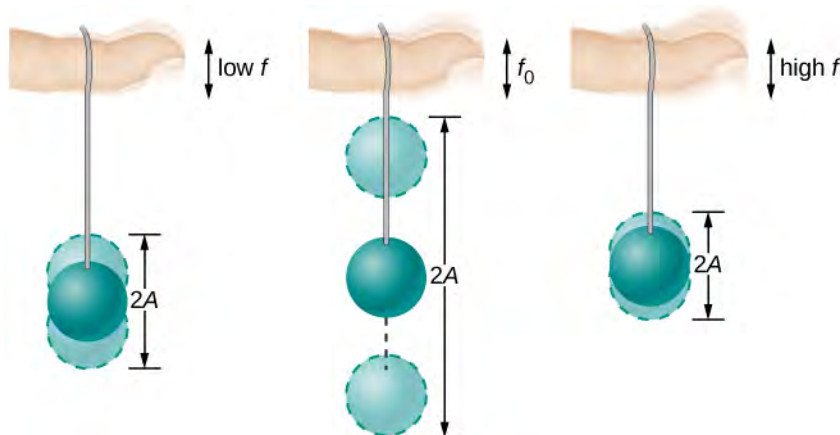


Figure 15.29 The paddle ball on its rubber band moves in response to the finger supporting it. If the finger moves with the natural frequency f_0 of the ball on the rubber band, then a resonance is achieved, and the amplitude of the ball's oscillations increases dramatically. At higher and lower driving frequencies, energy is transferred to the ball less efficiently, and it responds with lower-amplitude oscillations.

Consider a simple experiment. Attach a mass m to a spring in a viscous fluid, similar to the apparatus discussed in the damped harmonic oscillator. This time, instead of fixing the free end of the spring, attach the free end to a disk that is driven by a variable-speed motor. The motor turns with an angular driving frequency of ω . The rotating disk provides energy to the system by the work done by the driving force ($F_d = F_0 \sin(\omega t)$). The experimental apparatus is shown in [Figure 15.30](#).

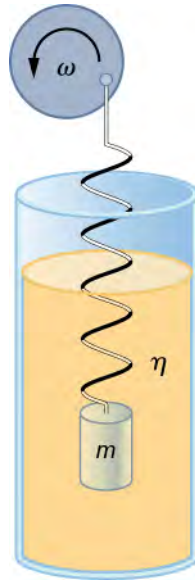


Figure 15.30 Forced, damped harmonic motion produced by driving a spring and mass with a disk driven by a variable-speed motor.

Using Newton's second law ($\vec{F}_{\text{net}} = m\vec{a}$), we can analyze the motion of the mass. The resulting equation is similar to the force equation for the damped harmonic oscillator, with the addition of the driving force:

$$-kx - b\frac{dx}{dt} + F_0 \sin(\omega t) = m\frac{d^2x}{dt^2}. \quad 15.27$$

When an oscillator is forced with a periodic driving force, the motion may seem chaotic. The motions of the oscillator is known as transients. After the transients die out, the oscillator reaches a steady state, where the motion is periodic. After some time, the steady state solution to this differential equation is

$$x(t) = A \cos(\omega t + \phi). \quad 15.28$$

Once again, it is left as an exercise to prove that this equation is a solution. Taking the first and second time derivative of $x(t)$ and substituting them into the force equation shows that $x(t) = A \sin(\omega t + \phi)$ is a solution as long as the amplitude is equal to

$$A = \frac{F_0}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + b^2\omega^2}} \quad 15.29$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of the driving force. Recall that the angular frequency, and therefore the frequency, of the motor can be adjusted. Looking at the denominator of the equation for the amplitude, when the driving frequency is much smaller, or much larger, than the natural frequency, the square of the difference of the two angular frequencies $(\omega^2 - \omega_0^2)^2$ is positive and large, making the denominator large, and the result is a small amplitude for the oscillations of the mass. As the frequency of the driving force approaches the natural frequency of the system, the denominator becomes small and the amplitude of the oscillations becomes large. The maximum amplitude results when the frequency of the driving force equals the natural frequency of the system ($A_{\text{max}} = \frac{F_0}{b\omega}$).

[Figure 15.31](#) shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. Each of the three curves on the graph represents a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less

energy is removed by the damping force. Note that since the amplitude grows as the damping decreases, taking this to the limit where there is no damping ($b = 0$), the amplitude becomes infinite.

Note that a small-amplitude driving force can produce a large-amplitude response. This phenomenon is known as resonance. A common example of resonance is a parent pushing a small child on a swing. When the child wants to go higher, the parent does not move back and then, getting a running start, slam into the child, applying a great force in a short interval. Instead, the parent applies small pushes to the child at just the right frequency, and the amplitude of the child's swings increases.

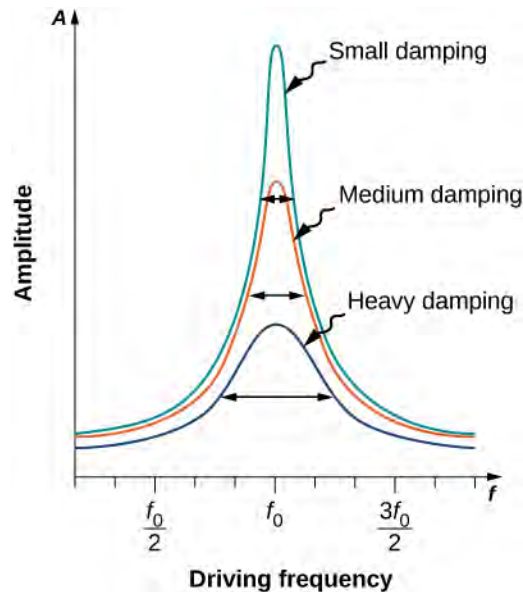


Figure 15.31 Amplitude of a harmonic oscillator as a function of the frequency of the driving force. The curves represent the same oscillator with the same natural frequency but with different amounts of damping. Resonance occurs when the driving frequency equals the natural frequency, and the greatest response is for the least amount of damping. The narrowest response is also for the least damping.

It is interesting to note that the widths of the resonance curves shown in [Figure 15.31](#) depend on damping: the less the damping, the narrower the resonance. The consequence is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible. For instance, a radio has a circuit that is used to choose a particular radio station. In this case, the forced damped oscillator consists of a resistor, capacitor, and inductor, which will be discussed later in this course. The circuit is “tuned” to pick a particular radio station. Here it is desirable to have the resonance curve be very narrow, to pick out the exact frequency of the radio station chosen. The narrowness of the graph, and the ability to pick out a certain frequency, is known as the quality of the system. The quality is defined as the spread of the angular frequency, or equivalently, the spread in the frequency, at half the maximum amplitude, divided by the natural frequency ($Q = \frac{\omega_0}{\Delta\omega}$) as shown in [Figure 15.32](#). For a small damping, the quality is approximately equal to $Q \approx \frac{2b}{m}$.

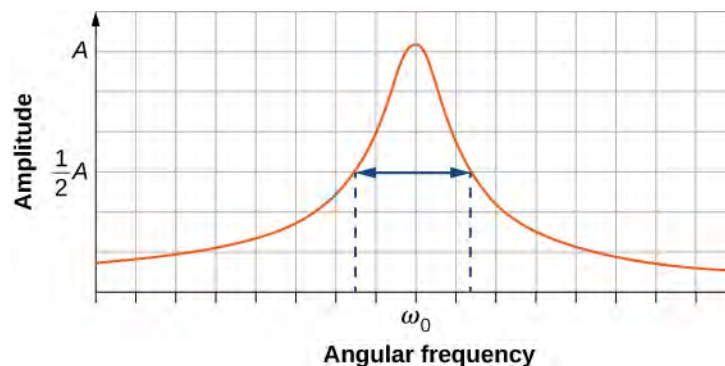


Figure 15.32 The quality of a system is defined as the spread in the frequencies at half the amplitude divided by the natural frequency.

These features of driven harmonic oscillators apply to a huge variety of systems. For instance, magnetic resonance imaging (MRI) is a widely used medical diagnostic tool in which atomic nuclei (mostly hydrogen nuclei or protons) are made to resonate by incoming radio waves (on the order of 100 MHz). In all of these cases, the efficiency of energy transfer from the driving force into the oscillator is best at resonance. [Figure 15.33](#) shows the London Millennium Footbridge that allows pedestrians to cross the River Thames in London. This bridge was nicknamed “Wobbly Bridge” when pedestrians experienced swaying motion while crossing it. The bridge was closed for roughly two years to get rid of this motion.



Figure 15.33 Initially when people crossed the London Millennium Footbridge, they experienced a swaying motion. People continuing to cross reinforced the oscillation's amplitude, thereby increasing the problematic swaying. (credit: Adrian Pingstone/Wikimedia Commons)

✓ CHECK YOUR UNDERSTANDING 15.6

A famous magic trick involves a performer singing a note toward a crystal glass until the glass shatters. Explain why the trick works in terms of resonance and natural frequency.

CHAPTER REVIEW

Key Terms

amplitude (A) maximum displacement from the equilibrium position of an object oscillating around the equilibrium position

critically damped condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

elastic potential energy potential energy stored as a result of deformation of an elastic object, such as the stretching of a spring

equilibrium position position where the spring is neither stretched nor compressed

force constant (k) characteristic of a spring which is defined as the ratio of the force applied to the spring to the displacement caused by the force

frequency (f) number of events per unit of time

natural angular frequency angular frequency of a system oscillating in SHM

oscillation single fluctuation of a quantity, or repeated and regular fluctuations of a quantity, between two extreme values around an equilibrium or average value

overdamped condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system

period (T) time taken to complete one oscillation

periodic motion motion that repeats itself at regular time intervals

phase shift angle, in radians, that is used in a

cosine or sine function to shift the function left or right, used to match up the function with the initial conditions of data

physical pendulum any extended object that swings like a pendulum

resonance large amplitude oscillations in a system produced by a small amplitude driving force, which has a frequency equal to the natural frequency

restoring force force acting in opposition to the force caused by a deformation

simple harmonic motion (SHM) oscillatory motion in a system where the restoring force is proportional to the displacement, which acts in the direction opposite to the displacement

simple harmonic oscillator a device that oscillates in SHM where the restoring force is proportional to the displacement and acts in the direction opposite to the displacement

simple pendulum point mass, called a pendulum bob, attached to a near massless string

stable equilibrium point point where the net force on a system is zero, but a small displacement of the mass will cause a restoring force that points toward the equilibrium point

torsional pendulum any suspended object that oscillates by twisting its suspension

underdamped condition in which damping of an oscillator causes the amplitude of oscillations of a damped harmonic oscillator to decrease over time, eventually approaching zero

Key Equations

Relationship between frequency and period

$$f = \frac{1}{T}$$

Position in SHM with $\phi = 0.00$

$$x(t) = A \cos(\omega t)$$

General position in SHM

$$x(t) = A \cos(\omega t + \phi)$$

General velocity in SHM

$$v(t) = -A\omega \sin(\omega t + \phi)$$

General acceleration in SHM

$$a(t) = -A\omega^2 \cos(\omega t + \phi)$$

Maximum displacement (amplitude) of SHM

$$x_{\max} = A$$

Maximum velocity of SHM

$$|v_{\max}| = A\omega$$

Maximum acceleration of SHM

$$|a_{\max}| = A\omega^2$$

Angular frequency of a mass-spring system in SHM

$$\omega = \sqrt{\frac{k}{m}}$$

Period of a mass-spring system in SHM

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Frequency of a mass-spring system in SHM

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Energy in a mass-spring system in SHM

$$E_{\text{Total}} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

The velocity of the mass in a spring-mass system in SHM

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

The x-component of the radius of a rotating disk

$$x(t) = A\cos(\omega t + \phi)$$

The x-component of the velocity of the edge of a rotating disk

$$v(t) = -v_{\max}\sin(\omega t + \phi)$$

The x-component of the acceleration of the edge of a rotating disk

$$a(t) = -a_{\max}\cos(\omega t + \phi)$$

Force equation for a simple pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

Angular frequency for a simple pendulum

$$\omega = \sqrt{\frac{g}{L}}$$

Period of a simple pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Angular frequency of a physical pendulum

$$\omega = \sqrt{\frac{mgL}{I}}$$

Period of a physical pendulum

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

Period of a torsional pendulum

$$T = 2\pi\sqrt{\frac{I}{k}}$$

Newton's second law for harmonic motion

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Solution for underdamped harmonic motion

$$x(t) = A_0e^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

Natural angular frequency of a mass-spring system

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Angular frequency of underdamped harmonic motion

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Newton's second law for forced,
damped oscillation

Solution to Newton's second law for forced,
damped oscillations

Amplitude of system undergoing forced,
damped oscillations

$$-kx - b\frac{dx}{dt} + F_o \sin(\omega t) = m\frac{d^2x}{dt^2}$$

$$x(t) = A \cos(\omega t + \phi)$$

$$A = \frac{F_o}{\sqrt{m^2(\omega^2 - \omega_o^2)^2 + b^2\omega^2}}$$

Summary

15.1 Simple Harmonic Motion

- Periodic motion is a repeating oscillation. The time for one oscillation is the period T and the number of oscillations per unit time is the frequency f . These quantities are related by $f = \frac{1}{T}$.
- Simple harmonic motion (SHM) is oscillatory motion for a system where the restoring force is proportional to the displacement and acts in the direction opposite to the displacement.
- Maximum displacement is the amplitude A . The angular frequency ω , period T , and frequency f of a simple harmonic oscillator are given by $\omega = \sqrt{\frac{k}{m}}$, $T = 2\pi\sqrt{\frac{m}{k}}$, and $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where m is the mass of the system and k is the force constant.
- Displacement as a function of time in SHM is given by $x(t) = A \cos\left(\frac{2\pi}{T}t + \phi\right) = A \cos(\omega t + \phi)$.
- The velocity is given by $v(t) = -A\omega \sin(\omega t + \phi) = -v_{\max} \sin(\omega t + \phi)$, where $v_{\max} = A\omega = A\sqrt{\frac{k}{m}}$.
- The acceleration is $a(t) = -A\omega^2 \cos(\omega t + \phi) = -a_{\max} \cos(\omega t + \phi)$, where $a_{\max} = A\omega^2 = A\frac{k}{m}$.

15.2 Energy in Simple Harmonic Motion

- The simplest type of oscillations are related to systems that can be described by Hooke's law, $F = -kx$, where F is the restoring force, x is the displacement from equilibrium or deformation, and k is the force constant of the system.
- Elastic potential energy U stored in the deformation of a system that can be described by Hooke's law is given by $U = \frac{1}{2}kx^2$.
- Energy in the simple harmonic oscillator is shared between elastic potential energy and kinetic energy, with the total being constant:

$$E_{\text{Total}} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant.}$$

- The magnitude of the velocity as a function of position for the simple harmonic oscillator can be found by using

$$|v| = \sqrt{\frac{k}{m}(A^2 - x^2)}.$$

15.3 Comparing Simple Harmonic Motion and Circular Motion

- A projection of uniform circular motion undergoes simple harmonic oscillation.
- Consider a circle with a radius A , moving at a constant angular speed ω . A point on the edge of the circle moves at a constant tangential speed of $v_{\max} = A\omega$. The projection of the radius onto the x -axis is $x(t) = A \cos(\omega t + \phi)$, where (ϕ) is the phase shift. The x -component of the tangential velocity is $v(t) = -A\omega \sin(\omega t + \phi)$.

15.4 Pendulums

- A mass m suspended by a wire of length L and negligible mass is a simple pendulum and undergoes SHM for amplitudes less than about 15° . The period of a simple pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the string and g is the acceleration due to gravity.
- The period of a physical pendulum $T = 2\pi\sqrt{\frac{I}{mgL}}$ can be found if the moment of inertia is known. The length between the point of rotation and the center of mass is L .
- The period of a torsional pendulum $T = 2\pi\sqrt{\frac{I}{\kappa}}$ can be found if the moment of inertia and torsion constant are known.

15.5 Damped Oscillations

- Damped harmonic oscillators have non-conservative forces that dissipate their energy.

- Critical damping returns the system to equilibrium as fast as possible without overshooting.
- An underdamped system will oscillate through the equilibrium position.
- An overdamped system moves more slowly toward equilibrium than one that is critically damped.

15.6 Forced Oscillations

- A system's natural frequency is the frequency at

which the system oscillates if not affected by driving or damping forces.

- A periodic force driving a harmonic oscillator at its natural frequency produces resonance. The system is said to resonate.
- The less damping a system has, the higher the amplitude of the forced oscillations near resonance. The more damping a system has, the broader response it has to varying driving frequencies.

Conceptual Questions

15.1 Simple Harmonic Motion

1. What conditions must be met to produce SHM?
2. (a) If frequency is not constant for some oscillation, can the oscillation be SHM? (b) Can you think of any examples of harmonic motion where the frequency may depend on the amplitude?
3. Give an example of a simple harmonic oscillator, specifically noting how its frequency is independent of amplitude.
4. Explain why you expect an object made of a stiff material to vibrate at a higher frequency than a similar object made of a more pliable material.
5. As you pass a freight truck with a trailer on a highway, you notice that its trailer is bouncing up and down slowly. Is it more likely that the trailer is heavily loaded or nearly empty? Explain your answer.
6. Some people modify cars to be much closer to the ground than when manufactured. Should they install stiffer springs? Explain your answer.

15.2 Energy in Simple Harmonic Motion

7. Describe a system in which elastic potential energy is stored.
8. Explain in terms of energy how dissipative forces such as friction reduce the amplitude of a harmonic oscillator. Also explain how a driving mechanism can compensate. (A pendulum clock is such a system.)
9. The temperature of the atmosphere oscillates from a maximum near noontime and a minimum near sunrise. Would you consider the atmosphere to be in stable or unstable equilibrium?

15.3 Comparing Simple Harmonic Motion and Circular Motion

10. Can this analogy of SHM to circular motion be carried out with an object oscillating on a spring vertically hung from the ceiling? Why or why not? If given the choice, would you prefer to use a sine function or a cosine function to model the motion?
11. If the maximum speed of the mass attached to a spring, oscillating on a frictionless table, was increased, what characteristics of the rotating disk would need to be changed?

15.4 Pendulums

12. Pendulum clocks are made to run at the correct rate by adjusting the pendulum's length. Suppose you move from one city to another where the acceleration due to gravity is slightly greater, taking your pendulum clock with you, will you have to lengthen or shorten the pendulum to keep the correct time, other factors remaining constant? Explain your answer.
13. A pendulum clock works by measuring the period of a pendulum. In the springtime the clock runs with perfect time, but in the summer and winter the length of the pendulum changes. When most materials are heated, they expand. Does the clock run too fast or too slow in the summer? What about the winter?
14. With the use of a phase shift, the position of an object may be modeled as a cosine or sine function. If given the option, which function would you choose? Assuming that the phase shift is zero, what are the initial conditions of function; that is, the initial position, velocity, and acceleration, when using a sine function? How about when a cosine function is used?

15.5 Damped Oscillations

15. Give an example of a damped harmonic oscillator. (They are more common than undamped or simple harmonic oscillators.)
16. How would a car bounce after a bump under each of these conditions?
 - (a) overdamping
 - (b) underdamping
 - (c) critical damping
17. Most harmonic oscillators are damped and, if undriven, eventually come to a stop. Why?

15.6 Forced Oscillations

18. Why are soldiers in general ordered to “route step” (walk out of step) across a bridge?

Problems

15.1 Simple Harmonic Motion

21. Prove that using $x(t) = A \sin(\omega t + \phi)$ will produce the same results for the period for the oscillations of a mass and a spring. Why do you think the cosine function was chosen?
22. What is the period of 60.0 Hz of electrical power?
23. If your heart rate is 150 beats per minute during strenuous exercise, what is the time per beat in units of seconds?
24. Find the frequency of a tuning fork that takes 2.50×10^{-3} s to complete one oscillation.
25. A stroboscope is set to flash every 8.00×10^{-5} s. What is the frequency of the flashes?
26. A tire has a tread pattern with a crevice every 2.00 cm. Each crevice makes a single vibration as the tire moves. What is the frequency of these vibrations if the car moves at 30.0 m/s?
27. Each piston of an engine makes a sharp sound every other revolution of the engine. (a) How fast is a race car going if its eight-cylinder engine emits a sound of frequency 750 Hz, given that the engine makes 2000 revolutions per kilometer? (b) At how many revolutions per minute is the engine rotating?
28. A type of cuckoo clock keeps time by having a mass bouncing on a spring, usually something cute like a cherub in a chair. What force constant is needed to produce a period of 0.500 s for a 0.0150-kg mass?
29. A mass m_0 is attached to a spring and hung vertically. The mass is raised a short distance in the vertical direction and released. The mass

19. Do you think there is any harmonic motion in the physical world that is not damped harmonic motion? Try to make a list of five examples of undamped harmonic motion and damped harmonic motion. Which list was easier to make?
20. Some engineers use sound to diagnose performance problems with car engines. Occasionally, a part of the engine is designed that resonates at the frequency of the engine. The unwanted oscillations can cause noise that irritates the driver or could lead to the part failing prematurely. In one case, a part was located that had a length L made of a material with a mass M . What can be done to correct this problem?

oscillates with a frequency f_0 . If the mass is replaced with a mass nine times as large, and the experiment was repeated, what would be the frequency of the oscillations in terms of f_0 ?

30. A 0.500-kg mass suspended from a spring oscillates with a period of 1.50 s. How much mass must be added to the object to change the period to 2.00 s?
31. How much leeway (both percentage and mass) would you have in the selection of the mass of the object in the previous problem if you did not wish the new period to be greater than 2.01 s or less than 1.99 s?

15.2 Energy in Simple Harmonic Motion

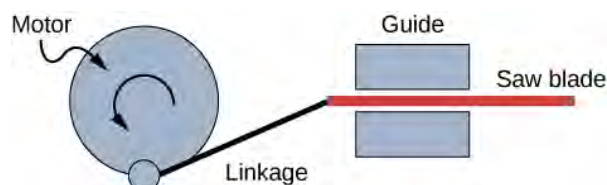
32. Fish are hung on a spring scale to determine their mass. (a) What is the force constant of the spring in such a scale if the spring stretches 8.00 cm for a 10.0 kg load? (b) What is the mass of a fish that stretches the spring 5.50 cm? (c) How far apart are the half-kilogram marks on the scale?
33. It is weigh-in time for the local under-85-kg rugby team. The bathroom scale used to assess eligibility can be described by Hooke's law and is depressed 0.75 cm by its maximum load of 120 kg. (a) What is the spring's effective force constant? (b) A player stands on the scales and depresses it by 0.48 cm. Is he eligible to play on this under-85-kg team?
34. One type of BB gun uses a spring-driven plunger to blow the BB from its barrel. (a) Calculate the force constant of its plunger's spring if you must compress it 0.150 m to drive

the 0.0500-kg plunger to a top speed of 20.0 m/s. (b) What force must be exerted to compress the spring?

35. When an 80.0-kg man stands on a pogo stick, the spring is compressed 0.120 m. (a) What is the force constant of the spring? (b) Will the spring be compressed more when he hops down the road?
36. A spring has a length of 0.200 m when a 0.300-kg mass hangs from it, and a length of 0.750 m when a 1.95-kg mass hangs from it. (a) What is the force constant of the spring? (b) What is the unloaded length of the spring?
37. The length of nylon rope from which a mountain climber is suspended has an effective force constant of 1.40×10^4 N/m. (a) What is the frequency at which he bounces, given his mass plus and the mass of his equipment are 90.0 kg? (b) How much would this rope stretch to break the climber's fall if he free-falls 2.00 m before the rope runs out of slack? (*Hint:* Use conservation of energy.) (c) Repeat both parts of this problem in the situation where twice this length of nylon rope is used.

15.3 Comparing Simple Harmonic Motion and Circular Motion

38. The motion of a mass on a spring hung vertically, where the mass oscillates up and down, can also be modeled using the rotating disk. Instead of the lights being placed horizontally along the top and pointing down, place the lights vertically and have the lights shine on the side of the rotating disk. A shadow will be produced on a nearby wall, and will move up and down. Write the equations of motion for the shadow taking the position at $t = 0.0$ s to be $y = 0.0$ m with the mass moving in the positive y -direction.
39. (a) A novelty clock has a 0.0100-kg-mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position? (b) How many joules of kinetic energy does the object have at its maximum velocity?
40. Reciprocating motion uses the rotation of a motor to produce linear motion up and down or back and forth. This is how a reciprocating saw operates, as shown below.



If the motor rotates at 60 Hz and has a radius of 3.0 cm, estimate the maximum speed of the saw blade as it moves left and right. This design is known as a scotch yoke.

41. A student stands on the edge of a merry-go-round which rotates five times a minute and has a radius of two meters one evening as the sun is setting. The student produces a shadow on the nearby building. (a) Write an equation for the position of the shadow. (b) Write an equation for the velocity of the shadow.

15.4 Pendulums

42. What is the length of a pendulum that has a period of 0.500 s?
43. Some people think a pendulum with a period of 1.00 s can be driven with “mental energy” or psycho kinetically, because its period is the same as an average heartbeat. True or not, what is the length of such a pendulum?
44. What is the period of a 1.00-m-long pendulum?
45. How long does it take a child on a swing to complete one swing if her center of gravity is 4.00 m below the pivot?
46. The pendulum on a cuckoo clock is 5.00-cm long. What is its frequency?
47. Two parakeets sit on a swing with their combined CMs 10.0 cm below the pivot. At what frequency do they swing?
48. (a) A pendulum that has a period of 3.00000 s and that is located where the acceleration due to gravity is 9.79 m/s^2 is moved to a location where the acceleration due to gravity is 9.82 m/s^2 . What is its new period? (b) Explain why so many digits are needed in the value for the period, based on the relation between the period and the acceleration due to gravity.
49. A pendulum with a period of 2.00000 s in one location ($g = 9.80 \text{ m/s}^2$) is moved to a new location where the period is now 1.99796 s. What is the acceleration due to gravity at its new location?
50. (a) What is the effect on the period of a pendulum if you double its length? (b) What is the effect on the period of a pendulum if you decrease its length by 5.00%?

15.5 Damped Oscillations

51. The amplitude of a lightly damped oscillator decreases by 3.0% during each cycle. What percentage of the mechanical energy of the oscillator is lost in each cycle?

15.6 Forced Oscillations

52. How much energy must the shock absorbers of a 1200-kg car dissipate in order to damp a bounce that initially has a velocity of 0.800 m/s at the equilibrium position? Assume the car returns to its original vertical position.
53. If a car has a suspension system with a force constant of 5.00×10^4 N/m, how much energy must the car's shocks remove to dampen an oscillation starting with a maximum displacement of 0.0750 m?
54. (a) How much will a spring that has a force constant of 40.0 N/m be stretched by an object with a mass of 0.500 kg when hung motionless

from the spring? (b) Calculate the decrease in gravitational potential energy of the 0.500-kg object when it descends this distance. (c) Part of this gravitational energy goes into the spring. Calculate the energy stored in the spring by this stretch, and compare it with the gravitational potential energy. Explain where the rest of the energy might go.

55. Suppose you have a 0.750-kg object on a horizontal surface connected to a spring that has a force constant of 150 N/m. There is simple friction between the object and surface with a static coefficient of friction $\mu_s = 0.100$. (a) How far can the spring be stretched without moving the mass? (b) If the object is set into oscillation with an amplitude twice the distance found in part (a), and the kinetic coefficient of friction is $\mu_k = 0.0850$, what total distance does it travel before stopping? Assume it starts at the maximum amplitude.

Additional Problems

56. Suppose you attach an object with mass m to a vertical spring originally at rest, and let it bounce up and down. You release the object from rest at the spring's original rest length, the length of the spring in equilibrium, without the mass attached. The amplitude of the motion is the distance between the equilibrium position of the spring without the mass attached and the equilibrium position of the spring with the mass attached. (a) Show that the spring exerts an upward force of $2.00mg$ on the object at its lowest point. (b) If the spring has a force constant of 10.0 N/m, is hung horizontally, and the position of the free end of the spring is marked as $y = 0.00$ m, where is the new equilibrium position if a 0.25-kg-mass object is hung from the spring? (c) If the spring has a force constant of 10.0 N/m and a 0.25-kg-mass object is set in motion as described, find the amplitude of the oscillations. (d) Find the maximum velocity.
57. A diver on a diving board is undergoing SHM. Her mass is 55.0 kg and the period of her motion is 0.800 s. The next diver is a male whose period of simple harmonic oscillation is 1.05 s. What is his mass if the mass of the board is negligible?
58. Suppose a diving board with no one on it bounces up and down in a SHM with a frequency of 4.00 Hz. The board has an effective mass of 10.0 kg. What is the frequency of the SHM of a 75.0-kg diver on the board?

59. The device pictured in the following figure entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring. (a) If the spring stretches 0.250 m while supporting an 8.0-kg child, what is its force constant? (b) What is the time for one complete bounce of this child? (c) What is the child's maximum velocity if the amplitude of her bounce is 0.200 m?



Figure 15.34 (credit: Lisa Doehnert)

60. A mass is placed on a frictionless, horizontal table. A spring ($k = 100 \text{ N/m}$), which can be stretched or compressed, is placed on the table. A 5.00-kg mass is attached to one end of the spring, the other end is anchored to the wall. The equilibrium position is marked at zero. A student moves the mass out to $x = 4.00 \text{ cm}$ and releases it from rest. The mass oscillates in SHM. (a) Determine the equations of motion. (b) Find the position, velocity, and acceleration of the mass at time $t = 3.00 \text{ s}$.
61. Find the ratio of the new/old periods of a pendulum if the pendulum were transported from Earth to the Moon, where the acceleration due to gravity is 1.63 m/s^2 .
62. At what rate will a pendulum clock run on the Moon, where the acceleration due to gravity is 1.63 m/s^2 , if it keeps time accurately on Earth? That is, find the time (in hours) it takes the clock's hour hand to make one revolution on the Moon.
63. If a pendulum-driven clock gains 5.00 s/day, what fractional change in pendulum length must be made for it to keep perfect time?
64. A 2.00-kg object hangs, at rest, on a 1.00-m-long string attached to the ceiling. A 100-g mass is fired with a speed of 20 m/s at the 2.00-kg mass, and the 100.00-g mass collides perfectly elastically with the 2.00-kg mass. Write an equation for the motion of the hanging mass after the collision. Assume air resistance is negligible.
65. A 2.00-kg object hangs, at rest, on a 1.00-m-long string attached to the ceiling. A 100-g object is fired with a speed of 20 m/s at the 2.00-kg object, and the two objects collide and stick together in a totally inelastic collision. Write an equation for the motion of the system after the collision. Assume air resistance is negligible.
66. Assume that a pendulum used to drive a grandfather clock has a length $L_0 = 1.00 \text{ m}$ and a mass M at temperature $T = 20.00^\circ\text{C}$. It can be modeled as a physical pendulum as a rod oscillating around one end. By what percentage will the period change if the temperature increases by 10°C ? Assume the length of the rod changes linearly with temperature, where $L = L_0 (1 + \alpha\Delta T)$ and the rod is made of brass ($\alpha = 18 \times 10^{-6}^\circ\text{C}^{-1}$).
67. A 2.00-kg block lies at rest on a frictionless table. A spring, with a spring constant of 100 N/m is attached to the wall and to the block. A second block of 0.50 kg is placed on top of the first block. The 2.00-kg block is gently pulled to a position $x = +A$ and released from rest. There is a coefficient of friction of 0.45 between the two blocks. (a) What is the period of the oscillations? (b) What is the largest amplitude of motion that will allow the blocks to oscillate without the 0.50-kg block sliding off?

Challenge Problems

- 68.** A suspension bridge oscillates with an effective force constant of $1.00 \times 10^8 \text{ N/m}$. (a) How much energy is needed to make it oscillate with an amplitude of 0.100 m? (b) If soldiers march across the bridge with a cadence equal to the bridge's natural frequency and impart $1.00 \times 10^4 \text{ J}$ of energy each second, how long does it take for the bridge's oscillations to go from 0.100 m to 0.500 m amplitude.
- 69.** Near the top of the Citigroup Center building in New York City, there is an object with mass of $4.00 \times 10^5 \text{ kg}$ on springs that have adjustable force constants. Its function is to dampen wind-driven oscillations of the building by oscillating at the same frequency as the building is being driven—the driving force is transferred to the object, which oscillates instead of the entire building. (a) What effective force constant should the springs have to make the object oscillate with a period of 2.00 s? (b) What energy is stored in the springs for a 2.00-m displacement from equilibrium?
- 70.** Parcels of air (small volumes of air) in a stable atmosphere (where the temperature increases with height) can oscillate up and down, due to the restoring force provided by the buoyancy of the air parcel. The frequency of the oscillations are a measure of the stability of the atmosphere. Assuming that the acceleration of an air parcel can be modeled as $\frac{\partial^2 z'}{\partial t^2} = \frac{g}{\rho_0} \frac{\partial \rho(z)}{\partial z} z'$, prove that $z' = z_0' e^{t\sqrt{-N^2}}$ is a solution, where N is known as the Brunt-Väisälä frequency. Note that in a stable atmosphere, the density decreases with height and parcel oscillates up and down.
- 71.** Consider the van der Waals potential $U(r) = U_o \left[\left(\frac{R_o}{r} \right)^{12} - 2 \left(\frac{R_o}{r} \right)^6 \right]$, used to model the potential energy function of two molecules, where the minimum potential is at $r = R_o$. Find the force as a function of r . Consider a small displacement $r = R_o + r'$ and use the binomial theorem:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots,$$
to show that the force does approximate a Hooke's law force.
- 72.** Suppose the length of a clock's pendulum is changed by 1.000%, exactly at noon one day. What time will the clock read 24.00 hours later, assuming it the pendulum has kept perfect time before the change? Note that there are two answers, and perform the calculation to four-digit precision.
- 73.** (a) The springs of a pickup truck act like a single spring with a force constant of $1.30 \times 10^5 \text{ N/m}$. By how much will the truck be depressed by its maximum load of 1000 kg? (b) If the pickup truck has four identical springs, what is the force constant of each?

CHAPTER 16

Waves

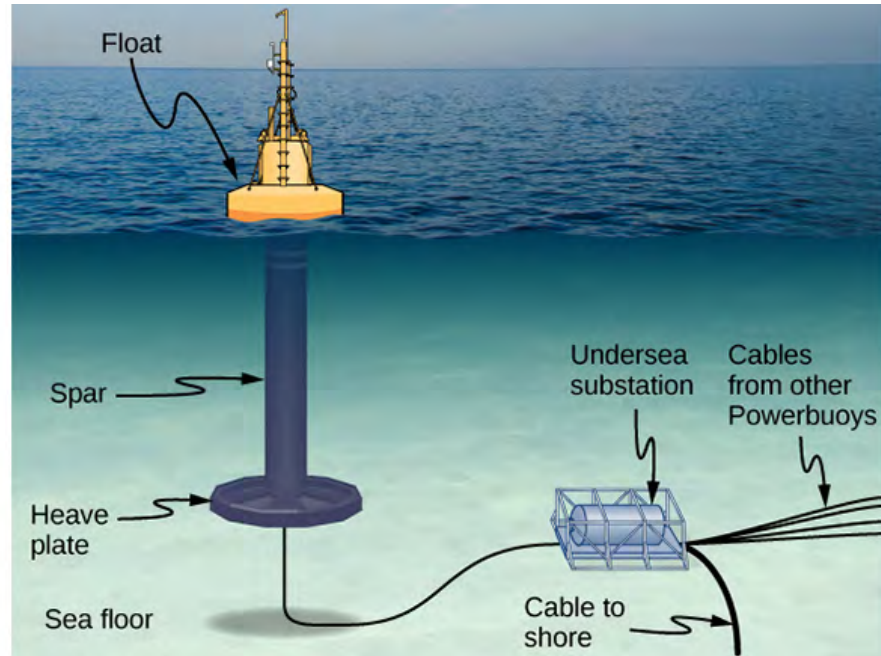


Figure 16.1 From the world of renewable energy sources comes the electric power-generating buoy. Although there are many versions, this one converts the up-and-down motion, as well as side-to-side motion, of the buoy into rotational motion in order to turn an electric generator, which stores the energy in batteries.

Chapter Outline

[16.1 Traveling Waves](#)

[16.2 Mathematics of Waves](#)

[16.3 Wave Speed on a Stretched String](#)

[16.4 Energy and Power of a Wave](#)

[16.5 Interference of Waves](#)

[16.6 Standing Waves and Resonance](#)

INTRODUCTION In this chapter, we study the physics of wave motion. We concentrate on mechanical waves, which are disturbances that move through a medium such as air or water. Like simple harmonic motion studied in the preceding chapter, the energy transferred through the medium is proportional to the amplitude squared. Surface water waves in the ocean are transverse waves in which the energy of the wave travels horizontally while the water oscillates up and down due to some restoring force. In the picture above, a buoy is used to convert the awesome power of ocean waves into electricity. The up-and-down motion of the buoy generated as the waves pass is converted into rotational motion that turns a rotor in an electric generator. The generator charges batteries, which are in turn used to provide a consistent energy source for the end user. This model was successfully tested by the US Navy in a project to provide power to coastal security networks and

was able to provide an average power of 350 W. The buoy survived the difficult ocean environment, including operation off the New Jersey coast through Hurricane Irene in 2011.

The concepts presented in this chapter will be the foundation for many interesting topics, from the transmission of information to the concepts of quantum mechanics.

16.1 Traveling Waves

Learning Objectives

By the end of this section, you will be able to:

- Describe the basic characteristics of wave motion
- Define the terms wavelength, amplitude, period, frequency, and wave speed
- Explain the difference between longitudinal and transverse waves, and give examples of each type
- List the different types of waves

We saw in [Oscillations](#) that oscillatory motion is an important type of behavior that can be used to model a wide range of physical phenomena. Oscillatory motion is also important because oscillations can generate waves, which are of fundamental importance in physics. Many of the terms and equations we studied in the chapter on oscillations apply equally well to wave motion ([Figure 16.2](#)).



Figure 16.2 An ocean wave is probably the first picture that comes to mind when you hear the word “wave.” Although this breaking wave, and ocean waves in general, have apparent similarities to the basic wave characteristics we will discuss, the mechanisms driving ocean waves are highly complex and beyond the scope of this chapter. It may seem natural, and even advantageous, to apply the concepts in this chapter to ocean waves, but ocean waves are nonlinear, and the simple models presented in this chapter do not fully explain them. (credit: Steve Jurvetson)

Types of Waves

A **wave** is a disturbance that propagates, or moves from the place it was created. There are three basic types of waves: mechanical waves, electromagnetic waves, and matter waves.

Basic **mechanical waves** are governed by Newton’s laws and require a medium. A medium is the substance a mechanical wave propagates through, and the medium produces an elastic restoring force when it is deformed. Mechanical waves transfer energy and momentum, without transferring mass. Some examples of mechanical waves are water waves, sound waves, and seismic waves. The medium for water waves is water; for sound waves, the medium is usually air. (Sound waves can travel in other media as well; we will look at that in more detail in [Sound](#).) For surface water waves, the disturbance occurs on the surface of the water, perhaps

created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker or a vibrating tuning fork. In both cases, the disturbance is the oscillation of the molecules of the fluid. In mechanical waves, energy and momentum transfer with the motion of the wave, whereas the mass oscillates around an equilibrium point. (We discuss this in [Energy and Power of a Wave](#).) Earthquakes generate seismic waves from several types of disturbances, including the disturbance of Earth's surface and pressure disturbances under the surface. Seismic waves travel through the solids and liquids that form Earth. In this chapter, we focus on mechanical waves.

Electromagnetic waves are associated with oscillations in electric and magnetic fields and do not require a medium. Examples include gamma rays, X-rays, ultraviolet waves, visible light, infrared waves, microwaves, and radio waves. Electromagnetic waves can travel through a vacuum at the speed of light, $v = c = 2.99792458 \times 10^8$ m/s. For example, light from distant stars travels through the vacuum of space and reaches Earth. Electromagnetic waves have some characteristics that are similar to mechanical waves; they are covered in more detail in [Electromagnetic Waves](#).

Matter waves are a central part of the branch of physics known as quantum mechanics. These waves are associated with protons, electrons, neutrons, and other fundamental particles found in nature. The theory that all types of matter have wave-like properties was first proposed by Louis de Broglie in 1924. Matter waves are discussed in [Photons and Matter Waves](#).

Mechanical Waves

Mechanical waves exhibit characteristics common to all waves, such as amplitude, wavelength, period, frequency, and energy. All wave characteristics can be described by a small set of underlying principles.

The simplest mechanical waves repeat themselves for several cycles and are associated with simple harmonic motion. These simple harmonic waves can be modeled using some combination of sine and cosine functions. For example, consider the simplified surface water wave that moves across the surface of water as illustrated in [Figure 16.3](#). Unlike complex ocean waves, in surface water waves, the medium, in this case water, moves vertically, oscillating up and down, whereas the disturbance of the wave moves horizontally through the medium. In [Figure 16.3](#), the waves causes a seagull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The crest is the highest point of the wave, and the trough is the lowest part of the wave. The time for one complete oscillation of the up-and-down motion is the wave's period T . The wave's frequency is the number of waves that pass through a point per unit time and is equal to $f = 1/T$. The period can be expressed using any convenient unit of time but is usually measured in seconds; frequency is usually measured in hertz (Hz), where $1 \text{ Hz} = 1 \text{ s}^{-1}$.

The length of the wave is called the **wavelength** and is represented by the Greek letter lambda (λ), which is measured in any convenient unit of length, such as a centimeter or meter. The wavelength can be measured between any two similar points along the medium that have the same height and the same slope. In [Figure 16.3](#), the wavelength is shown measured between two crests. As stated above, the period of the wave is equal to the time for one oscillation, but it is also equal to the time for one wavelength to pass through a point along the wave's path.

The amplitude of the wave (A) is a measure of the maximum displacement of the medium from its equilibrium position. In the figure, the equilibrium position is indicated by the dotted line, which is the height of the water if there were no waves moving through it. In this case, the wave is symmetrical, the crest of the wave is a distance $+A$ above the equilibrium position, and the trough is a distance $-A$ below the equilibrium position. The units for the amplitude can be centimeters or meters, or any convenient unit of distance.

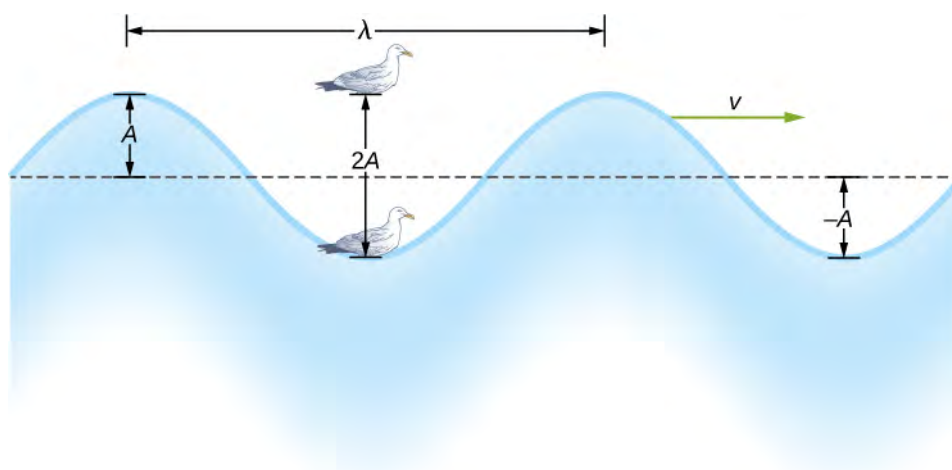


Figure 16.3 An idealized surface water wave passes under a seagull that bobs up and down in simple harmonic motion. The wave has a wavelength λ , which is the distance between adjacent identical parts of the wave. The amplitude A of the wave is the maximum displacement of the wave from the equilibrium position, which is indicated by the dotted line. In this example, the medium moves up and down, whereas the disturbance of the surface propagates parallel to the surface at a speed v .

The water wave in the figure moves through the medium with a propagation velocity $\vec{v}$. The magnitude of the **wave velocity** is the distance the wave travels in a given time, which is one wavelength in the time of one period, and the **wave speed** is the magnitude of wave velocity. In equation form, this is

$$v = \frac{\lambda}{T} = \lambda f. \quad 16.1$$

This fundamental relationship holds for all types of waves. For water waves, v is the speed of a surface wave; for sound, v is the speed of sound; and for visible light, v is the speed of light.

Transverse and Longitudinal Waves

We have seen that a simple mechanical wave consists of a periodic disturbance that propagates from one place to another through a medium. In [Figure 16.4\(a\)](#), the wave propagates in the horizontal direction, whereas the medium is disturbed in the vertical direction. Such a wave is called a **transverse wave**. In a transverse wave, the wave may propagate in any direction, but the disturbance of the medium is perpendicular to the direction of propagation. In contrast, in a **longitudinal wave** or compressional wave, the disturbance is parallel to the direction of propagation. [Figure 16.4\(b\)](#) shows an example of a longitudinal wave. The size of the disturbance is its amplitude A and is completely independent of the speed of propagation v .

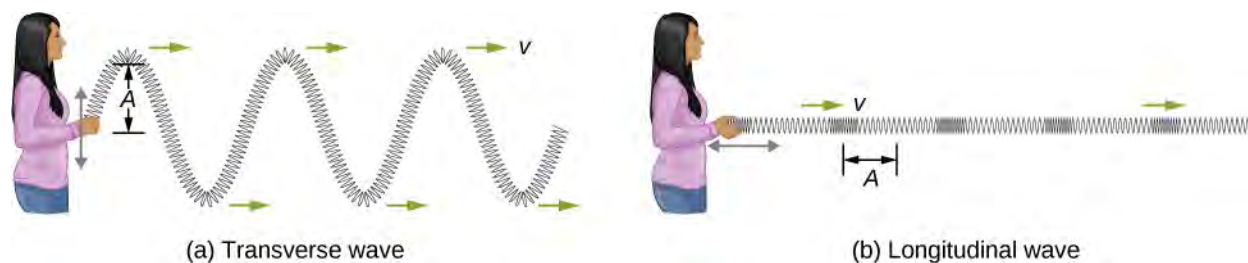


Figure 16.4 (a) In a transverse wave, the medium oscillates perpendicular to the wave velocity. Here, the spring moves vertically up and down, while the wave propagates horizontally to the right. (b) In a longitudinal wave, the medium oscillates parallel to the propagation of the wave. In this case, the spring oscillates back and forth, while the wave propagates to the right.

A simple graphical representation of a section of the spring shown in [Figure 16.4\(b\)](#) is shown in [Figure 16.5](#). [Figure 16.5\(a\)](#) shows the equilibrium position of the spring before any waves move down it. A point on the spring is marked with a blue dot. [Figure 16.5\(b\)](#) through (g) show snapshots of the spring taken one-quarter of a period apart, sometime after the end of the spring is oscillated back and forth in the x -direction at a

constant frequency. The disturbance of the wave is seen as the compressions and the expansions of the spring. Note that the blue dot oscillates around its equilibrium position a distance A , as the longitudinal wave moves in the positive x -direction with a constant speed. The distance A is the amplitude of the wave. The y -position of the dot does not change as the wave moves through the spring. The wavelength of the wave is measured in part (d). The wavelength depends on the speed of the wave and the frequency of the driving force.

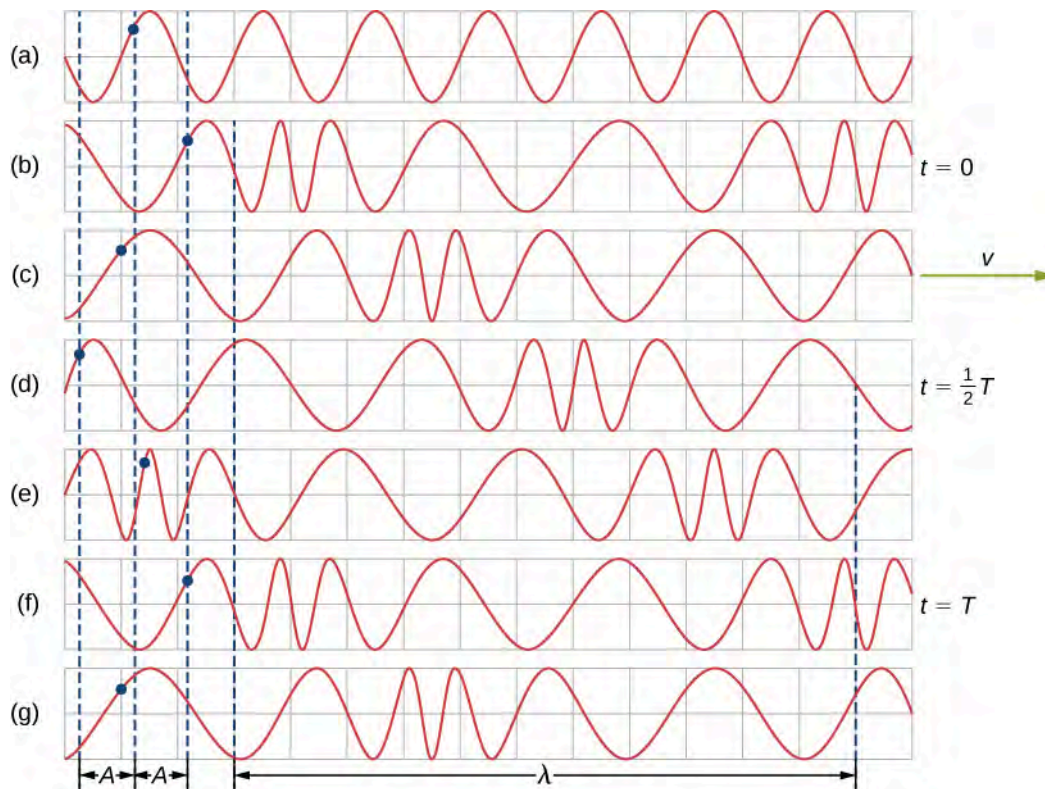


Figure 16.5 (a) This is a simple, graphical representation of a section of the stretched spring shown in Figure 16.4(b), representing the spring's equilibrium position before any waves are induced on the spring. A point on the spring is marked by a blue dot. (b–g) Longitudinal waves are created by oscillating the end of the spring (not shown) back and forth along the x -axis. The longitudinal wave, with a wavelength λ , moves along the spring in the $+x$ -direction with a wave speed v . For convenience, the wavelength is measured in (d). Note that the point on the spring that was marked with the blue dot moves back and forth a distance A from the equilibrium position, oscillating around the equilibrium position of the point.

Waves may be transverse, longitudinal, or a combination of the two. Examples of transverse waves are the waves on stringed instruments or surface waves on water, such as ripples moving on a pond. Sound waves in air and water are longitudinal. With sound waves, the disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and for this reason, the sound waves in them are longitudinal waves. Sound in solids can have both longitudinal and transverse components, such as those in a seismic wave. Earthquakes generate seismic waves under Earth's surface with both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). The components of seismic waves have important individual characteristics—they propagate at different speeds, for example. Earthquakes also have surface waves that are similar to surface waves on water. Ocean waves also have both transverse and longitudinal components.



EXAMPLE 16.1

Wave on a String

A student takes a 30.00-m-long string and attaches one end to the wall in the physics lab. The student then holds the free end of the rope, keeping the tension constant in the rope. The student then begins to send waves

down the string by moving the end of the string up and down with a frequency of 2.00 Hz. The maximum displacement of the end of the string is 20.00 cm. The first wave hits the lab wall 6.00 s after it was created. (a) What is the speed of the wave? (b) What is the period of the wave? (c) What is the wavelength of the wave?

Strategy

- The speed of the wave can be derived by dividing the distance traveled by the time.
- The period of the wave is the inverse of the frequency of the driving force.
- The wavelength can be found from the speed and the period $v = \lambda/T$.

Solution

- The first wave traveled 30.00 m in 6.00 s:

$$v = \frac{30.00 \text{ m}}{6.00 \text{ s}} = 5.00 \frac{\text{m}}{\text{s}}.$$

- The period is equal to the inverse of the frequency:

$$T = \frac{1}{f} = \frac{1}{2.00 \text{ s}^{-1}} = 0.50 \text{ s}.$$

- The wavelength is equal to the velocity times the period:

$$\lambda = vT = 5.00 \frac{\text{m}}{\text{s}} (0.50 \text{ s}) = 2.50 \text{ m}.$$

Significance

The frequency of the wave produced by an oscillating driving force is equal to the frequency of the driving force.

CHECK YOUR UNDERSTANDING 16.1

When a guitar string is plucked, the guitar string oscillates as a result of waves moving through the string. The vibrations of the string cause the air molecules to oscillate, forming sound waves. The frequency of the sound waves is equal to the frequency of the vibrating string. Is the wavelength of the sound wave always equal to the wavelength of the waves on the string?

EXAMPLE 16.2

Characteristics of a Wave

A transverse mechanical wave propagates in the positive x -direction through a spring (as shown in [Figure 16.4\(a\)](#)) with a constant wave speed, and the medium oscillates between $+A$ and $-A$ around an equilibrium position. The graph in [Figure 16.6](#) shows the height of the spring (y) versus the position (x), where the x -axis points in the direction of propagation. The figure shows the height of the spring versus the x -position at $t = 0.00 \text{ s}$ as a dotted line and the wave at $t = 3.00 \text{ s}$ as a solid line. Assume the wave has not traveled more than 1 wavelength in this time. (a) Determine the wavelength and amplitude of the wave. (b) Find the propagation velocity of the wave. (c) Calculate the period and frequency of the wave.

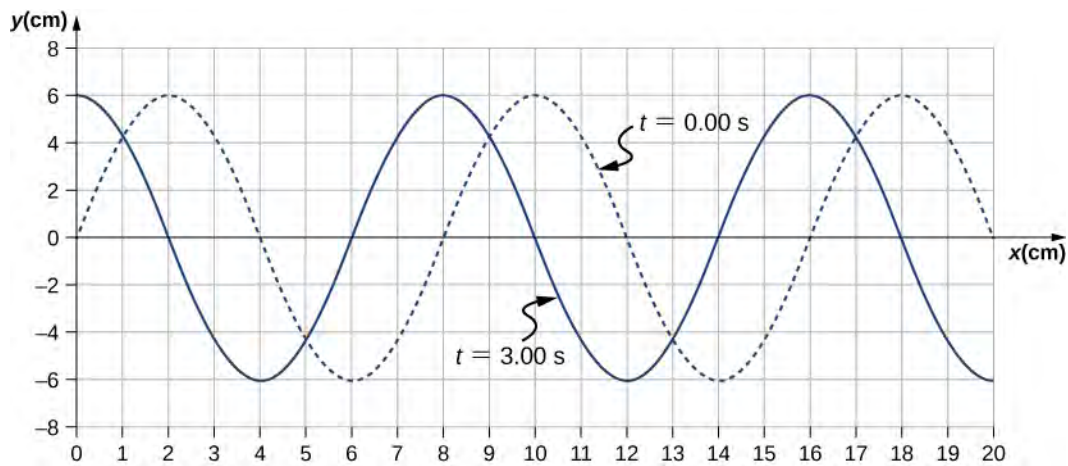


Figure 16.6 A transverse wave shown at two instants of time.

Strategy

- The amplitude and wavelength can be determined from the graph.
- Since the velocity is constant, the velocity of the wave can be found by dividing the distance traveled by the wave by the time it took the wave to travel the distance.
- The period can be found from $v = \frac{\lambda}{T}$ and the frequency from $f = \frac{1}{T}$.

Solution

- Read the wavelength from the graph, looking at the purple arrow in Figure 16.7. Read the amplitude by looking at the green arrow. The wavelength is $\lambda = 8.00$ cm and the amplitude is $A = 6.00$ cm.

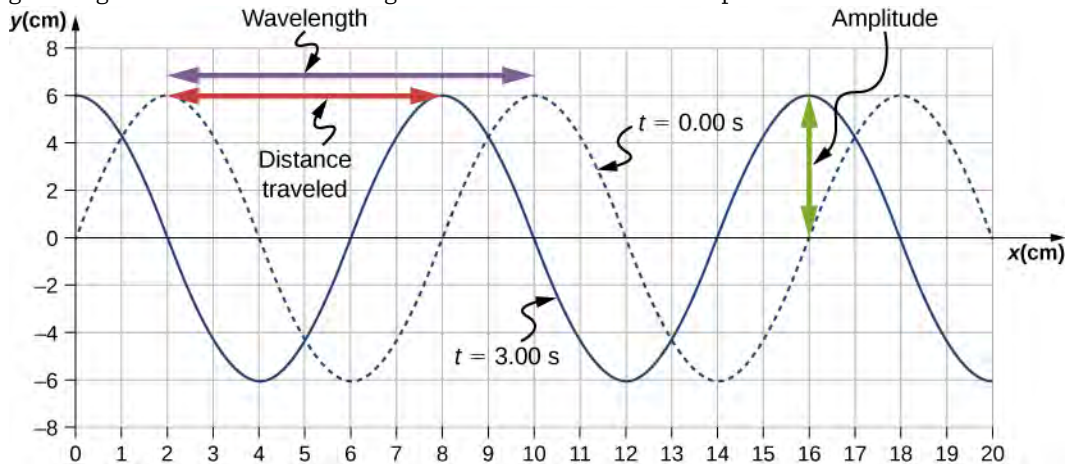


Figure 16.7 Characteristics of the wave marked on a graph of its displacement.

- The distance the wave traveled from time $t = 0.00$ s to time $t = 3.00$ s can be seen in the graph. Consider the red arrow, which shows the distance the crest has moved in 3 s. The distance is

8.00 cm – 2.00 cm = 6.00 cm. The velocity is

$$v = \frac{\Delta x}{\Delta t} = \frac{8.00 \text{ cm} - 2.00 \text{ cm}}{3.00 \text{ s} - 0.00 \text{ s}} = 2.00 \text{ cm/s}.$$

- The period is $T = \frac{\lambda}{v} = \frac{8.00 \text{ cm}}{2.00 \text{ cm/s}} = 4.00$ s and the frequency is $f = \frac{1}{T} = \frac{1}{4.00 \text{ s}} = 0.25$ Hz.

Significance

Note that the wavelength can be found using any two successive identical points that repeat, having the same height and slope. You should choose two points that are most convenient. The displacement can also be found using any convenient point.

✓ CHECK YOUR UNDERSTANDING 16.2

The propagation velocity of a transverse or longitudinal mechanical wave may be constant as the wave disturbance moves through the medium. Consider a transverse mechanical wave: Is the velocity of the medium also constant?

16.2 Mathematics of Waves

Learning Objectives

By the end of this section, you will be able to:

- Model a wave, moving with a constant wave velocity, with a mathematical expression
- Calculate the velocity and acceleration of the medium
- Show how the velocity of the medium differs from the wave velocity (propagation velocity)

In the previous section, we described periodic waves by their characteristics of wavelength, period, amplitude, and wave speed of the wave. Waves can also be described by the motion of the particles of the medium through which the waves move. The position of particles of the medium can be mathematically modeled as **wave functions**, which can be used to find the position, velocity, and acceleration of the particles of the medium of the wave at any time.

Pulses

A **pulse** can be described as wave consisting of a single disturbance that moves through the medium with a constant amplitude. The pulse moves as a pattern that maintains its shape as it propagates with a constant wave speed. Because the wave speed is constant, the distance the pulse moves in a time Δt is equal to $\Delta x = v\Delta t$ (Figure 16.8).

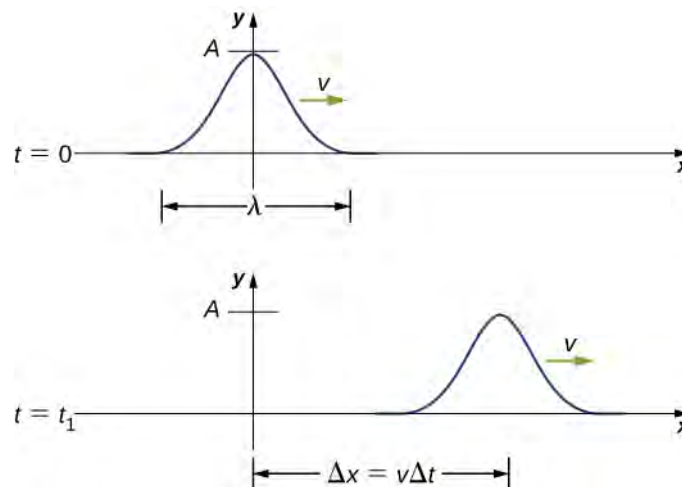


Figure 16.8 The pulse at time $t = 0$ is centered on $x = 0$ with amplitude A . The pulse moves as a pattern with a constant shape, with a constant maximum value A . The velocity is constant and the pulse moves a distance $\Delta x = v\Delta t$ in a time Δt . The distance traveled is measured with any convenient point on the pulse. In this figure, the crest is used.

Modeling a One-Dimensional Sinusoidal Wave using a Wave Function

Consider a string kept at a constant tension F_T where one end is fixed and the free end is oscillated between $y = +A$ and $y = -A$ by a mechanical device at a constant frequency. Figure 16.9 shows snapshots of the wave at an interval of an eighth of a period, beginning after one period ($t = T$).

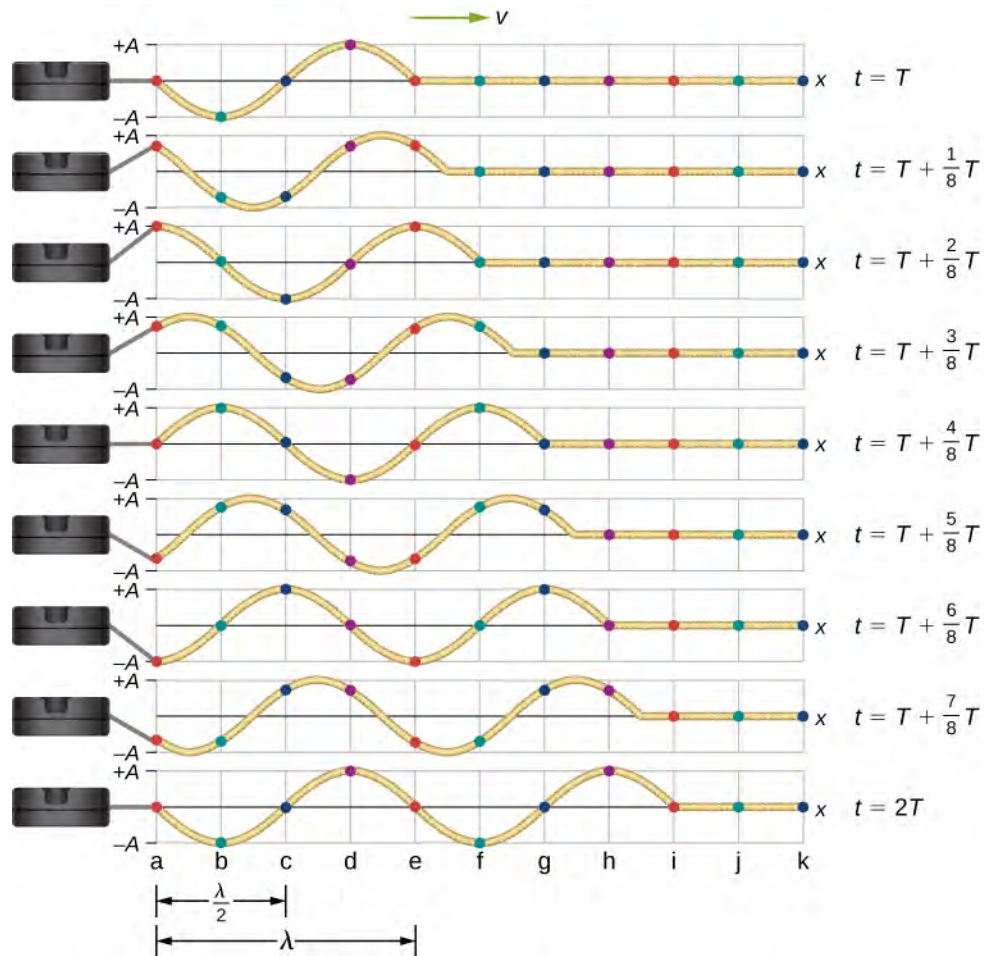


Figure 16.9 Snapshots of a transverse wave moving through a string under tension, beginning at time $t = T$ and taken at intervals of $\frac{1}{8}T$. Colored dots are used to highlight points on the string. Points that are a wavelength apart in the x -direction are highlighted with the same color dots.

Notice that each select point on the string (marked by colored dots) oscillates up and down in simple harmonic motion, between $y = +A$ and $y = -A$, with a period T . The wave on the string is sinusoidal and is translating in the positive x -direction as time progresses.

At this point, it is useful to recall from your study of algebra that if $f(x)$ is some function, then $f(x - d)$ is the same function translated in the positive x -direction by a distance d . The function $f(x + d)$ is the same function translated in the negative x -direction by a distance d . We want to define a wave function that will give the y -position of each segment of the string for every position x along the string for every time t .

Looking at the first snapshot in [Figure 16.9](#), the y -position of the string between $x = 0$ and $x = \lambda$ can be modeled as a sine function. This wave propagates down the string one wavelength in one period, as seen in the last snapshot. The wave therefore moves with a constant wave speed of $v = \lambda/T$.

Recall that a sine function is a function of the angle θ , oscillating between $+1$ and -1 , and repeating every 2π radians ([Figure 16.10](#)). However, the y -position of the medium, or the wave function, oscillates between $+A$ and $-A$, and repeats every wavelength λ .

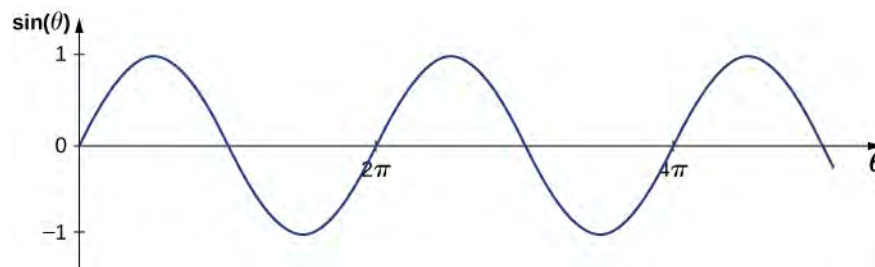


Figure 16.10 A sine function oscillates between +1 and −1 every 2π radians.

To construct our model of the wave using a periodic function, consider the ratio of the angle and the position,

$$\begin{aligned}\frac{\theta}{x} &= \frac{2\pi}{\lambda}, \\ \theta &= \frac{2\pi}{\lambda}x.\end{aligned}$$

Using $\theta = \frac{2\pi}{\lambda}x$ and multiplying the sine function by the amplitude A , we can now model the y -position of the string as a function of the position x :

$$y(x) = A \sin\left(\frac{2\pi}{\lambda}x\right).$$

The wave on the string travels in the positive x -direction with a constant velocity v , and moves a distance vt in a time t . The wave function can now be defined by

$$y(x, t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right).$$

It is often convenient to rewrite this wave function in a more compact form. Multiplying through by the ratio $\frac{2\pi}{\lambda}$ leads to the equation

$$y(x, t) = A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}vt\right).$$

The value $\frac{2\pi}{\lambda}$ is defined as the **wave number**. The symbol for the wave number is k and has units of inverse meters, m^{-1} :

$$k \equiv \frac{2\pi}{\lambda} \quad 16.2$$

Recall from [Oscillations](#) that the angular frequency is defined as $\omega \equiv \frac{2\pi}{T}$. The second term of the wave function becomes

$$\frac{2\pi}{\lambda}vt = \frac{2\pi}{\lambda}\left(\frac{\lambda}{T}\right)t = \frac{2\pi}{T}t = \omega t.$$

The wave function for a simple harmonic wave on a string reduces to

$$y(x, t) = A \sin(kx \mp \omega t),$$

where A is the amplitude, $k = \frac{2\pi}{\lambda}$ is the wave number, $\omega = \frac{2\pi}{T}$ is the angular frequency, the minus sign is for waves moving in the positive x -direction, and the plus sign is for waves moving in the negative x -direction. The velocity of the wave is equal to

$$v = \frac{\lambda}{T} = \frac{\lambda}{T}\left(\frac{2\pi}{2\pi}\right) = \frac{\omega}{k}. \quad 16.3$$

Think back to our discussion of a mass on a spring, when the position of the mass was modeled as

$x(t) = A \cos(\omega t + \phi)$. The angle ϕ is a phase shift, added to allow for the fact that the mass may have initial conditions other than $x = +A$ and $v = 0$. For similar reasons, the initial phase is added to the wave function. The wave function modeling a sinusoidal wave, allowing for an initial phase shift ϕ , is

$$y(x, t) = A \sin(kx \mp \omega t + \phi) \quad 16.4$$

The value

$$(kx \mp \omega t + \phi) \quad 16.5$$

is known as the phase of the wave, where ϕ is the initial phase of the wave function. Whether the temporal term ωt is negative or positive depends on the direction of the wave. First consider the minus sign for a wave with an initial phase equal to zero ($\phi = 0$). The phase of the wave would be $(kx - \omega t)$. Consider following a point on a wave, such as a crest. A crest will occur when $\sin(kx - \omega t) = 1.00$, that is, when $kx - \omega t = n\pi + \frac{\pi}{2}$, for any integral value of n . For instance, one particular crest occurs at $kx - \omega t = \frac{\pi}{2}$. As the wave moves, time increases and x must also increase to keep the phase equal to $\frac{\pi}{2}$. Therefore, the minus sign is for a wave moving in the positive x -direction. Using the plus sign, $kx + \omega t = \frac{\pi}{2}$. As time increases, x must decrease to keep the phase equal to $\frac{\pi}{2}$. The plus sign is used for waves moving in the negative x -direction. In summary, $y(x, t) = A \sin(kx - \omega t + \phi)$ models a wave moving in the positive x -direction and $y(x, t) = A \sin(kx + \omega t + \phi)$ models a wave moving in the negative x -direction.

Equation 16.4 is known as a simple harmonic wave function. A wave function is any function such that $f(x, t) = f(x - vt)$. Later in this chapter, we will see that it is a solution to the linear wave equation. Note that $y(x, t) = A \cos(kx + \omega t + \phi')$ works equally well because it corresponds to a different phase shift $\phi' = \phi - \frac{\pi}{2}$.



PROBLEM-SOLVING STRATEGY

Finding the Characteristics of a Sinusoidal Wave

1. To find the amplitude, wavelength, period, and frequency of a sinusoidal wave, write down the wave function in the form $y(x, t) = A \sin(kx - \omega t + \phi)$.
2. The amplitude can be read straight from the equation and is equal to A .
3. The period of the wave can be derived from the angular frequency ($T = \frac{2\pi}{\omega}$).
4. The frequency can be found using $f = \frac{1}{T}$.
5. The wavelength can be found using the wave number ($\lambda = \frac{2\pi}{k}$).



EXAMPLE 16.3

Characteristics of a Traveling Wave on a String

A transverse wave on a taut string is modeled with the wave function

$$y(x, t) = A \sin(kx - \omega t) = 0.2 \text{ m} \sin(6.28 \text{ m}^{-1}x - 1.57 \text{ s}^{-1}t).$$

Find the amplitude, wavelength, period, and speed of the wave.

Strategy

All these characteristics of the wave can be found from the constants included in the equation or from simple combinations of these constants.

Solution

1. The amplitude, wave number, and angular frequency can be read directly from the wave equation:

$$y(x, t) = A \sin(kx - \omega t) = 0.2 \text{ m} \sin(6.28 \text{ m}^{-1}x - 1.57 \text{ s}^{-1}t).$$

$$(A = 0.2 \text{ m}; k = 6.28 \text{ m}^{-1}; \omega = 1.57 \text{ s}^{-1})$$

2. The wave number can be used to find the wavelength:

$$k = \frac{2\pi}{\lambda}.$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{6.28 \text{ m}^{-1}} = 1.0 \text{ m}.$$

3. The period of the wave can be found using the angular frequency:

$$\omega = \frac{2\pi}{T}.$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1.57 \text{ s}^{-1}} = 4 \text{ s}.$$

4. The speed of the wave can be found using the wave number and the angular frequency. The direction of the wave can be determined by considering the sign of $kx \mp \omega t$: A negative sign suggests that the wave is moving in the positive x -direction:

$$|v| = \frac{\omega}{k} = \frac{1.57 \text{ s}^{-1}}{6.28 \text{ m}^{-1}} = 0.25 \text{ m/s}.$$

Significance

All of the characteristics of the wave are contained in the wave function. Note that the wave speed is the speed of the wave in the direction parallel to the motion of the wave. Plotting the height of the medium y versus the position x for two times $t = 0.00 \text{ s}$ and $t = 0.80 \text{ s}$ can provide a graphical visualization of the wave ([Figure 16.11](#)).

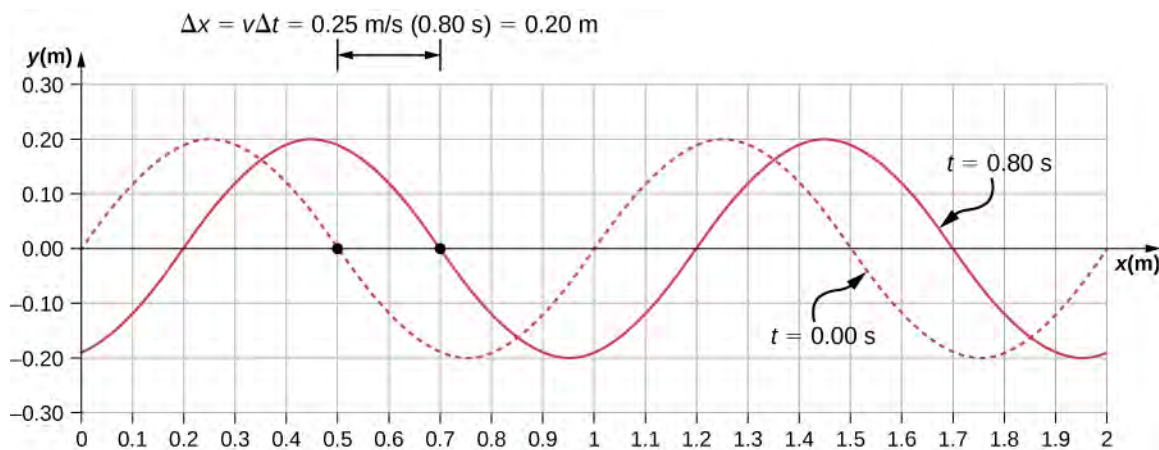


Figure 16.11 A graph of height of the wave y as a function of position x for snapshots of the wave at two times. The dotted line represents the wave at time $t = 0.00 \text{ s}$ and the solid line represents the wave at $t = 0.80 \text{ s}$. Since the wave velocity is constant, the distance the wave travels is the wave velocity times the time interval. The black dots indicate the points used to measure the displacement of the wave. The medium moves up and down, whereas the wave moves to the right.

There is a second velocity to the motion. In this example, the wave is transverse, moving horizontally as the medium oscillates up and down perpendicular to the direction of motion. The graph in [Figure 16.12](#) shows the motion of the medium at point $x = 0.60 \text{ m}$ as a function of time. Notice that the medium of the wave oscillates up and down between $y = +0.20 \text{ m}$ and $y = -0.20 \text{ m}$ every period of 4.0 seconds.

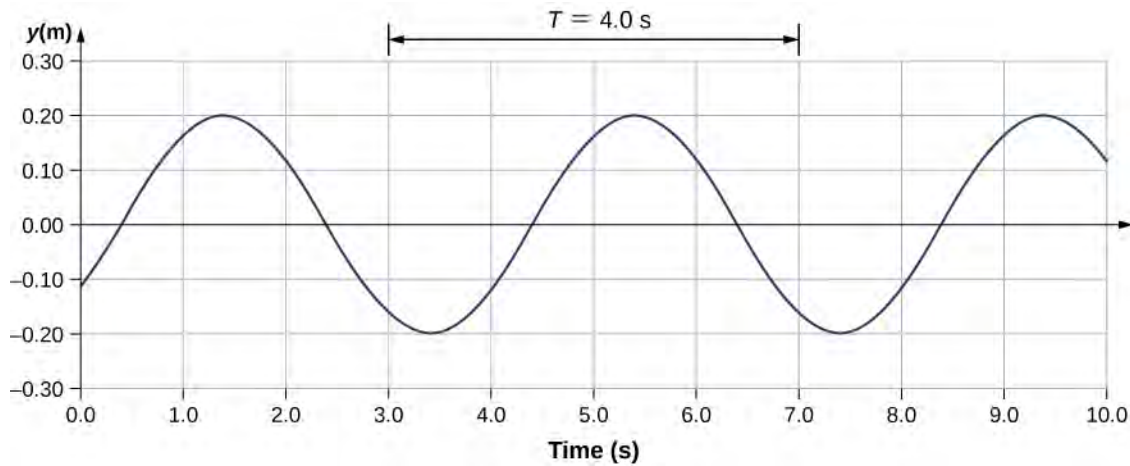


Figure 16.12 A graph of height of the wave y as a function of time t for the position $x = 0.6$ m. The medium oscillates between $y = +0.20$ m and $y = -0.20$ m every period. The period represented picks two convenient points in the oscillations to measure the period. The period can be measured between any two adjacent points with the same amplitude and the same velocity, $(\partial y / \partial t)$. The velocity can be found by looking at the slope tangent to the point on a y -versus- t plot. Notice that at times $t = 3.00$ s and $t = 7.00$ s, the heights and the velocities are the same and the period of the oscillation is 4.00 s.

✓ CHECK YOUR UNDERSTANDING 16.3

The wave function above is derived using a sine function. Can a cosine function be used instead?

Velocity and Acceleration of the Medium

As seen in [Example 16.4](#), the wave speed is constant and represents the speed of the wave as it propagates through the medium, not the speed of the particles that make up the medium. The particles of the medium oscillate around an equilibrium position as the wave propagates through the medium. In the case of the transverse wave propagating in the x -direction, the particles oscillate up and down in the y -direction, perpendicular to the motion of the wave. The velocity of the particles of the medium is not constant, which means there is an acceleration. The velocity of the medium, which is perpendicular to the wave velocity in a transverse wave, can be found by taking the partial derivative of the position equation with respect to time. The partial derivative is found by taking the derivative of the function, treating all variables as constants, except for the variable in question. In the case of the partial derivative with respect to time t , the position x is treated as a constant. Although this may sound strange if you haven't seen it before, the object of this exercise is to find the transverse velocity at a point, so in this sense, the x -position is not changing. We have

$$\begin{aligned} y(x, t) &= A \sin(kx - \omega t + \phi) \\ v_y(x, t) &= \frac{\partial y(x, t)}{\partial t} = \frac{\partial}{\partial t} (A \sin(kx - \omega t + \phi)) \\ &= -A\omega \cos(kx - \omega t + \phi) \\ &= -v_{y \max} \cos(kx - \omega t + \phi). \end{aligned}$$

The magnitude of the maximum velocity of the medium is $|v_{y \max}| = A\omega$. This may look familiar from the [Oscillations](#) and a mass on a spring.

We can find the acceleration of the medium by taking the partial derivative of the velocity equation with respect to time,

$$\begin{aligned} a_y(x, t) &= \frac{\partial v_y}{\partial t} = \frac{\partial}{\partial t} (-A\omega \cos(kx - \omega t + \phi)) \\ &= -A\omega^2 \sin(kx - \omega t + \phi) \\ &= -a_{y \max} \sin(kx - \omega t + \phi). \end{aligned}$$

The magnitude of the maximum acceleration is $|a_{y_{\max}}| = A\omega^2$. The particles of the medium, or the mass elements, oscillate in simple harmonic motion for a mechanical wave.

The Linear Wave Equation

We have just determined the velocity of the medium at a position x by taking the partial derivative, with respect to time, of the position y . For a transverse wave, this velocity is perpendicular to the direction of propagation of the wave. We found the acceleration by taking the partial derivative, with respect to time, of the velocity, which is the second time derivative of the position:

$$a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = \frac{\partial^2}{\partial t^2} (A \sin(kx - \omega t + \phi)) = -A\omega^2 \sin(kx - \omega t + \phi).$$

Now consider the partial derivatives with respect to the other variable, the position x , holding the time constant. The first derivative is the slope of the wave at a point x at a time t ,

$$\text{slope} = \frac{\partial y(x, t)}{\partial x} = \frac{\partial}{\partial x} (A \sin(kx - \omega t + \phi)) = Ak \cos(kx - \omega t + \phi).$$

The second partial derivative expresses how the slope of the wave changes with respect to position—in other words, the curvature of the wave, where

$$\text{curvature} = \frac{\partial^2 y(x, t)}{\partial x^2} = \frac{\partial^2}{\partial x^2} (A \sin(kx - \omega t + \phi)) = -Ak^2 \sin(kx - \omega t + \phi).$$

The ratio of the acceleration and the curvature leads to a very important relationship in physics known as the **linear wave equation**. Taking the ratio and using the equation $v = \omega/k$ yields the linear wave equation (also known simply as the wave equation or the equation of a vibrating string),

$$\begin{aligned} \frac{\frac{\partial^2 y(x, t)}{\partial t^2}}{\frac{\partial^2 y(x, t)}{\partial x^2}} &= \frac{-A\omega^2 \sin(kx - \omega t + \phi)}{-Ak^2 \sin(kx - \omega t + \phi)} \\ &= \frac{\omega^2}{k^2} = v^2, \end{aligned}$$

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}.$$

16.6

[Equation 16.6](#) is the linear wave equation, which is one of the most important equations in physics and engineering. We derived it here for a transverse wave, but it is equally important when investigating longitudinal waves. This relationship was also derived using a sinusoidal wave, but it successfully describes any wave or pulse that has the form $y(x, t) = f(x \mp vt)$. These waves result due to a linear restoring force of the medium—thus, the name linear wave equation. Any wave function that satisfies this equation is a linear wave function.

An interesting aspect of the linear wave equation is that if two wave functions are individually solutions to the linear wave equation, then the sum of the two linear wave functions is also a solution to the wave equation. Consider two transverse waves that propagate along the x -axis, occupying the same medium. Assume that the individual waves can be modeled with the wave functions $y_1(x, t) = f(x \mp vt)$ and $y_2(x, t) = g(x \mp vt)$, which are solutions to the linear wave equations and are therefore linear wave functions. The sum of the wave functions is the wave function

$$y_1(x, t) + y_2(x, t) = f(x \mp vt) + g(x \mp vt).$$

Consider the linear wave equation:

$$\begin{aligned}\frac{\partial^2(f+g)}{\partial x^2} &= \frac{1}{v^2} \frac{\partial^2(f+g)}{\partial t^2} \\ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} &= \frac{1}{v^2} \left[\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 g}{\partial t^2} \right].\end{aligned}$$

This has shown that if two linear wave functions are added algebraically, the resulting wave function is also linear. This wave function models the displacement of the medium of the resulting wave at each position along the x -axis. If two linear waves occupy the same medium, they are said to interfere. If these waves can be modeled with a linear wave function, these wave functions add to form the wave equation of the wave resulting from the interference of the individual waves. The displacement of the medium at every point of the resulting wave is the algebraic sum of the displacements due to the individual waves.

Taking this analysis a step further, if wave functions $y_1(x, t) = f(x \mp vt)$ and $y_2(x, t) = g(x \mp vt)$ are solutions to the linear wave equation, then $Ay_1(x, t) + By_2(x, t)$, where A and B are constants, is also a solution to the linear wave equation. This property is known as the principle of superposition. Interference and superposition are covered in more detail in [Interference of Waves](#).



EXAMPLE 16.4

Interference of Waves on a String

Consider a very long string held taut by two students, one on each end. Student A oscillates the end of the string producing a wave modeled with the wave function $y_1(x, t) = A \sin(kx - \omega t)$ and student B oscillates the string producing at twice the frequency, moving in the opposite direction. Both waves move at the same speed $v = \frac{\omega}{k}$. The two waves interfere to form a resulting wave whose wave function is $y_R(x, t) = y_1(x, t) + y_2(x, t)$.

Find the velocity of the resulting wave using the linear wave equation $\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}$.

Strategy

First, write the wave function for the wave created by the second student. Note that the angular frequency of the second wave is twice the frequency of the first wave (2ω), and since the velocity of the two waves are the same, the wave number of the second wave is twice that of the first wave ($2k$). Next, write the wave equation for the resulting wave function, which is the sum of the two individual wave functions. Then find the second partial derivative with respect to position and the second partial derivative with respect to time. Use the linear wave equation to find the velocity of the resulting wave.

Solution

1. Write the wave function of the second wave: $y_2(x, t) = A \sin(2kx + 2\omega t)$.
2. Write the resulting wave function:

$$y_R(x, t) = y_1(x, t) + y_2(x, t) = A \sin(kx - \omega t) + A \sin(2kx + 2\omega t).$$

3. Find the partial derivatives:

$$\begin{aligned}\frac{\partial y_R(x, t)}{\partial x} &= -Ak \cos(kx - \omega t) + 2Ak \cos(2kx + 2\omega t), \\ \frac{\partial^2 y_R(x, t)}{\partial x^2} &= -Ak^2 \sin(kx - \omega t) - 4Ak^2 \sin(2kx + 2\omega t), \\ \frac{\partial y_R(x, t)}{\partial t} &= -A\omega \cos(kx - \omega t) + 2A\omega \cos(2kx + 2\omega t), \\ \frac{\partial^2 y_R(x, t)}{\partial t^2} &= -A\omega^2 \sin(kx - \omega t) - 4A\omega^2 \sin(2kx + 2\omega t).\end{aligned}$$

4. Use the wave equation to find the velocity of the resulting wave:

$$\begin{aligned}
 \frac{\partial^2 y(x,t)}{\partial x^2} &= \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2}, \\
 -Ak^2 \sin(kx - \omega t) - 4Ak^2 \sin(2kx + 2\omega t) &= \frac{1}{v^2} (-A\omega^2 \sin(kx - \omega t) - 4A\omega^2 \sin(2kx + 2\omega t)), \\
 k^2 (-A \sin(kx - \omega t) - 4A \sin(2kx + 2\omega t)) &= \frac{\omega^2}{v^2} (-A \sin(kx - \omega t) - 4A \sin(2kx + 2\omega t)), \\
 k^2 &= \frac{\omega^2}{v^2}, |v| = \frac{\omega}{k}.
 \end{aligned}$$

Significance

The speed of the resulting wave is equal to the speed of the original waves ($v = \frac{\omega}{k}$). We will show in the next section that the speed of a simple harmonic wave on a string depends on the tension in the string and the mass per length of the string. For this reason, it is not surprising that the component waves as well as the resultant wave all travel at the same speed.

✓ CHECK YOUR UNDERSTANDING 16.4

The wave equation $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2}$ works for any wave of the form $y(x, t) = f(x \mp vt)$. In the previous section, we stated that a cosine function could also be used to model a simple harmonic mechanical wave. Check if the wave

$$y(x, t) = 0.50 \text{ m} \cos\left(0.20\pi \text{ m}^{-1}x - 4.00\pi \text{ s}^{-1}t + \frac{\pi}{10}\right)$$

is a solution to the wave equation.

Any disturbance that complies with the wave equation can propagate as a wave moving along the x -axis with a wave speed v . It works equally well for waves on a string, sound waves, and electromagnetic waves. This equation is extremely useful. For example, it can be used to show that electromagnetic waves move at the speed of light.

16.3 Wave Speed on a Stretched String

Learning Objectives

By the end of this section, you will be able to:

- Determine the factors that affect the speed of a wave on a string
- Write a mathematical expression for the speed of a wave on a string and generalize these concepts for other media

The speed of a wave depends on the characteristics of the medium. For example, in the case of a guitar, the strings vibrate to produce the sound. The speed of the waves on the strings, and the wavelength, determine the frequency of the sound produced. The strings on a guitar have different thickness but may be made of similar material. They have different *linear densities*, where the linear density is defined as the mass per length,

$$\mu = \frac{\text{mass of string}}{\text{length of string}} = \frac{m}{l}. \quad 16.7$$

In this chapter, we consider only string with a constant linear density. If the linear density is constant, then the mass (Δm) of a small length of string (Δx) is $\Delta m = \mu \Delta x$. For example, if the string has a length of 2.00 m and a mass of 0.06 kg, then the linear density is $\mu = \frac{0.06 \text{ kg}}{2.00 \text{ m}} = 0.03 \frac{\text{kg}}{\text{m}}$. If a 1.00-mm section is cut from the string, the mass of the 1.00-mm length is $\Delta m = \mu \Delta x = \left(0.03 \frac{\text{kg}}{\text{m}}\right) 0.001 \text{ m} = 3.00 \times 10^{-5} \text{ kg}$. The guitar also has a method to change the tension of the strings. The tension of the strings is adjusted by turning spindles, called the tuning pegs, around which the strings are wrapped. For the guitar, the linear density of the string and the tension in the string determine the speed of the waves in the string and the frequency of the sound produced is

proportional to the wave speed.

Wave Speed on a String under Tension

To see how the speed of a wave on a string depends on the tension and the linear density, consider a pulse sent down a taut string (Figure 16.13). When the taut string is at rest at the equilibrium position, the tension in the string F_T is constant. Consider a small element of the string with a mass equal to $\Delta m = \mu \Delta x$. The mass element is at rest and in equilibrium and the force of tension of either side of the mass element is equal and opposite.



Figure 16.13 Mass element of a string kept taut with a tension F_T . The mass element is in static equilibrium, and the force of tension acting on either side of the mass element is equal in magnitude and opposite in direction.

If you pluck a string under tension, a transverse wave moves in the positive x -direction, as shown in Figure 16.14. The mass element is small but is enlarged in the figure to make it visible. The small mass element oscillates perpendicular to the wave motion as a result of the restoring force provided by the string and does not move in the x -direction. The tension F_T in the string, which acts in the positive and negative x -direction, is approximately constant and is independent of position and time.

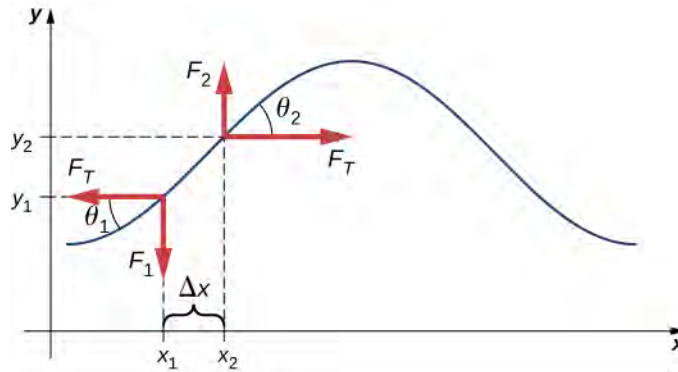


Figure 16.14 A string under tension is plucked, causing a pulse to move along the string in the positive x -direction.

Assume that the inclination of the displaced string with respect to the horizontal axis is small. The net force on the element of the string, acting parallel to the string, is the sum of the tension in the string and the restoring force. The x -components of the force of tension cancel, so the net force is equal to the sum of the y -components of the force. The magnitude of the x -component of the force is equal to the horizontal force of tension of the string F_T as shown in Figure 16.14. To obtain the y -components of the force, note that $\tan \theta_1 = \frac{-F_1}{F_T}$ and $\tan \theta_2 = \frac{F_2}{F_T}$. The $\tan \theta$ is equal to the slope of a function at a point, which is equal to the partial derivative of y with respect to x at that point. Therefore, $\frac{F_1}{F_T}$ is equal to the negative slope of the string at x_1 and $\frac{F_2}{F_T}$ is equal to the slope of the string at x_2 :

$$\frac{F_1}{F_T} = -\left(\frac{\partial y}{\partial x}\right)_{x_1} \quad \text{and} \quad \frac{F_2}{F_T} = \left(\frac{\partial y}{\partial x}\right)_{x_2}.$$

The net force on the small mass element can be written as

$$F_{\text{net}} = F_1 + F_2 = F_T \left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1} \right].$$

Using Newton's second law, the net force is equal to the mass times the acceleration. The linear density of the string μ is the mass per length of the string, and the mass of the portion of the string is $\mu \Delta x$,

$$F_T \left[\left(\frac{\partial y}{\partial x} \right)_{x_2} - \left(\frac{\partial y}{\partial x} \right)_{x_1} \right] = \Delta m a,$$

$$F_T \left[\left(\frac{\partial y}{\partial x} \right)_{x_2} - \left(\frac{\partial y}{\partial x} \right)_{x_1} \right] = \mu \Delta x \frac{\partial^2 y}{\partial t^2}.$$

Dividing by $F_T \Delta x$ and taking the limit as Δx approaches zero,

$$\frac{\left[\left(\frac{\partial y}{\partial x} \right)_{x_2} - \left(\frac{\partial y}{\partial x} \right)_{x_1} \right]}{\Delta x} = \frac{\mu}{F_T} \frac{\partial^2 y}{\partial t^2}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\left[\left(\frac{\partial y}{\partial x} \right)_{x_2} - \left(\frac{\partial y}{\partial x} \right)_{x_1} \right]}{\Delta x} = \frac{\mu}{F_T} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F_T} \frac{\partial^2 y}{\partial t^2}.$$

Recall that the linear wave equation is

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}.$$

Therefore,

$$\frac{1}{v^2} = \frac{\mu}{F_T}.$$

Solving for v , we see that the speed of the wave on a string depends on the tension and the linear density.

Speed of a Wave on a String Under Tension

The speed of a pulse or wave on a string under tension can be found with the equation

$$|v| = \sqrt{\frac{F_T}{\mu}}$$

16.8

where F_T is the tension in the string and μ is the mass per length of the string.



EXAMPLE 16.5

The Wave Speed of a Guitar Spring

On a six-string guitar, the high E string has a linear density of $\mu_{\text{High E}} = 3.09 \times 10^{-4} \text{ kg/m}$ and the low E string has a linear density of $\mu_{\text{Low E}} = 5.78 \times 10^{-3} \text{ kg/m}$. (a) If the high E string is plucked, producing a wave in the string, what is the speed of the wave if the tension of the string is 56.40 N? (b) The linear density of the low E string is approximately 20 times greater than that of the high E string. For waves to travel through the low E string at the same wave speed as the high E, would the tension need to be larger or smaller than the high E string? What would be the approximate tension? (c) Calculate the tension of the low E string needed for the same wave speed.

Strategy

- The speed of the wave can be found from the linear density and the tension $v = \sqrt{\frac{F_T}{\mu}}$.
- From the equation $v = \sqrt{\frac{F_T}{\mu}}$, if the linear density is increased by a factor of almost 20, the tension would

need to be increased by a factor of 20.

- c. Knowing the velocity and the linear density, the velocity equation can be solved for the force of tension $F_T = \mu v^2$.

Solution

- a. Use the velocity equation to find the speed:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{56.40 \text{ N}}{3.09 \times 10^{-4} \text{ kg/m}}} = 427.23 \text{ m/s}.$$

- b. The tension would need to be increased by a factor of approximately 20. The tension would be slightly less than 1128 N.
- c. Use the velocity equation to find the actual tension:

$$F_T = \mu v^2 = 5.78 \times 10^{-3} \text{ kg/m} (427.23 \text{ m/s})^2 = 1055.00 \text{ N}.$$

This solution is within 7% of the approximation.

Significance

The standard notes of the six string (high E, B, G, D, A, low E) are tuned to vibrate at the fundamental frequencies (329.63 Hz, 246.94 Hz, 196.00 Hz, 146.83 Hz, 110.00 Hz, and 82.41 Hz) when plucked. The frequencies depend on the speed of the waves on the string and the wavelength of the waves. The six strings have different linear densities and are “tuned” by changing the tensions in the strings. We will see in [Interference of Waves](#) that the wavelength depends on the length of the strings and the boundary conditions. To play notes other than the fundamental notes, the lengths of the strings are changed by pressing down on the strings.

✓ CHECK YOUR UNDERSTANDING 16.5

The wave speed of a wave on a string depends on the tension and the linear mass density. If the tension is doubled, what happens to the speed of the waves on the string?

Speed of Compression Waves in a Fluid

The speed of a wave on a string depends on the square root of the tension divided by the mass per length, the linear density. In general, the speed of a wave through a medium depends on the elastic property of the medium and the inertial property of the medium.

$$|v| = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}$$

The elastic property describes the tendency of the particles of the medium to return to their initial position when perturbed. The inertial property describes the tendency of the particle to resist changes in velocity.

The speed of a longitudinal wave through a liquid or gas depends on the density of the fluid and the bulk modulus of the fluid,

$$v = \sqrt{\frac{B}{\rho}}.$$

16.9

Here the bulk modulus is defined as $B = -\frac{\Delta P}{\Delta V/V_0}$, where ΔP is the change in the pressure and the denominator is the ratio of the change in volume to the initial volume, and $\rho \equiv \frac{m}{V}$ is the mass per unit volume. For example, sound is a mechanical wave that travels through a fluid or a solid. The speed of sound in air with an atmospheric pressure of $1.013 \times 10^5 \text{ Pa}$ and a temperature of 20°C is $v_s \approx 343.00 \text{ m/s}$. Because the density depends on temperature, the speed of sound in air depends on the temperature of the air. This will be

discussed in detail in [Sound](#).

16.4 Energy and Power of a Wave

Learning Objectives

By the end of this section, you will be able to:

- Explain how energy travels with a pulse or wave
- Describe, using a mathematical expression, how the energy in a wave depends on the amplitude of the wave

All waves carry energy, and sometimes this can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls ([Figure 16.15](#)). Loud sounds can pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.



Figure 16.15 The destructive effect of an earthquake is observable evidence of the energy carried in these waves. The Richter scale rating of earthquakes is a logarithmic scale related to both their amplitude and the energy they carry.

In this section, we examine the quantitative expression of energy in waves. This will be of fundamental importance in later discussions of waves, from sound to light to quantum mechanics.

Energy in Waves

The amount of energy in a wave is related to its amplitude and its frequency. Large-amplitude earthquakes produce large ground displacements. Loud sounds have high-pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. Consider the example of the seagull and the water wave earlier in the chapter ([Figure 16.3](#)). Work is done on the seagull by the wave as the seagull is moved up, changing its potential energy. The larger the amplitude, the higher the seagull is lifted by the wave and the larger the change in potential energy.

The energy of the wave depends on both the amplitude and the frequency. If the energy of each wavelength is considered to be a discrete packet of energy, a high-frequency wave will deliver more of these packets per unit time than a low-frequency wave. We will see that the average rate of energy transfer in mechanical waves is proportional to both the square of the amplitude and the square of the frequency. If two mechanical waves have equal amplitudes, but one wave has a frequency equal to twice the frequency of the other, the higher-frequency wave will have a rate of energy transfer a factor of four times as great as the rate of energy transfer of

the lower-frequency wave. It should be noted that although the rate of energy transport is proportional to both the square of the amplitude and square of the frequency in mechanical waves, the rate of energy transfer in electromagnetic waves is proportional to the square of the amplitude, but independent of the frequency.

Power in Waves

Consider a sinusoidal wave on a string that is produced by a string vibrator, as shown in [Figure 16.16](#). The string vibrator is a device that vibrates a rod up and down. A string of uniform linear mass density is attached to the rod, and the rod oscillates the string, producing a sinusoidal wave. The rod does work on the string, producing energy that propagates along the string. Consider a mass element of the string with a mass Δm , as seen in [Figure 16.16](#). As the energy propagates along the string, each mass element of the string is driven up and down at the same frequency as the wave. Each mass element of the string can be modeled as a simple harmonic oscillator. Since the string has a constant linear density $\mu = \frac{\Delta m}{\Delta x}$, each mass element of the string has the mass $\Delta m = \mu \Delta x$.

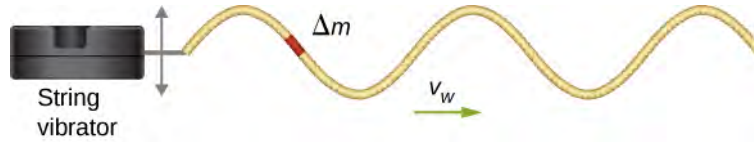


Figure 16.16 A string vibrator is a device that vibrates a rod. A string is attached to the rod, and the rod does work on the string, driving the string up and down. This produces a sinusoidal wave in the string, which moves with a wave velocity v . The wave speed depends on the tension in the string and the linear mass density of the string. A section of the string with mass Δm oscillates at the same frequency as the wave.

The total mechanical energy of the wave is the sum of its kinetic energy and potential energy. The kinetic energy $K = \frac{1}{2}mv^2$ of each mass element of the string of length Δx is $\Delta K = \frac{1}{2}(\Delta m)v_y^2$, as the mass element oscillates perpendicular to the direction of the motion of the wave. Using the constant linear mass density, the kinetic energy of each mass element of the string with length Δx is

$$\Delta K = \frac{1}{2}(\mu \Delta x) v_y^2.$$

A differential equation can be formed by letting the length of the mass element of the string approach zero,

$$dK = \lim_{\Delta x \rightarrow 0} \frac{1}{2}(\mu \Delta x) v_y^2 = \frac{1}{2}(\mu dx) v_y^2.$$

Since the wave is a sinusoidal wave with an angular frequency ω , the position of each mass element may be modeled as $y(x, t) = A \sin(kx - \omega t)$. Each mass element of the string oscillates with a velocity $v_y = \frac{\partial y(x, t)}{\partial t} = -A\omega \cos(kx - \omega t)$. The kinetic energy of each mass element of the string becomes

$$\begin{aligned} dK &= \frac{1}{2}(\mu dx) (-A\omega \cos(kx - \omega t))^2, \\ &= \frac{1}{2}(\mu dx) A^2 \omega^2 \cos^2(kx - \omega t). \end{aligned}$$

The wave can be very long, consisting of many wavelengths. To standardize the energy, consider the kinetic energy associated with a wavelength of the wave. This kinetic energy can be integrated over the wavelength to find the energy associated with each wavelength of the wave:

$$\begin{aligned} dK &= \frac{1}{2}(\mu dx) A^2 \omega^2 \cos^2(kx), \\ \int_0^{\lambda} dK &= \int_0^{\lambda} \frac{1}{2} \mu A^2 \omega^2 \cos^2(kx) dx = \frac{1}{2} \mu A^2 \omega^2 \int_0^{\lambda} \cos^2(kx) dx, \\ K_{\lambda} &= \frac{1}{2} \mu A^2 \omega^2 \left[\frac{1}{2}x + \frac{1}{4k} \sin(2kx) \right]_0^{\lambda} = \frac{1}{2} \mu A^2 \omega^2 \left[\frac{1}{2}\lambda + \frac{1}{4k} \sin(2k\lambda) - \frac{1}{4k} \sin(0) \right], \\ K_{\lambda} &= \frac{1}{4} \mu A^2 \omega^2 \lambda. \end{aligned}$$

There is also potential energy associated with the wave. Much like the mass oscillating on a spring, there is a conservative restoring force that, when the mass element is displaced from the equilibrium position, drives the mass element back to the equilibrium position. The potential energy of the mass element can be found by considering the linear restoring force of the string. In [Oscillations](#), we saw that the potential energy stored in a spring with a linear restoring force is equal to $U = \frac{1}{2}k_s x^2$, where the equilibrium position is defined as $x = 0.00$ m. When a mass attached to the spring oscillates in simple harmonic motion, the angular frequency is equal to $\omega = \sqrt{\frac{k_s}{m}}$. As each mass element oscillates in simple harmonic motion, the spring constant is equal to $k_s = \Delta m \omega^2$. The potential energy of the mass element is equal to

$$\Delta U = \frac{1}{2}k_s x^2 = \frac{1}{2}\Delta m \omega^2 x^2.$$

Note that k_s is the spring constant and not the wave number $k = \frac{2\pi}{\lambda}$. This equation can be used to find the energy over a wavelength. Integrating over the wavelength, we can compute the potential energy over a wavelength:

$$\begin{aligned} dU &= \frac{1}{2}k_s x^2 = \frac{1}{2}\mu \omega^2 x^2 dx, \\ U_\lambda &= \frac{1}{2}\mu \omega^2 A^2 \int_0^\lambda \sin^2(kx) dx = \frac{1}{4}\mu A^2 \omega^2 \lambda. \end{aligned}$$

The potential energy associated with a wavelength of the wave is equal to the kinetic energy associated with a wavelength.

The total energy associated with a wavelength is the sum of the potential energy and the kinetic energy:

$$\begin{aligned} E_\lambda &= U_\lambda + K_\lambda, \\ E_\lambda &= \frac{1}{4}\mu A^2 \omega^2 \lambda + \frac{1}{4}\mu A^2 \omega^2 \lambda = \frac{1}{2}\mu A^2 \omega^2 \lambda. \end{aligned}$$

The time-averaged power of a sinusoidal mechanical wave, which is the average rate of energy transfer associated with a wave as it passes a point, can be found by taking the total energy associated with the wave divided by the time it takes to transfer the energy. If the velocity of the sinusoidal wave is constant, the time for one wavelength to pass by a point is equal to the period of the wave, which is also constant. For a sinusoidal mechanical wave, the time-averaged power is therefore the energy associated with a wavelength divided by the period of the wave. The wavelength of the wave divided by the period is equal to the velocity of the wave,

$$P_{\text{ave}} = \frac{E_\lambda}{T} = \frac{1}{2}\mu A^2 \omega^2 \frac{\lambda}{T} = \frac{1}{2}\mu A^2 \omega^2 v. \quad 16.10$$

Note that this equation for the time-averaged power of a sinusoidal mechanical wave shows that the power is proportional to the square of the amplitude of the wave and to the square of the angular frequency of the wave. Recall that the angular frequency is equal to $\omega = 2\pi f$, so the power of a mechanical wave is equal to the square of the amplitude and the square of the frequency of the wave.



EXAMPLE 16.6

Power Supplied by a String Vibrator

Consider a two-meter-long string with a mass of 70.00 g attached to a string vibrator as illustrated in [Figure 16.16](#). The tension in the string is 90.0 N. When the string vibrator is turned on, it oscillates with a frequency of 60 Hz and produces a sinusoidal wave on the string with an amplitude of 4.00 cm and a constant wave speed. What is the time-averaged power supplied to the wave by the string vibrator?

Strategy

The power supplied to the wave should equal the time-averaged power of the wave on the string. We know the

mass of the string (m_s), the length of the string (L_s), and the tension (F_T) in the string. The speed of the wave on the string can be derived from the linear mass density and the tension. The string oscillates with the same frequency as the string vibrator, from which we can find the angular frequency.

Solution

1. Begin with the equation of the time-averaged power of a sinusoidal wave on a string:

$$P = \frac{1}{2} \mu A^2 \omega^2 v.$$

The amplitude is given, so we need to calculate the linear mass density of the string, the angular frequency of the wave on the string, and the speed of the wave on the string.

2. We need to calculate the linear density to find the wave speed:

$$\mu = \frac{m_s}{L_s} = \frac{0.070 \text{ kg}}{2.00 \text{ m}} = 0.035 \text{ kg/m}.$$

3. The wave speed can be found using the linear mass density and the tension of the string:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{90.00 \text{ N}}{0.035 \text{ kg/m}}} = 50.71 \text{ m/s}.$$

4. The angular frequency can be found from the frequency:

$$\omega = 2\pi f = 2\pi (60 \text{ s}^{-1}) = 376.80 \text{ s}^{-1}.$$

5. Calculate the time-averaged power:

$$P = \frac{1}{2} \mu A^2 \omega^2 v = \frac{1}{2} \left(0.035 \frac{\text{kg}}{\text{m}} \right) (0.040 \text{ m})^2 (376.80 \text{ s}^{-1})^2 \left(50.71 \frac{\text{m}}{\text{s}} \right) = 201.59 \text{ W}.$$

Significance

The time-averaged power of a sinusoidal wave is proportional to the square of the amplitude of the wave and the square of the angular frequency of the wave. This is true for most mechanical waves. If either the angular frequency or the amplitude of the wave were doubled, the power would increase by a factor of four. The time-averaged power of the wave on a string is also proportional to the speed of the sinusoidal wave on the string. If the speed were doubled, by increasing the tension by a factor of four, the power would also be doubled.

✓ CHECK YOUR UNDERSTANDING 16.6

Is the time-averaged power of a sinusoidal wave on a string proportional to the linear density of the string?

The equations for the energy of the wave and the time-averaged power were derived for a sinusoidal wave on a string. In general, the energy of a mechanical wave and the power are proportional to the amplitude squared and to the angular frequency squared (and therefore the frequency squared).

Another important characteristic of waves is the intensity of the waves. Waves can also be concentrated or spread out. Waves from an earthquake, for example, spread out over a larger area as they move away from a source, so they do less damage the farther they get from the source. Changing the area the waves cover has important effects. All these pertinent factors are included in the definition of **intensity (I)** as power per unit area:

$$I = \frac{P}{A},$$

16.11

where P is the power carried by the wave through area A . The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter (W/m^2). Many waves are spherical waves that move out from a source as a sphere. For example, a sound speaker mounted on a post above the ground may produce sound waves that move away from the source as a spherical wave. Sound waves are discussed in more detail in the next chapter, but in general, the farther you are from the speaker, the less

intense the sound you hear. As a spherical wave moves out from a source, the surface area of the wave increases as the radius increases ($A = 4\pi r^2$). The intensity for a spherical wave is therefore

$$I = \frac{P}{4\pi r^2}. \quad 16.12$$

If there are no dissipative forces, the energy will remain constant as the spherical wave moves away from the source, but the intensity will decrease as the surface area increases.

In the case of the two-dimensional circular wave, the wave moves out, increasing the circumference of the wave as the radius of the circle increases. If you toss a pebble in a pond, the surface ripple moves out as a circular wave. As the ripple moves away from the source, the amplitude decreases. The energy of the wave spreads around a larger circumference and the intensity decreases proportional to $\frac{1}{r}$, which is also the same in the case of a spherical wave, since intensity is proportional to the amplitude squared.

16.5 Interference of Waves

Learning Objectives

By the end of this section, you will be able to:

- Explain how mechanical waves are reflected and transmitted at the boundaries of a medium
- Define the terms interference and superposition
- Find the resultant wave of two identical sinusoidal waves that differ only by a phase shift

Up to now, we have been studying mechanical waves that propagate continuously through a medium, but we have not discussed what happens when waves encounter the boundary of the medium or what happens when a wave encounters another wave propagating through the same medium. Waves do interact with boundaries of the medium, and all or part of the wave can be reflected. For example, when you stand some distance from a rigid cliff face and yell, you can hear the sound waves reflect off the rigid surface as an echo. Waves can also interact with other waves propagating in the same medium. If you throw two rocks into a pond some distance from one another, the circular ripples that result from the two stones seem to pass through one another as they propagate out from where the stones entered the water. This phenomenon is known as interference. In this section, we examine what happens to waves encountering a boundary of a medium or another wave propagating in the same medium. We will see that their behavior is quite different from the behavior of particles and rigid bodies. Later, when we study modern physics, we will see that only at the scale of atoms do we see similarities in the properties of waves and particles.

Reflection and Transmission

When a wave propagates through a medium, it reflects when it encounters the boundary of the medium. The wave before hitting the boundary is known as the incident wave. The wave after encountering the boundary is known as the reflected wave. How the wave is reflected at the boundary of the medium depends on the boundary conditions; waves will react differently if the boundary of the medium is fixed in place or free to move ([Figure 16.17](#)). A **fixed boundary condition** exists when the medium at a boundary is fixed in place so it cannot move. A **free boundary condition** exists when the medium at the boundary is free to move.

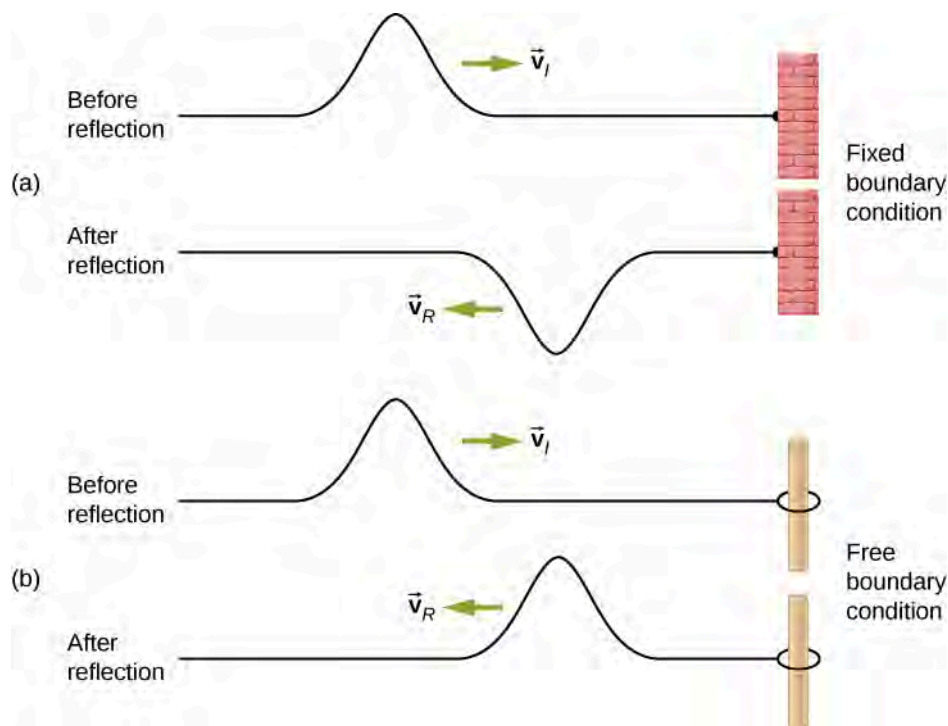


Figure 16.17 (a) One end of a string is fixed so that it cannot move. A wave propagating on the string, encountering this *fixed boundary condition*, is reflected $180^\circ (\pi \text{ rad})$ out of phase with respect to the incident wave. (b) One end of a string is tied to a solid ring of negligible mass on a frictionless lab pole, where the ring is free to move. A wave propagating on the string, encountering this *free boundary condition*, is reflected in phase $0^\circ (0 \text{ rad})$ with respect to the wave.

Part (a) of the [Figure 16.17](#) shows a fixed boundary condition. Here, one end of the string is fixed to a wall so the end of the string is fixed in place and the medium (the string) at the boundary cannot move. When the wave is reflected, the amplitude of the reflected wave is exactly the same as the amplitude of the incident wave, but the reflected wave is reflected $180^\circ (\pi \text{ rad})$ out of phase with respect to the incident wave. The phase change can be explained using Newton's third law: Recall that Newton's third law states that when object *A* exerts a force on object *B*, then object *B* exerts an equal and opposite force on object *A*. As the incident wave encounters the wall, the string exerts an upward force on the wall and the wall reacts by exerting an equal and opposite force on the string. The reflection at a fixed boundary is inverted. Note that the figure shows a crest of the incident wave reflected as a trough. If the incident wave were a trough, the reflected wave would be a crest.

Part (b) of the figure shows a free boundary condition. Here, one end of the string is tied to a solid ring of negligible mass on a frictionless pole, so the end of the string is free to move up and down. As the incident wave encounters the boundary of the medium, it is also reflected. In the case of a free boundary condition, the reflected wave is in phase with respect to the incident wave. In this case, the wave encounters the free boundary applying an upward force on the ring, accelerating the ring up. The ring travels up to the maximum height equal to the amplitude of the wave and then accelerates down towards the equilibrium position due to the tension in the string. The figure shows the crest of an incident wave being reflected in phase with respect to the incident wave as a crest. If the incident wave were a trough, the reflected wave would also be a trough. The amplitude of the reflected wave would be equal to the amplitude of the incident wave.

In some situations, the boundary of the medium is neither fixed nor free. Consider [Figure 16.18\(a\)](#), where a low-linear mass density string is attached to a string of a higher linear mass density. In this case, the reflected wave is out of phase with respect to the incident wave. There is also a transmitted wave that is in phase with respect to the incident wave. Both the transmitted and the reflected waves have amplitudes less than the amplitude of the incident wave. If the tension is the same in both strings, the wave speed is higher in the string with the lower linear mass density.

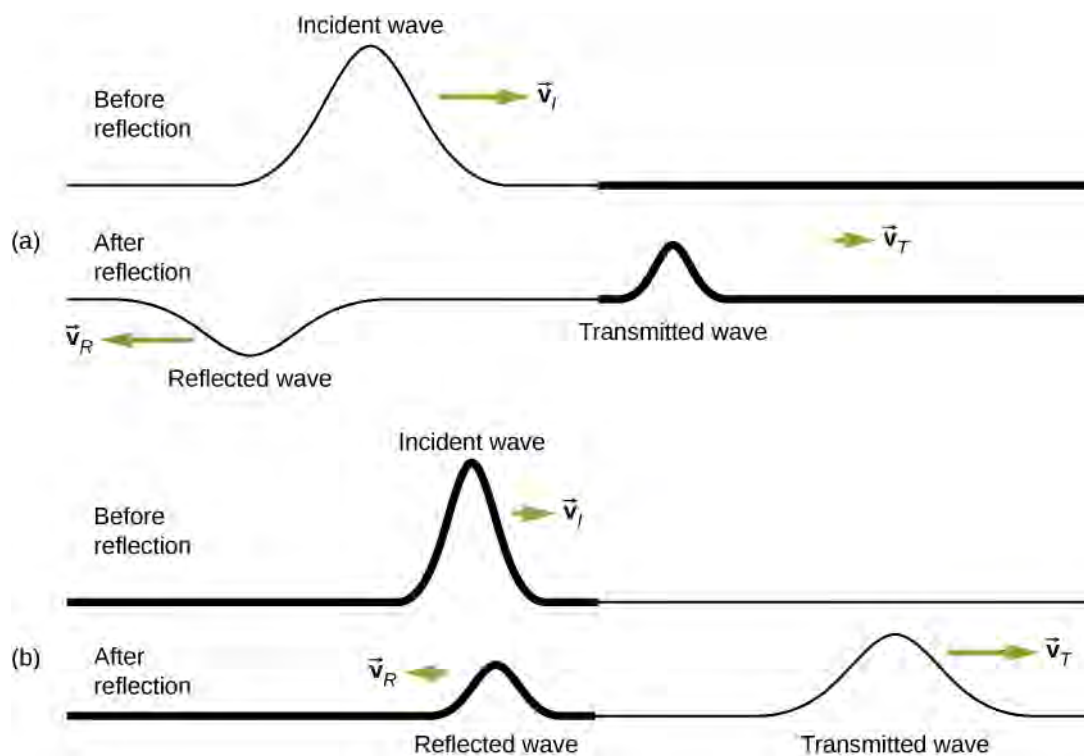


Figure 16.18 Waves traveling along two types of strings: a thick string with a high linear density and a thin string with a low linear density. Both strings are under the same tension, so a wave moves faster on the low-density string than on the high-density string. (a) A wave moving from a low-density to a high-density medium results in a reflected wave that is 180° (π rad) out of phase with respect to the incident pulse (or wave) and a transmitted wave that is in phase with the incident wave. (b) When a wave moves from a high-density medium to a low-density medium, both the reflected and transmitted wave are in phase with respect to the incident wave.

Part (b) of the figure shows a high-linear mass density string is attached to a string of a lower linear density. In this case, the reflected wave is in phase with respect to the incident wave. There is also a transmitted wave that is in phase with respect to the incident wave. Both the incident and the reflected waves have amplitudes less than the amplitude of the incident wave. Here you may notice that if the tension is the same in both strings, the wave speed is higher in the string with the lower linear mass density.

Superposition and Interference

Most waves do not look very simple. Complex waves are more interesting, even beautiful, but they look formidable. Most interesting mechanical waves consist of a combination of two or more traveling waves propagating in the same medium. The principle of superposition can be used to analyze the combination of waves.

Consider two simple pulses of the same amplitude moving toward one another in the same medium, as shown in [Figure 16.19](#). Eventually, the waves overlap, producing a wave that has twice the amplitude, and then continue on unaffected by the encounter. The pulses are said to interfere, and this phenomenon is known as **interference**.

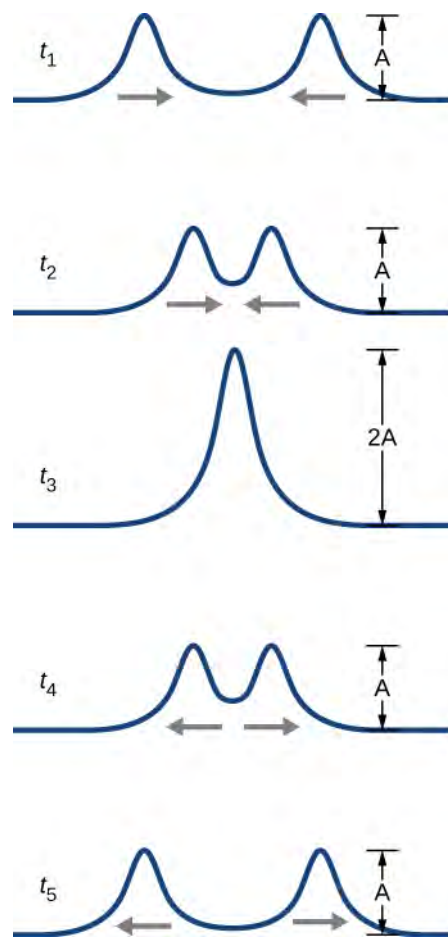


Figure 16.19 Two pulses moving toward one another experience interference. The term interference refers to what happens when two waves overlap.

To analyze the interference of two or more waves, we use the principle of superposition. For mechanical waves, the principle of **superposition** states that if two or more traveling waves combine at the same point, the resulting position of the mass element of the medium, at that point, is the algebraic sum of the position due to the individual waves. This property is exhibited by many waves observed, such as waves on a string, sound waves, and surface water waves. Electromagnetic waves also obey the superposition principle, but the electric and magnetic fields of the combined wave are added instead of the displacement of the medium. Waves that obey the superposition principle are linear waves; waves that do not obey the superposition principle are said to be nonlinear waves. In this chapter, we deal with linear waves, in particular, sinusoidal waves.

The superposition principle can be understood by considering the linear wave equation. In [Mathematics of a Wave](#), we defined a linear wave as a wave whose mathematical representation obeys the linear wave equation. For a transverse wave on a string with an elastic restoring force, the linear wave equation is

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}.$$

Any wave function $y(x, t) = y(x \mp vt)$, where the argument of the function is linear ($x \mp vt$) is a solution to the linear wave equation and is a linear wave function. If wave functions $y_1(x, t)$ and $y_2(x, t)$ are solutions to the linear wave equation, the sum of the two functions $y_1(x, t) + y_2(x, t)$ is also a solution to the linear wave equation. Mechanical waves that obey superposition are normally restricted to waves with amplitudes that are small with respect to their wavelengths. If the amplitude is too large, the medium is distorted past the region where the restoring force of the medium is linear.

Waves can interfere constructively or destructively. [Figure 16.20](#) shows two identical sinusoidal waves that arrive at the same point exactly in phase. [Figure 16.20\(a\)](#) and (b) show the two individual waves, [Figure](#)

16.20(c) shows the resultant wave that results from the algebraic sum of the two linear waves. The crests of the two waves are precisely aligned, as are the troughs. This superposition produces **constructive interference**. Because the disturbances add, constructive interference produces a wave that has twice the amplitude of the individual waves, but has the same wavelength.

Figure 16.21 shows two identical waves that arrive exactly 180° out of phase, producing **destructive interference**. Figure 16.21(a) and (b) show the individual waves, and Figure 16.21(c) shows the superposition of the two waves. Because the troughs of one wave add the crest of the other wave, the resulting amplitude is zero for destructive interference—the waves completely cancel.

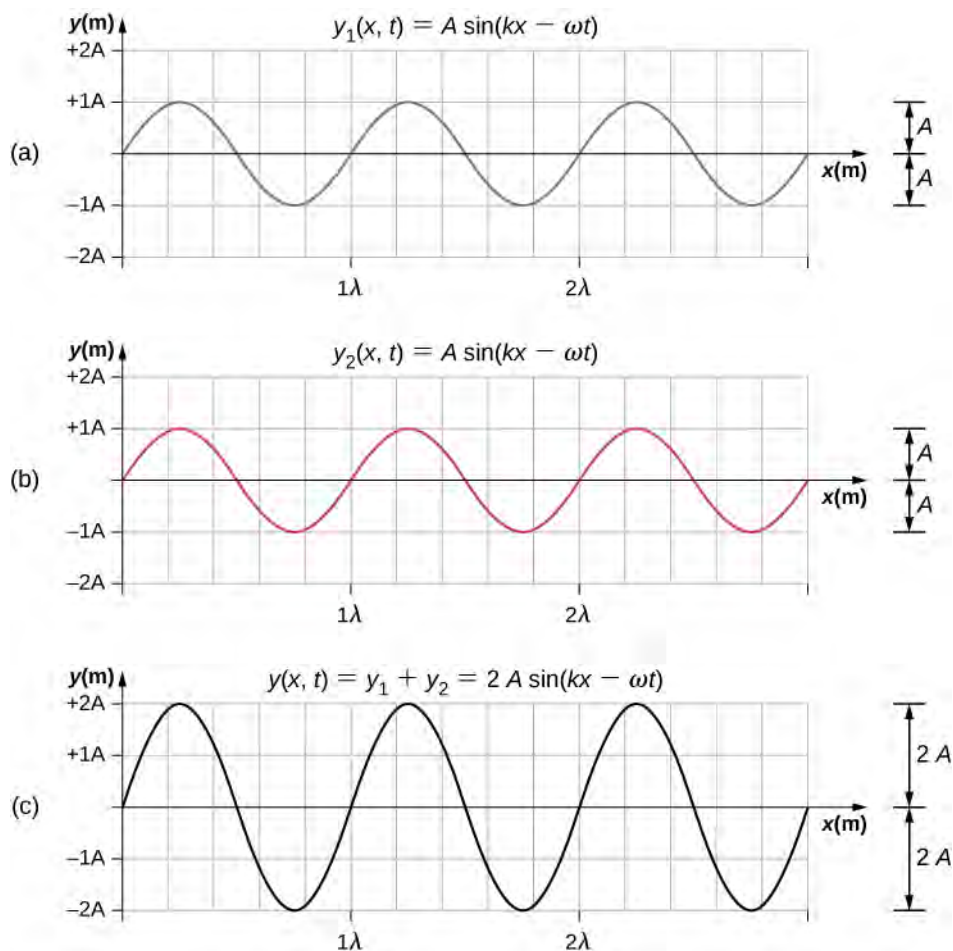


Figure 16.20 Constructive interference of two identical waves produces a wave with twice the amplitude, but the same wavelength.

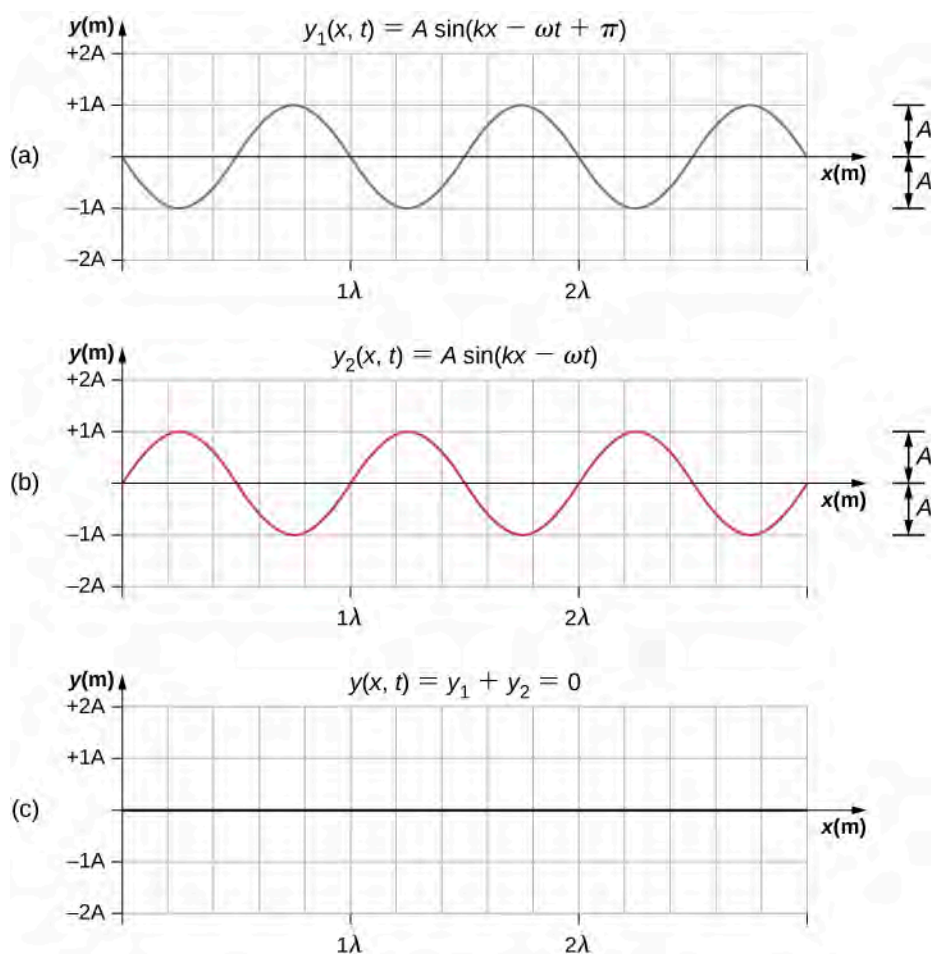


Figure 16.21 Destructive interference of two identical waves, one with a phase shift of 180° (π rad), produces zero amplitude, or complete cancellation.

When linear waves interfere, the resultant wave is just the algebraic sum of the individual waves as stated in the principle of superposition. [Figure 16.22](#) shows two waves (red and blue) and the resultant wave (black). The resultant wave is the algebraic sum of the two individual waves.

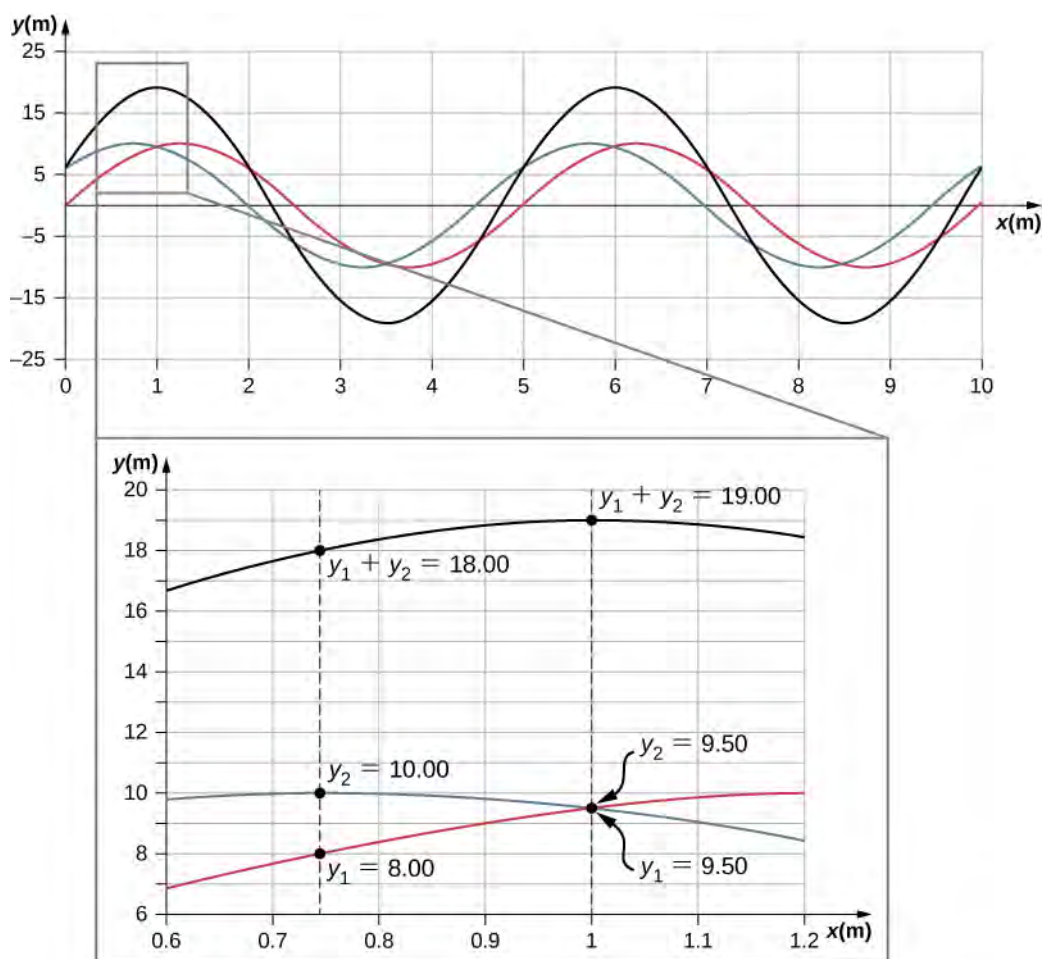


Figure 16.22 When two linear waves in the same medium interfere, the height of resulting wave is the sum of the heights of the individual waves, taken point by point. This plot shows two waves (red and blue) added together, along with the resulting wave (black). These graphs represent the height of the wave at each point. The waves may be any linear wave, including ripples on a pond, disturbances on a string, sound, or electromagnetic waves.

The superposition of most waves produces a combination of constructive and destructive interference, and can vary from place to place and time to time. Sound from a stereo, for example, can be loud in one spot and quiet in another. Varying loudness means the sound waves add partially constructively and partially destructively at different locations. A stereo has at least two speakers creating sound waves, and waves can reflect from walls. All these waves interfere, and the resulting wave is the superposition of the waves.

We have shown several examples of the superposition of waves that are similar. [Figure 16.23](#) illustrates an example of the superposition of two dissimilar waves. Here again, the disturbances add, producing a resultant wave.

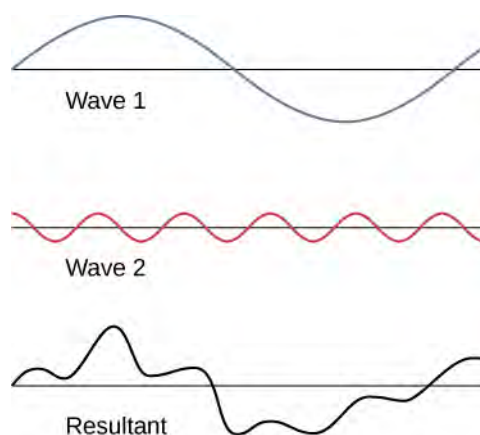


Figure 16.23 Superposition of nonidentical waves exhibits both constructive and destructive interference.

At times, when two or more mechanical waves interfere, the pattern produced by the resulting wave can be rich in complexity, some without any readily discernable patterns. For example, plotting the sound wave of your favorite music can look quite complex and is the superposition of the individual sound waves from many instruments; it is the complexity that makes the music interesting and worth listening to. At other times, waves can interfere and produce interesting phenomena, which are complex in their appearance and yet beautiful in simplicity of the physical principle of superposition, which formed the resulting wave. One example is the phenomenon known as standing waves, produced by two identical waves moving in different directions. We will look more closely at this phenomenon in the next section.

INTERACTIVE

Try this [simulation \(https://openstax.org/l/21waveinterfer\)](https://openstax.org/l/21waveinterfer) to make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern. You can observe one source or two sources. Using two sources, you can observe the interference patterns that result from varying the frequencies and the amplitudes of the sources.

Superposition of Sinusoidal Waves that Differ by a Phase Shift

Many examples in physics consist of two sinusoidal waves that are identical in amplitude, wave number, and angular frequency, but differ by a phase shift:

$$y_1(x, t) = A \sin(kx - \omega t + \phi),$$

$$y_2(x, t) = A \sin(kx - \omega t).$$

When these two waves exist in the same medium, the resultant wave resulting from the superposition of the two individual waves is the sum of the two individual waves:

$$y_R(x, t) = y_1(x, t) + y_2(x, t) = A \sin(kx - \omega t + \phi) + A \sin(kx - \omega t).$$

The resultant wave can be better understood by using the trigonometric identity:

$$\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right),$$

where $u = kx - \omega t + \phi$ and $v = kx - \omega t$. The resulting wave becomes

$$\begin{aligned} y_R(x, t) &= y_1(x, t) + y_2(x, t) = A \sin(kx - \omega t + \phi) + A \sin(kx - \omega t) \\ &= 2A \sin\left(\frac{(kx - \omega t + \phi) + (kx - \omega t)}{2}\right) \cos\left(\frac{(kx - \omega t + \phi) - (kx - \omega t)}{2}\right) \\ &= 2A \sin\left(kx - \omega t + \frac{\phi}{2}\right) \cos\left(\frac{\phi}{2}\right). \end{aligned}$$

This equation is usually written as

$$y_R(x, t) = \left[2A \cos\left(\frac{\phi}{2}\right) \right] \sin\left(kx - \omega t + \frac{\phi}{2}\right). \quad 16.13$$

The resultant wave has the same wave number and angular frequency, an amplitude of $A_R = \left[2A \cos\left(\frac{\phi}{2}\right) \right]$, and a phase shift equal to half the original phase shift. Examples of waves that differ only in a phase shift are shown in [Figure 16.24](#). The red and blue waves each have the same amplitude, wave number, and angular frequency, and differ only in a phase shift. They therefore have the same period, wavelength, and frequency. The green wave is the result of the superposition of the two waves. When the two waves have a phase difference of zero, the waves are in phase, and the resultant wave has the same wave number and angular frequency, and an amplitude equal to twice the individual amplitudes (part (a)). This is constructive interference. If the phase difference is 180° , the waves interfere in destructive interference (part (c)). The resultant wave has an amplitude of zero. Any other phase difference results in a wave with the same wave number and angular frequency as the two incident waves but with a phase shift of $\phi/2$ and an amplitude equal to $2A \cos(\phi/2)$. Examples are shown in parts (b) and (d).

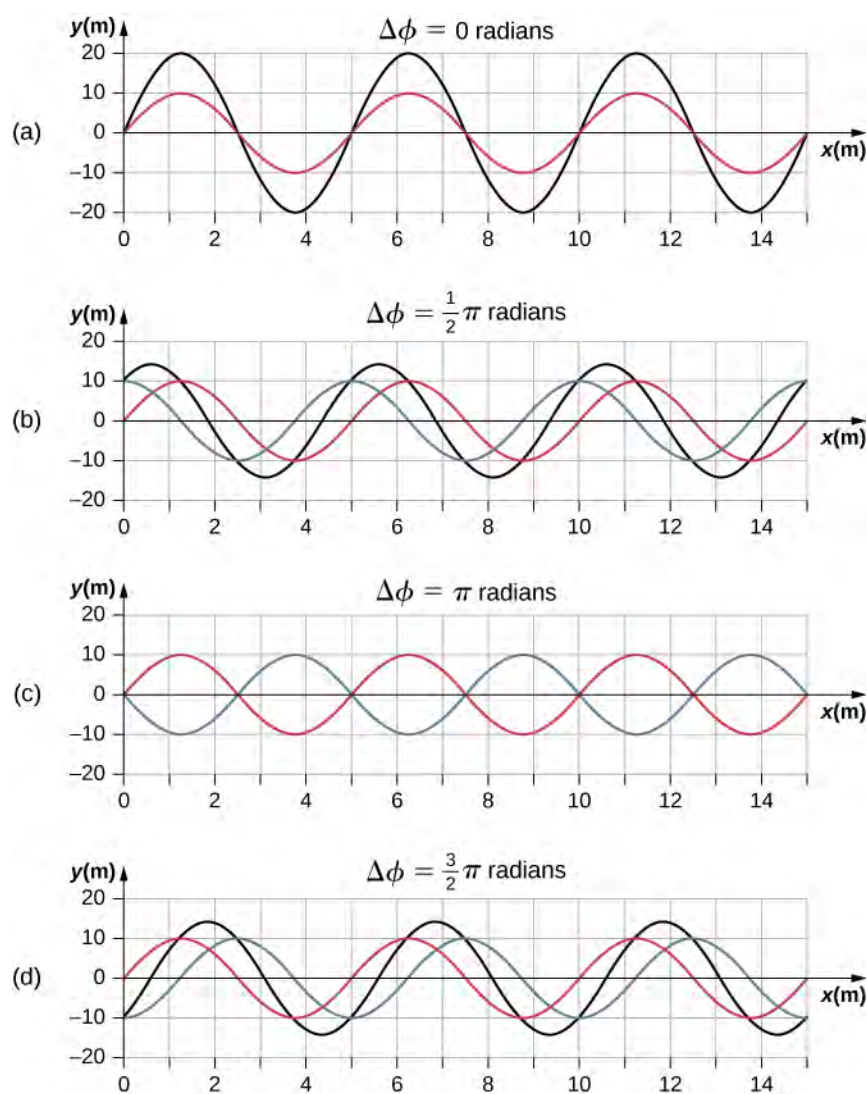


Figure 16.24 Superposition of two waves with identical amplitudes, wavelengths, and frequency, but that differ in a phase shift. The red wave is defined by the wave function $y_1(x, t) = A \sin(kx - \omega t)$ and the blue wave is defined by the wave function $y_2(x, t) = A \sin(kx - \omega t + \phi)$. The black line shows the result of adding the two waves. The phase difference between the two waves are (a) 0.00 rad, (b) $\pi/2$ rad, (c) π rad, and (d) $3\pi/2$ rad.

16.6 Standing Waves and Resonance

Learning Objectives

By the end of this section, you will be able to:

- Describe standing waves and explain how they are produced
- Describe the modes of a standing wave on a string
- Provide examples of standing waves beyond the waves on a string

Throughout this chapter, we have been studying traveling waves, or waves that transport energy from one place to another. Under certain conditions, waves can bounce back and forth through a particular region, effectively becoming stationary. These are called **standing waves**.

Another related effect is known as resonance. In [Oscillations](#), we defined resonance as a phenomenon in which a small-amplitude driving force could produce large-amplitude motion. Think of a child on a swing, which can be modeled as a physical pendulum. Relatively small-amplitude pushes by a parent can produce large-amplitude swings. Sometimes this resonance is good—for example, when producing music with a stringed instrument. At other times, the effects can be devastating, such as the collapse of a building during an earthquake. In the case of standing waves, the relatively large amplitude standing waves are produced by the superposition of smaller amplitude component waves.

Standing Waves

Sometimes waves do not seem to move; rather, they just vibrate in place. You can see unmoving waves on the surface of a glass of milk in a refrigerator, for example. Vibrations from the refrigerator motor create waves on the milk that oscillate up and down but do not seem to move across the surface. [Figure 16.25](#) shows an experiment you can try at home. Take a bowl of milk and place it on a common box fan. Vibrations from the fan will produce circular standing waves in the milk. The waves are visible in the photo due to the reflection from a lamp. These waves are formed by the superposition of two or more traveling waves, such as illustrated in [Figure 16.26](#) for two identical waves moving in opposite directions. The waves move through each other with their disturbances adding as they go by. If the two waves have the same amplitude and wavelength, then they alternate between constructive and destructive interference. The resultant looks like a wave standing in place and, thus, is called a standing wave.



Figure 16.25 Standing waves are formed on the surface of a bowl of milk sitting on a box fan. The vibrations from the fan causes the surface of the milk to oscillate. The waves are visible due to the reflection of light from a lamp. (credit: David Chelton)

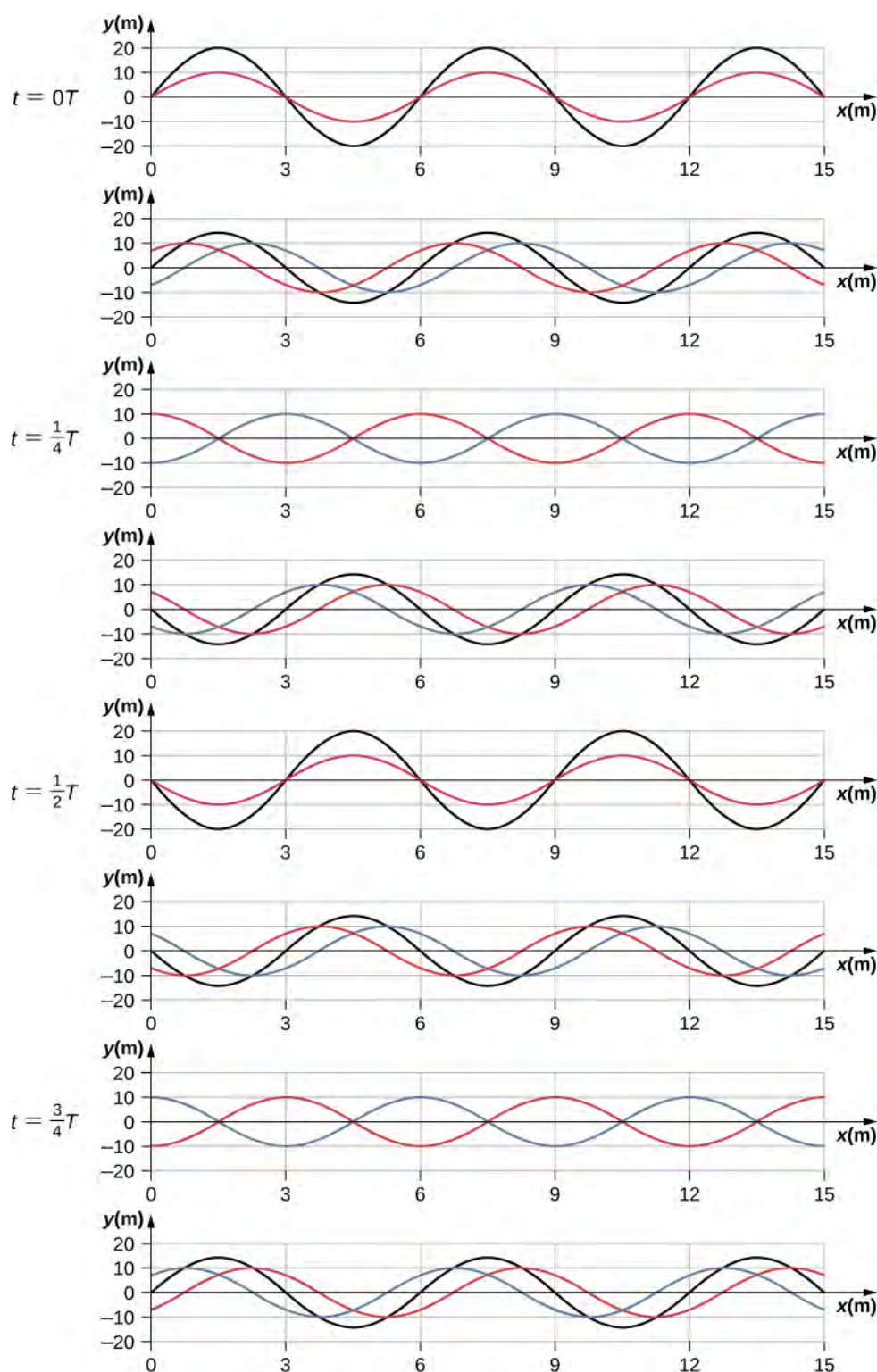


Figure 16.26 Time snapshots of two sine waves. The red wave is moving in the $-x$ -direction and the blue wave is moving in the $+x$ -direction. The resulting wave is shown in black. Consider the resultant wave at the points $x = 0 \text{ m}, 3 \text{ m}, 6 \text{ m}, 9 \text{ m}, 12 \text{ m}, 15 \text{ m}$ and notice that the resultant wave always equals zero at these points, no matter what the time is. These points are known as fixed points (nodes). In between each two nodes is an antinode, a place where the medium oscillates with an amplitude equal to the sum of the amplitudes of the individual waves.

Consider two identical waves that move in opposite directions. The first wave has a wave function of $y_1(x, t) = A \sin(kx - \omega t)$ and the second wave has a wave function $y_2(x, t) = A \sin(kx + \omega t)$. The waves

interfere and form a resultant wave

$$y(x, t) = y_1(x, t) + y_2(x, t),$$

$$y(x, t) = A \sin(kx - \omega t) + A \sin(kx + \omega t).$$

This can be simplified using the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta,$$

where $\alpha = kx$ and $\beta = \omega t$, giving us

$$y(x, t) = A[\sin(kx) \cos(\omega t) - \cos(kx) \sin(\omega t) + \sin(kx) \cos(\omega t) + \cos(kx) \sin(\omega t)],$$

which simplifies to

$$y(x, t) = [2A \sin(kx)] \cos(\omega t).$$

16.14

Notice that the resultant wave is a sine wave that is a function only of position, multiplied by a cosine function that is a function only of time. Graphs of $y(x, t)$ as a function of x for various times are shown in [Figure 16.26](#). The red wave moves in the negative x -direction, the blue wave moves in the positive x -direction, and the black wave is the sum of the two waves. As the red and blue waves move through each other, they move in and out of constructive interference and destructive interference.

Initially, at time $t = 0$, the two waves are in phase, and the result is a wave that is twice the amplitude of the individual waves. The waves are also in phase at the time $t = \frac{T}{2}$. In fact, the waves are in phase at any integer multiple of half of a period:

$$t = n \frac{T}{2} \text{ where } n = 0, 1, 2, 3, \dots \text{ (in phase).}$$

At other times, the two waves are $180^\circ (\pi \text{ radians})$ out of phase, and the resulting wave is equal to zero. This happens at

$$t = \frac{1}{4}T, \frac{3}{4}T, \frac{5}{4}T, \dots, \frac{n}{4}T \text{ where } n = 1, 3, 5, \dots \text{ (out of phase).}$$

Notice that some x -positions of the resultant wave are always zero no matter what the phase relationship is. These positions are called **nodes**. Where do the nodes occur? Consider the solution to the sum of the two waves

$$y(x, t) = [2A \sin(kx)] \cos(\omega t).$$

Finding the positions where the sine function equals zero provides the positions of the nodes.

$$\begin{aligned} \sin(kx) &= 0 \\ kx &= 0, \pi, 2\pi, 3\pi, \dots \\ \frac{2\pi}{\lambda}x &= 0, \pi, 2\pi, 3\pi, \dots \\ x &= 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \dots = n \frac{\lambda}{2} \quad n = 0, 1, 2, 3, \dots \end{aligned}$$

There are also positions where y oscillates between $y = \pm A$. These are the **antinodes**. We can find them by considering which values of x result in $\sin(kx) = \pm 1$.

$$\begin{aligned} \sin(kx) &= \pm 1 \\ kx &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \\ \frac{2\pi}{\lambda}x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \\ x &= \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots = n \frac{\lambda}{4} \quad n = 1, 3, 5, \dots \end{aligned}$$

What results is a standing wave as shown in [Figure 16.27](#), which shows snapshots of the resulting wave of two identical waves moving in opposite directions. The resulting wave appears to be a sine wave with nodes at integer multiples of half wavelengths. The antinodes oscillate between $y = \pm 2A$ due to the cosine term, $\cos(\omega t)$, which oscillates between ± 1 .

The resultant wave appears to be standing still, with no apparent movement in the x -direction, although it is composed of one wave function moving in the positive, whereas the second wave is moving in the negative x -direction. [Figure 16.27](#) shows various snapshots of the resulting wave. The nodes are marked with red dots while the antinodes are marked with blue dots.

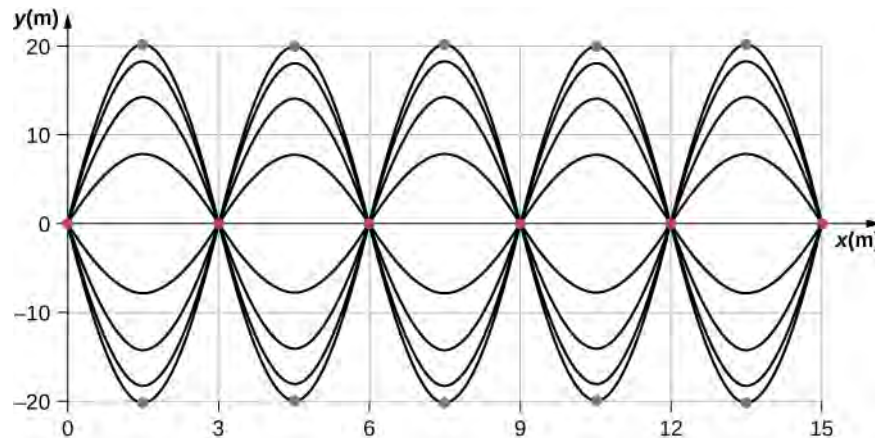


Figure 16.27 When two identical waves are moving in opposite directions, the resultant wave is a standing wave. Nodes appear at integer multiples of half wavelengths. Antinodes appear at odd multiples of quarter wavelengths, where they oscillate between $y = \pm A$. The nodes are marked with red dots and the antinodes are marked with blue dots.

A common example of standing waves are the waves produced by stringed musical instruments. When the string is plucked, pulses travel along the string in opposite directions. The ends of the strings are fixed in place, so nodes appear at the ends of the strings—the boundary conditions of the system, regulating the resonant frequencies in the strings. The resonance produced on a string instrument can be modeled in a physics lab using the apparatus shown in [Figure 16.28](#).

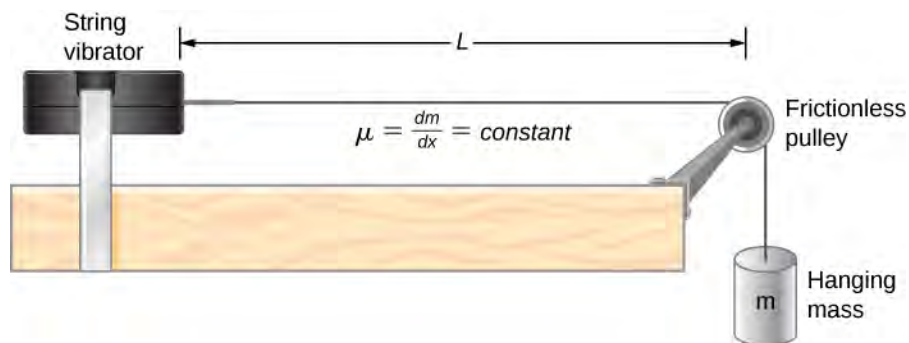


Figure 16.28 A lab setup for creating standing waves on a string. The string has a node on each end and a constant linear density. The length between the fixed boundary conditions is L . The hanging mass provides the tension in the string, and the speed of the waves on the string is proportional to the square root of the tension divided by the linear mass density.

The lab setup shows a string attached to a string vibrator, which oscillates the string with an adjustable frequency f . The other end of the string passes over a frictionless pulley and is tied to a hanging mass. The magnitude of the tension in the string is equal to the weight of the hanging mass. The string has a constant linear density (mass per length) μ and the speed at which a wave travels down the string equals

$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{mg}{\mu}}$ [Equation 16.7](#). The symmetrical boundary conditions (a node at each end) dictate the possible frequencies that can excite standing waves. Starting from a frequency of zero and slowly increasing the frequency, the first mode $n = 1$ appears as shown in [Figure 16.29](#). The first mode, also called the fundamental mode or the first harmonic, shows half of a wavelength has formed, so the wavelength is equal to twice the length between the nodes $\lambda_1 = 2L$. The **fundamental frequency**, or first harmonic frequency, that drives this mode is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L},$$

where the speed of the wave is $v = \sqrt{\frac{F_T}{\mu}}$. Keeping the tension constant and increasing the frequency leads to the second harmonic or the $n = 2$ mode. This mode is a full wavelength $\lambda_2 = L$ and the frequency is twice the fundamental frequency:

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1.$$

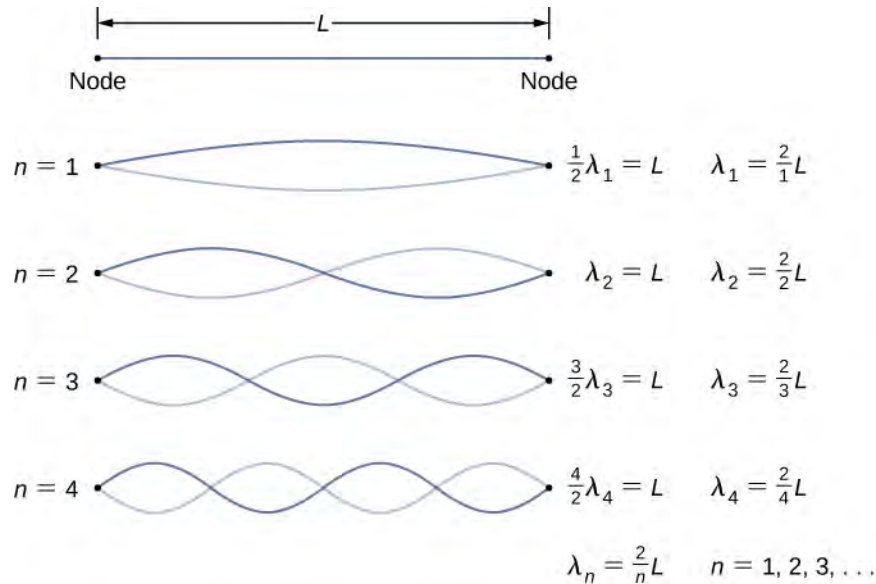


Figure 16.29 Standing waves created on a string of length L . A node occurs at each end of the string. The nodes are boundary conditions that limit the possible frequencies that excite standing waves. (Note that the amplitudes of the oscillations have been kept constant for visualization. The standing wave patterns possible on the string are known as the normal modes. Conducting this experiment in the lab would result in a decrease in amplitude as the frequency increases.)

The next two modes, or the third and fourth harmonics, have wavelengths of $\lambda_3 = \frac{2}{3}L$ and $\lambda_4 = \frac{2}{4}L$, driven by frequencies of $f_3 = \frac{3v}{2L} = 3f_1$ and $f_4 = \frac{4v}{2L} = 4f_1$. All frequencies above the frequency f_1 are known as the **overtone**s. The equations for the wavelength and the frequency can be summarized as:

$$\lambda_n = \frac{2}{n}L \quad n = 1, 2, 3, 4, 5, \dots \quad 16.15$$

$$f_n = n \frac{v}{2L} = nf_1 \quad n = 1, 2, 3, 4, 5, \dots \quad 16.16$$

The standing wave patterns that are possible for a string, the first four of which are shown in [Figure 16.29](#), are known as the **normal modes**, with frequencies known as the normal frequencies. In summary, the first frequency to produce a normal mode is called the fundamental frequency (or first harmonic). Any frequencies above the fundamental frequency are overtones. The second frequency of the $n = 2$ normal mode of the string is the first overtone (or second harmonic). The frequency of the $n = 3$ normal mode is the second overtone (or third harmonic) and so on.

The solutions shown as [Equation 16.15](#) and [Equation 16.16](#) are for a string with the boundary condition of a node on each end. When the boundary condition on either side is the same, the system is said to have symmetric boundary conditions. [Equation 16.15](#) and [Equation 16.16](#) are good for any symmetric boundary conditions, that is, nodes at both ends or antinodes at both ends.

EXAMPLE 16.7

Standing Waves on a String

Consider a string of $L = 2.00$ m, attached to an adjustable-frequency string vibrator as shown in [Figure 16.30](#). The waves produced by the vibrator travel down the string and are reflected by the fixed boundary condition at the pulley. The string, which has a linear mass density of $\mu = 0.006$ kg/m, is passed over a frictionless pulley of a negligible mass, and the tension is provided by a 2.00-kg hanging mass. (a) What is the velocity of the waves on the string? (b) Draw a sketch of the first three normal modes of the standing waves that can be produced on the string and label each with the wavelength. (c) List the frequencies that the string vibrator must be tuned to in order to produce the first three normal modes of the standing waves.

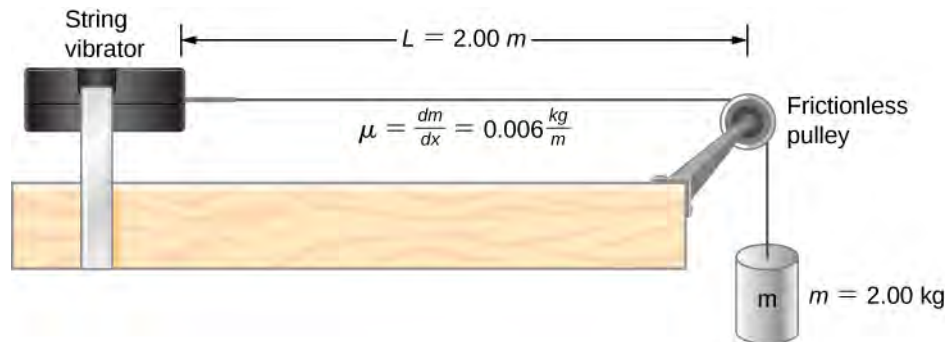


Figure 16.30 A string attached to an adjustable-frequency string vibrator.

Strategy

- The velocity of the wave can be found using $v = \sqrt{\frac{F_T}{\mu}}$. The tension is provided by the weight of the hanging mass.
- The standing waves will depend on the boundary conditions. There must be a node at each end. The first mode will be one half of a wave. The second can be found by adding a half wavelength. That is the shortest length that will result in a node at the boundaries. For example, adding one quarter of a wavelength will result in an antinode at the boundary and is not a mode which would satisfy the boundary conditions. This is shown in [Figure 16.31](#).
- Since the wave speed velocity is the wavelength times the frequency, the frequency is wave speed divided by the wavelength.



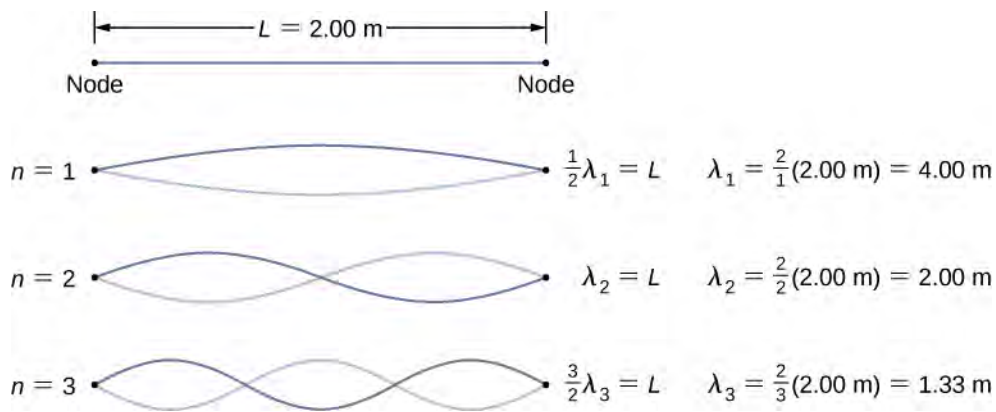
Figure 16.31 (a) The figure represents the second mode of the string that satisfies the boundary conditions of a node at each end of the string. (b) This figure could not possibly be a normal mode on the string because it does not satisfy the boundary conditions. There is a node on one end, but an antinode on the other.

Solution

- Begin with the velocity of a wave on a string. The tension is equal to the weight of the hanging mass. The linear mass density and mass of the hanging mass are given:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{mg}{\mu}} = \sqrt{\frac{2 \text{ kg} (9.8 \frac{\text{m}}{\text{s}^2})}{0.006 \frac{\text{kg}}{\text{m}}}} = 57.15 \text{ m/s}.$$

- The first normal mode that has a node on each end is a half wavelength. The next two modes are found by adding a half of a wavelength.



c. The frequencies of the first three modes are found by using $f = \frac{v\omega}{\lambda}$.

$$f_1 = \frac{v\omega}{\lambda_1} = \frac{57.15 \text{ m/s}}{4.00 \text{ m}} = 14.29 \text{ Hz}$$

$$f_2 = \frac{v\omega}{\lambda_2} = \frac{57.15 \text{ m/s}}{2.00 \text{ m}} = 28.58 \text{ Hz}$$

$$f_3 = \frac{v\omega}{\lambda_3} = \frac{57.15 \text{ m/s}}{1.333 \text{ m}} = 42.87 \text{ Hz}$$

Significance

The three standing modes in this example were produced by maintaining the tension in the string and adjusting the driving frequency. Keeping the tension in the string constant results in a constant velocity. The same modes could have been produced by keeping the frequency constant and adjusting the speed of the wave in the string (by changing the hanging mass.)

INTERACTIVE

Visit this [simulation \(https://openstax.org/l/21normalmodes\)](https://openstax.org/l/21normalmodes) to play with a 1D or 2D system of coupled mass-spring oscillators. Vary the number of masses, set the initial conditions, and watch the system evolve. See the spectrum of normal modes for arbitrary motion. See longitudinal or transverse modes in the 1D system.

CHECK YOUR UNDERSTANDING 16.7

The equations for the wavelengths and the frequencies of the modes of a wave produced on a string:

$$\lambda_n = \frac{2}{n}L \quad n = 1, 2, 3, 4, 5, \dots \text{ and}$$

$$f_n = n \frac{v}{2L} = nf_1 \quad n = 1, 2, 3, 4, 5, \dots$$

were derived by considering a wave on a string where there were symmetric boundary conditions of a node at each end. These modes resulted from two sinusoidal waves with identical characteristics except they were moving in opposite directions, confined to a region L with nodes required at both ends. Will the same equations work if there were symmetric boundary conditions with antinodes at each end? What would the normal modes look like for a medium that was free to oscillate on each end? Don't worry for now if you cannot imagine such a medium, just consider two sinusoidal wave functions in a region of length L , with antinodes on each end.

The free boundary conditions shown in the last Check Your Understanding may seem hard to visualize. How can there be a system that is free to oscillate on each end? In [Figure 16.32](#) are shown two possible configuration of a metallic rods (shown in red) attached to two supports (shown in blue). In part (a), the rod is supported at the ends, and there are fixed boundary conditions at both ends. Given the proper frequency, the

rod can be driven into resonance with a wavelength equal to length of the rod, with nodes at each end. In part (b), the rod is supported at positions one quarter of the length from each end of the rod, and there are free boundary conditions at both ends. Given the proper frequency, this rod can also be driven into resonance with a wavelength equal to the length of the rod, but there are antinodes at each end. If you are having trouble visualizing the wavelength in this figure, remember that the wavelength may be measured between any two nearest identical points and consider [Figure 16.33](#).

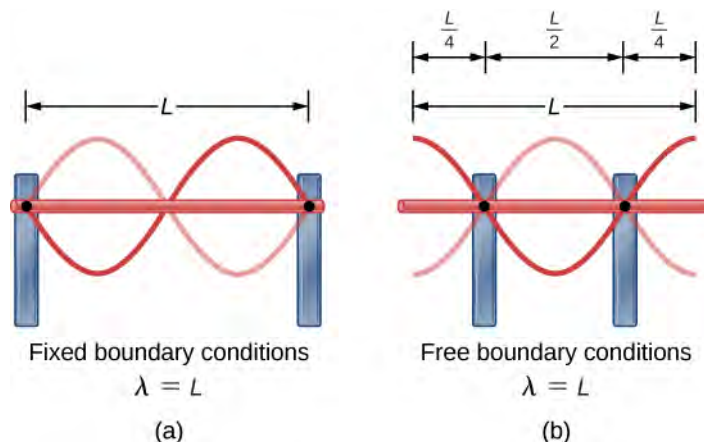


Figure 16.32 (a) A metallic rod of length L (red) supported by two supports (blue) on each end. When driven at the proper frequency, the rod can resonate with a wavelength equal to the length of the rod with a node on each end. (b) The same metallic rod of length L (red) supported by two supports (blue) at a position a quarter of the length of the rod from each end. When driven at the proper frequency, the rod can resonate with a wavelength equal to the length of the rod with an antinode on each end.

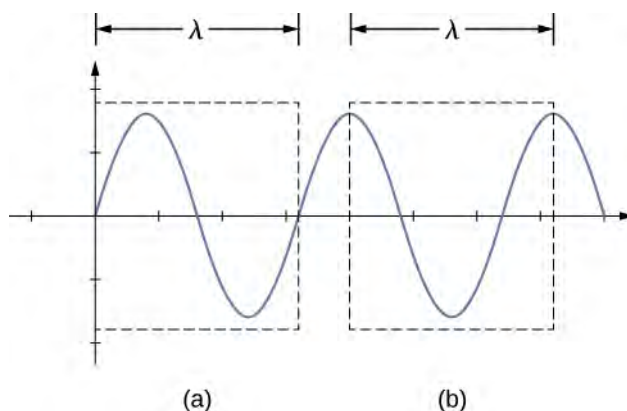


Figure 16.33 A wavelength may be measure between the nearest two repeating points. On the wave on a string, this means the same height and slope. (a) The wavelength is measured between the two nearest points where the height is zero and the slope is maximum and positive. (b) The wavelength is measured between two identical points where the height is maximum and the slope is zero.

Note that the study of standing waves can become quite complex. In [Figure 16.32\(a\)](#), the $n = 2$ mode of the standing wave is shown, and it results in a wavelength equal to L . In this configuration, the $n = 1$ mode would also have been possible with a standing wave equal to $2L$. Is it possible to get the $n = 1$ mode for the configuration shown in part (b)? The answer is no. In this configuration, there are additional conditions set beyond the boundary conditions. Since the rod is mounted at a point one quarter of the length from each side, a node must exist there, and this limits the possible modes of standing waves that can be created. We leave it as an exercise for the reader to consider if other modes of standing waves are possible. It should be noted that when a system is driven at a frequency that does not cause the system to resonate, vibrations may still occur, but the amplitude of the vibrations will be much smaller than the amplitude at resonance.

A field of mechanical engineering uses the sound produced by the vibrating parts of complex mechanical systems to troubleshoot problems with the systems. Suppose a part in an automobile is resonating at the frequency of the car's engine, causing unwanted vibrations in the automobile. This may cause the engine to fail

prematurely. The engineers use microphones to record the sound produced by the engine, then use a technique called Fourier analysis to find frequencies of sound produced with large amplitudes and then look at the parts list of the automobile to find a part that would resonate at that frequency. The solution may be as simple as changing the composition of the material used or changing the length of the part in question.

There are other numerous examples of resonance in standing waves in the physical world. The air in a tube, such as found in a musical instrument like a flute, can be forced into resonance and produce a pleasant sound, as we discuss in [Sound](#).

At other times, resonance can cause serious problems. A closer look at earthquakes provides evidence for conditions appropriate for resonance, standing waves, and constructive and destructive interference. A building may vibrate for several seconds with a driving frequency matching that of the natural frequency of vibration of the building—producing a resonance resulting in one building collapsing while neighboring buildings do not. Often, buildings of a certain height are devastated while other taller buildings remain intact. The building height matches the condition for setting up a standing wave for that particular height. The span of the roof is also important. Often it is seen that gymnasiums, supermarkets, and churches suffer damage when individual homes suffer far less damage. The roofs with large surface areas supported only at the edges resonate at the frequencies of the earthquakes, causing them to collapse. As the earthquake waves travel along the surface of Earth and reflect off denser rocks, constructive interference occurs at certain points. Often areas closer to the epicenter are not damaged, while areas farther away are damaged.

CHAPTER REVIEW

Key Terms

antinode location of maximum amplitude in standing waves

constructive interference when two waves arrive at the same point exactly in phase; that is, the crests of the two waves are precisely aligned, as are the troughs

destructive interference when two identical waves arrive at the same point exactly out of phase; that is, precisely aligned crest to trough

fixed boundary condition when the medium at a boundary is fixed in place so it cannot move

free boundary condition exists when the medium at the boundary is free to move

fundamental frequency lowest frequency that will produce a standing wave

intensity (I) power per unit area

interference overlap of two or more waves at the same point and time

linear wave equation equation describing waves that result from a linear restoring force of the medium; any function that is a solution to the wave equation describes a wave moving in the positive x -direction or the negative x -direction with a constant wave speed v

longitudinal wave wave in which the disturbance is parallel to the direction of propagation

mechanical wave wave that is governed by Newton's laws and requires a medium

node point where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave

normal mode possible standing wave pattern for a standing wave on a string

overtone frequency that produces standing waves and is higher than the fundamental frequency

pulse single disturbance that moves through a medium, transferring energy but not mass

standing wave wave that can bounce back and forth through a particular region, effectively becoming stationary

superposition phenomenon that occurs when two or more waves arrive at the same point

transverse wave wave in which the disturbance is perpendicular to the direction of propagation

wave disturbance that moves from its source and carries energy

wave function mathematical model of the position of particles of the medium

wave number $\frac{2\pi}{\lambda}$

wave speed magnitude of the wave velocity

wave velocity velocity at which the disturbance moves; also called the propagation velocity

wavelength distance between adjacent identical parts of a wave

Key Equations

Wave speed

$$v = \frac{\lambda}{T} = \lambda f$$

Linear mass density

$$\mu = \frac{\text{mass of the string}}{\text{length of the string}}$$

Speed of a wave or pulse on a string under tension

$$|v| = \sqrt{\frac{F_T}{\mu}}$$

Speed of a compression wave in a fluid

$$v = \sqrt{\frac{B}{\rho}}$$

Resultant wave from superposition of two sinusoidal waves that are identical except for a phase shift

$$y_R(x, t) = \left[2A \cos\left(\frac{\phi}{2}\right) \right] \sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

Wave number

$$k \equiv \frac{2\pi}{\lambda}$$

Wave speed

$$v = \frac{\omega}{k}$$

A periodic wave	$y(x, t) = A \sin(kx \mp \omega t + \phi)$
Phase of a wave	$kx \mp \omega t + \phi$
The linear wave equation	$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v_w^2} \frac{\partial^2 y(x, t)}{\partial t^2}$
Power averaged over a wavelength	$P_{\text{ave}} = \frac{E\lambda}{T} = \frac{1}{2} \mu A^2 \omega^2 \frac{\lambda}{T} = \frac{1}{2} \mu A^2 \omega^2 v$
Intensity	$I = \frac{P}{A}$
Intensity for a spherical wave	$I = \frac{P}{4\pi r^2}$
Equation of a standing wave	$y(x, t) = [2A \sin(kx)] \cos(\omega t)$
Wavelength for symmetric boundary conditions	$\lambda_n = \frac{2}{n} L, \quad n = 1, 2, 3, 4, 5 \dots$
Frequency for symmetric boundary conditions	$f_n = n \frac{v}{2L} = n f_1, \quad n = 1, 2, 3, 4, 5 \dots$

Summary

16.1 Traveling Waves

- A wave is a disturbance that moves from the point of origin with a wave velocity v .
- A wave has a wavelength λ , which is the distance between adjacent identical parts of the wave. Wave velocity and wavelength are related to the wave's frequency and period by $v = \frac{\lambda}{T} = \lambda f$.
- Mechanical waves are disturbances that move through a medium and are governed by Newton's laws.
- Electromagnetic waves are disturbances in the electric and magnetic fields, and do not require a medium.
- Matter waves are a central part of quantum mechanics and are associated with protons, electrons, neutrons, and other fundamental particles found in nature.
- A transverse wave has a disturbance perpendicular to the wave's direction of propagation, whereas a longitudinal wave has a disturbance parallel to its direction of propagation.

16.2 Mathematics of Waves

- A wave is an oscillation (of a physical quantity) that travels through a medium, accompanied by a transfer of energy. Energy transfers from one

point to another in the direction of the wave motion. The particles of the medium oscillate up and down, back and forth, or both up and down and back and forth, around an equilibrium position.

- A snapshot of a sinusoidal wave at time $t = 0.00$ s can be modeled as a function of position. Two examples of such functions are $y(x) = A \sin(kx + \phi)$ and $y(x) = A \cos(kx + \phi)$.
- Given a function of a wave that is a snapshot of the wave, and is only a function of the position x , the motion of the pulse or wave moving at a constant velocity can be modeled with the function, replacing x with $x \mp vt$. The minus sign is for motion in the positive direction and the plus sign for the negative direction.
- The wave function is given by $y(x, t) = A \sin(kx - \omega t + \phi)$ where $k = 2\pi/\lambda$ is defined as the wave number, $\omega = 2\pi/T$ is the angular frequency, and ϕ is the phase shift.
- The wave moves with a constant velocity v_w , where the particles of the medium oscillate about an equilibrium position. The constant velocity of a wave can be found by $v = \frac{\lambda}{T} = \frac{\omega}{k}$.

16.3 Wave Speed on a Stretched String

- The speed of a wave on a string depends on the

linear density of the string and the tension in the string. The linear density is mass per unit length of the string.

- In general, the speed of a wave depends on the square root of the ratio of the elastic property to the inertial property of the medium.
- The speed of a wave through a fluid is equal to the square root of the ratio of the bulk modulus of the fluid to the density of the fluid.
- The speed of sound through air at $T = 20^\circ\text{C}$ is approximately $v_s = 343.00\text{ m/s}$.

16.4 Energy and Power of a Wave

- The energy and power of a wave are proportional to the square of the amplitude of the wave and the square of the angular frequency of the wave.
- The time-averaged power of a sinusoidal wave on a string is found by $P_{\text{ave}} = \frac{1}{2}\mu A^2\omega^2 v$, where μ is the linear mass density of the string, A is the amplitude of the wave, ω is the angular frequency of the wave, and v is the speed of the wave.
- Intensity is defined as the power divided by the area. In a spherical wave, the area is $A = 4\pi r^2$ and the intensity is $I = \frac{P}{4\pi r^2}$. As the wave moves out from a source, the energy is conserved, but the intensity decreases as the area increases.

Conceptual Questions

16.1 Traveling Waves

1. Give one example of a transverse wave and one example of a longitudinal wave, being careful to note the relative directions of the disturbance and wave propagation in each.
2. A sinusoidal transverse wave has a wavelength of 2.80 m. It takes 0.10 s for a portion of the string at a position x to move from a maximum position of $y = 0.03\text{ m}$ to the equilibrium position $y = 0$. What are the period, frequency, and wave speed of the wave?
3. What is the difference between propagation speed and the frequency of a mechanical wave? Does one or both affect wavelength? If so, how?
4. Consider a stretched spring, such as a slinky. The stretched spring can support longitudinal waves and transverse waves. How can you produce transverse waves on the spring? How can you produce longitudinal waves on the spring?
5. Consider a wave produced on a stretched spring

16.5 Interference of Waves

- Superposition is the combination of two waves at the same location.
- Constructive interference occurs from the superposition of two identical waves that are in phase.
- Destructive interference occurs from the superposition of two identical waves that are 180° (π radians) out of phase.
- The wave that results from the superposition of two sine waves that differ only by a phase shift is a wave with an amplitude that depends on the value of the phase difference.

16.6 Standing Waves and Resonance

- A standing wave is the superposition of two waves which produces a wave that varies in amplitude but does not propagate.
- Nodes are points of no motion in standing waves.
- An antinode is the location of maximum amplitude of a standing wave.
- Normal modes of a wave on a string are the possible standing wave patterns. The lowest frequency that will produce a standing wave is known as the fundamental frequency. The higher frequencies which produce standing waves are called overtones.

by holding one end and shaking it up and down. Does the wavelength depend on the distance you move your hand up and down?

6. A sinusoidal, transverse wave is produced on a stretched spring, having a period T . Each section of the spring moves perpendicular to the direction of propagation of the wave, in simple harmonic motion with an amplitude A . Does each section oscillate with the same period as the wave or a different period? If the amplitude of the transverse wave were doubled but the period stays the same, would your answer be the same?
7. An electromagnetic wave, such as light, does not require a medium. Can you think of an example that would support this claim?

16.2 Mathematics of Waves

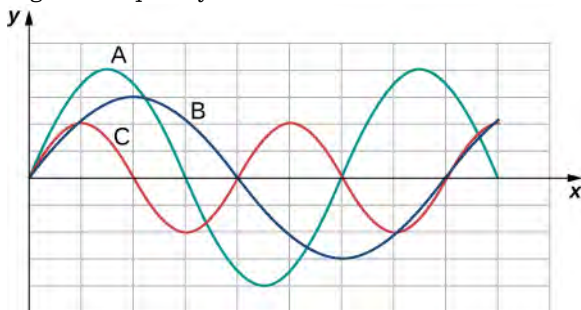
8. If you were to shake the end of a taut spring up and down 10 times a second, what would be the frequency and the period of the sinusoidal wave

produced on the spring?

9. If you shake the end of a stretched spring up and down with a frequency f , you can produce a sinusoidal, transverse wave propagating down the spring. Does the wave number depend on the frequency you are shaking the spring?
10. Does the vertical speed of a segment of a horizontal taut string through which a sinusoidal, transverse wave is propagating depend on the wave speed of the transverse wave?
11. In this section, we have considered waves that move at a constant wave speed. Does the medium accelerate?
12. If you drop a pebble in a pond you may notice that several concentric ripples are produced, not just a single ripple. Why do you think that is?

16.3 Wave Speed on a Stretched String

13. If the tension in a string were increased by a factor of four, by what factor would the wave speed of a wave on the string increase?
14. Does a sound wave move faster in seawater or fresh water, if both the sea water and fresh water are at the same temperature and the sound wave moves near the surface?
($\rho_w \approx 1000 \frac{\text{kg}}{\text{m}^3}$, $\rho_s \approx 1030 \frac{\text{kg}}{\text{m}^3}$, $B_w = 2.15 \times 10^9$ Pa, $B_s = 2.34 \times 10^9$ Pa)
15. Guitars have strings of different linear mass density. If the lowest density string and the highest density string are under the same tension, which string would support waves with the higher wave speed?
16. Shown below are three waves that were sent down a string at different times. The tension in the string remains constant. (a) Rank the waves from the smallest wavelength to the largest wavelength. (b) Rank the waves from the lowest frequency to the highest frequency.



17. Electrical power lines connected by two utility poles are sometimes heard to hum when driven

into oscillation by the wind. The speed of the waves on the power lines depend on the tension. What provides the tension in the power lines?

18. Two strings, one with a low mass density and one with a high linear density are spliced together. The higher density end is tied to a lab post and a student holds the free end of the low-mass density string. The student gives the string a flip and sends a pulse down the strings. If the tension is the same in both strings, does the pulse travel at the same wave velocity in both strings? If not, where does it travel faster, in the low density string or the high density string?

16.4 Energy and Power of a Wave

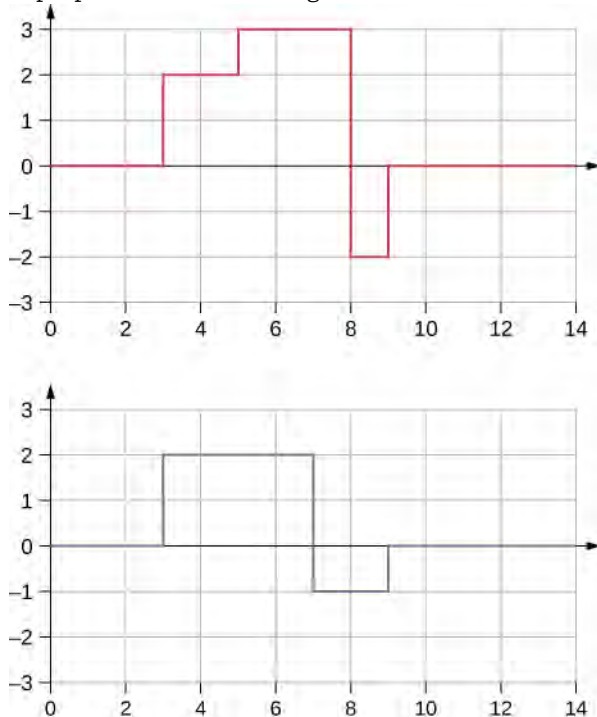
19. Consider a string with under tension with a constant linear mass density. A sinusoidal wave with an angular frequency and amplitude produced by some external driving force. If the frequency of the driving force is decreased to half of the original frequency, how is the time-averaged power of the wave affected? If the amplitude of the driving force is decreased by half, how is the time-averaged power affected? Explain your answer.
20. Circular water waves decrease in amplitude as they move away from where a rock is dropped. Explain why.
21. In a transverse wave on a string, the motion of the string is perpendicular to the motion of the wave. If this is so, how is possible to move energy along the length of the string?
22. The energy from the sun warms the portion of the earth facing the sun during the daylight hours. Why are the North and South Poles cold while the equator is quite warm?
23. The intensity of a spherical waves decreases as the wave moves away from the source. If the intensity of the wave at the source is I_0 , how far from the source will the intensity decrease by a factor of nine?

16.5 Interference of Waves

24. An incident sinusoidal wave is sent along a string that is fixed to the wall with a wave speed of v . The wave reflects off the end of the string. Describe the reflected wave.
25. A string of a length of 2.00 m with a linear mass density of $\mu = 0.006 \text{ kg/m}$ is attached to the end of a 2.00-m-long string with a linear mass

density of $\mu = 0.012 \text{ kg/m}$. The free end of the higher-density string is fixed to the wall, and a student holds the free end of the low-density string, keeping the tension constant in both strings. The student sends a pulse down the string. Describe what happens at the interface between the two strings.

26. A long, tight spring is held by two students, one student holding each end. Each student gives the end a flip sending one wavelength of a sinusoidal wave down the spring in opposite directions. When the waves meet in the middle, what does the wave look like?
27. Many of the topics discussed in this chapter are useful beyond the topics of mechanical waves. It is hard to conceive of a mechanical wave with sharp corners, but you could encounter such a wave form in your digital electronics class, as shown below. This could be a signal from a device known as an analog to digital converter, in which a continuous voltage signal is converted into a discrete signal or a digital recording of sound. What is the result of the superposition of the two signals?



Problems

16.1 Traveling Waves

34. Storms in the South Pacific can create waves that travel all the way to the California coast, 12,000 km away. How long does it take them to travel this distance if they travel at 15.0 m/s?
35. Waves on a swimming pool propagate at 0.75 m/s. You splash the water at one end of the pool and observe the wave go to the opposite end, reflect, and return in 30.00 s. How far away is the other end of the pool?
36. Wind gusts create ripples on the ocean that have a wavelength of 5.00 cm and propagate at
28. A string of a constant linear mass density is held taut by two students, each holding one end. The tension in the string is constant. The students each send waves down the string by wiggling the string. (a) Is it possible for the waves to have different wave speeds? (b) Is it possible for the waves to have different frequencies? (c) Is it possible for the waves to have different wavelengths?

16.6 Standing Waves and Resonance

29. A truck manufacturer finds that a strut in the engine is failing prematurely. A sound engineer determines that the strut resonates at the frequency of the engine and suspects that this could be the problem. What are two possible characteristics of the strut can be modified to correct the problem?
30. Why do roofs of gymnasiums and churches seem to fail more than family homes when an earthquake occurs?
31. Wine glasses can be set into resonance by moistening your finger and rubbing it around the rim of the glass. Why?
32. Air conditioning units are sometimes placed on the roof of homes in the city. Occasionally, the air conditioners cause an undesirable hum throughout the upper floors of the homes. Why does this happen? What can be done to reduce the hum?
33. Consider a standing wave modeled as $y(x, t) = 4.00 \text{ cm} \sin(3 \text{ m}^{-1} x) \cos(4 \text{ s}^{-1} t)$. Is there a node or an antinode at $x = 0.00 \text{ m}$? What about a standing wave modeled as $y(x, t) = 4.00 \text{ cm} \sin(3 \text{ m}^{-1} x + \frac{\pi}{2}) \cos(4 \text{ s}^{-1} t)$? Is there a node or an antinode at the $x = 0.00 \text{ m}$ position?

- 2.00 m/s. What is their frequency?
37. How many times a minute does a boat bob up and down on ocean waves that have a wavelength of 40.0 m and a propagation speed of 5.00 m/s?
 38. Scouts at a camp shake the rope bridge they have just crossed and observe the wave crests to be 8.00 m apart. If they shake the bridge twice per second, what is the propagation speed of the waves?
 39. What is the wavelength of the waves you create in a swimming pool if you splash your hand at a rate of 2.00 Hz and the waves propagate at a wave speed of 0.800 m/s?
 40. What is the wavelength of an earthquake that shakes you with a frequency of 10.0 Hz and gets to another city 84.0 km away in 12.0 s?
 41. Radio waves transmitted through empty space at the speed of light ($v = c = 3.00 \times 10^8$ m/s) by the *Voyager* spacecraft have a wavelength of 0.120 m. What is their frequency?
 42. Your ear is capable of differentiating sounds that arrive at each ear just 0.34 ms apart, which is useful in determining where low frequency sound is originating from. (a) Suppose a low-frequency sound source is placed to the right of a person, whose ears are approximately 18 cm apart, and the speed of sound generated is 340 m/s. How long is the interval between when the sound arrives at the right ear and the sound arrives at the left ear? (b) Assume the same person was scuba diving and a low-frequency sound source was to the right of the scuba diver. How long is the interval between when the sound arrives at the right ear and the sound arrives at the left ear, if the speed of sound in water is 1500 m/s? (c) What is significant about the time interval of the two situations?
 43. (a) Seismographs measure the arrival times of earthquakes with a precision of 0.100 s. To get the distance to the epicenter of the quake, geologists compare the arrival times of S- and P-waves, which travel at different speeds. If S- and P-waves arrive at 4.00 and 7.20 km/s, respectively, in the region considered, how precisely can the distance to the source of the earthquake be determined? (b) Seismic waves from underground detonations of nuclear bombs can be used to locate the test site and detect violations of test bans. Discuss whether your answer to (a) implies a serious limit to such detection. (Note also that the uncertainty is greater if there is an uncertainty in the propagation speeds of the S- and P-waves.)
 44. A Girl Scout is taking a 10.00-km hike to earn a merit badge. While on the hike, she sees a cliff some distance away. She wishes to estimate the time required to walk to the cliff. She knows that the speed of sound is approximately 343 meters per second. She yells and finds that the echo returns after approximately 2.00 seconds. If she can hike 1.00 km in 10 minutes, how long would it take her to reach the cliff?
 45. A quality assurance engineer at a frying pan company is asked to qualify a new line of nonstick-coated frying pans. The coating needs to be 1.00 mm thick. One method to test the thickness is for the engineer to pick a percentage of the pans manufactured, strip off the coating, and measure the thickness using a micrometer. This method is a destructive testing method. Instead, the engineer decides that every frying pan will be tested using a nondestructive method. An ultrasonic transducer is used that produces sound waves with a frequency of $f = 25$ kHz. The sound waves are sent through the coating and are reflected by the interface between the coating and the metal pan, and the time is recorded. The wavelength of the ultrasonic waves in the coating is 0.076 m. What should be the time recorded if the coating is the correct thickness (1.00 mm)?

16.2 Mathematics of Waves

46. A pulse can be described as a single wave disturbance that moves through a medium. Consider a pulse that is defined at time $t = 0.00$ s by the equation $y(x) = \frac{6.00 \text{ m}^3}{x^2 + 2.00 \text{ m}^2}$ centered around $x = 0.00$ m. The pulse moves with a velocity of $v = 3.00$ m/s in the positive x -direction. (a) What is the amplitude of the pulse? (b) What is the equation of the pulse as a function of position and time? (c) Where is the pulse centered at time $t = 5.00$ s?
47. A transverse wave on a string is modeled with the wave function $y(x, t) = (0.20 \text{ cm})\sin(2.00 \text{ m}^{-1}x - 3.00 \text{ s}^{-1}t + \frac{\pi}{16})$. What is the height of the string with respect to the equilibrium position at a position $x = 4.00$ m and a time $t = 10.00$ s?
48. Consider the wave function $y(x, t) = (3.00 \text{ cm})\sin(0.4 \text{ m}^{-1}x + 2.00 \text{ s}^{-1}t + \frac{\pi}{10})$. What are the period, wavelength, speed, and initial phase shift of the wave modeled by the wave

function?

49. A pulse is defined as

$y(x, t) = e^{-2.77 \left(\frac{2.00(x - 2.00 \text{ m/s}(t))}{5.00 \text{ m}} \right)^2}$. Use a spreadsheet, or other computer program, to plot the pulse as the height of medium y as a function of position x . Plot the pulse at times $t = 0.00 \text{ s}$ and $t = 3.00 \text{ s}$ on the same graph. Where is the pulse centered at time $t = 3.00 \text{ s}$? Use your spreadsheet to check your answer.

50. A wave is modeled at time $t = 0.00 \text{ s}$ with a wave function that depends on position. The equation is $y(x) = (0.30 \text{ m})\sin(6.28 \text{ m}^{-1}x)$. The wave travels a distance of 4.00 meters in 0.50 s in the positive x -direction. Write an equation for the wave as a function of position and time.

51. A wave is modeled with the function $y(x, t) = (0.25 \text{ m})\cos(0.30 \text{ m}^{-1}x - 0.90 \text{ s}^{-1}t + \frac{\pi}{3})$. Find the (a) amplitude, (b) wave number, (c) angular frequency, (d) wave speed, (e) initial phase shift, (f) wavelength, and (g) period of the wave.

52. A surface ocean wave has an amplitude of 0.60 m and the distance from trough to trough is 8.00 m. It moves at a constant wave speed of 1.50 m/s propagating in the positive x -direction. At $t = 0$, the water displacement at $x = 0$ is zero, and v_y is positive. (a) Assuming the wave can be modeled as a sine wave, write a wave function to model the wave. (b) Use a spreadsheet to plot the wave function at times $t = 0.00 \text{ s}$ and $t = 2.00 \text{ s}$ on the same graph. Verify that the wave moves 3.00 m in those 2.00 s.

53. A wave is modeled by the wave function $y(x, t) = (0.30 \text{ m})\sin\left[\frac{2\pi}{4.50 \text{ m}}(x - 18.00 \frac{\text{m}}{\text{s}}t)\right]$. What are the amplitude, wavelength, wave speed, period, and frequency of the wave?

54. A transverse wave on a string is described with the wave function $y(x, t) = (0.50 \text{ cm})\sin(1.57 \text{ m}^{-1}x - 6.28 \text{ s}^{-1}t)$. (a) What is the wave velocity of the wave? (b) What is the magnitude of the maximum velocity of the string perpendicular to the direction of the motion?

55. A swimmer in the ocean observes one day that the ocean surface waves are periodic and resemble a sine wave. The swimmer estimates that the vertical distance between the crest and the trough of each wave is approximately 0.45 m, and the distance between each crest is approximately 1.8 m. The swimmer counts that 12 waves pass every two minutes. Determine the simple harmonic wave function that would

describes these waves.

56. Consider a wave described by the wave function $y(x, t) = 0.3 \text{ m} \sin(2.00 \text{ m}^{-1}x - 628.00 \text{ s}^{-1}t)$. (a) How many crests pass by an observer at a fixed location in 2.00 minutes? (b) How far has the wave traveled in that time?

57. Consider two waves defined by the wave functions

$$y_1(x, t) = 0.50 \text{ m} \sin\left(\frac{2\pi}{3.00 \text{ m}}x + \frac{2\pi}{4.00 \text{ s}}t\right) \text{ and } y_2(x, t) = 0.50 \text{ m} \sin\left(\frac{2\pi}{6.00 \text{ m}}x - \frac{2\pi}{4.00 \text{ s}}t\right).$$

What are the similarities and differences between the two waves?

58. Consider two waves defined by the wave functions

$$y_1(x, t) = 0.20 \text{ m} \sin\left(\frac{2\pi}{6.00 \text{ m}}x - \frac{2\pi}{4.00 \text{ s}}t\right) \text{ and } y_2(x, t) = 0.20 \text{ m} \cos\left(\frac{2\pi}{6.00 \text{ m}}x - \frac{2\pi}{4.00 \text{ s}}t\right).$$

What are the similarities and differences between the two waves?

59. The speed of a transverse wave on a string is 300.00 m/s, its wavelength is 0.50 m, and the amplitude is 20.00 cm. How much time is required for a particle on the string to move through a distance of 5.00 km?

16.3 Wave Speed on a Stretched String

60. Transverse waves are sent along a 5.00-m-long string with a speed of 30.00 m/s. The string is under a tension of 10.00 N. What is the mass of the string?

61. A copper wire has a density of $\rho = 8920 \text{ kg/m}^3$, a radius of 1.20 mm, and a length L . The wire is held under a tension of 10.00 N. Transverse waves are sent down the wire. (a) What is the linear mass density of the wire? (b) What is the speed of the waves through the wire?

62. A piano wire has a linear mass density of $\mu = 4.95 \times 10^{-3} \text{ kg/m}$. Under what tension must the string be kept to produce waves with a wave speed of 500.00 m/s?

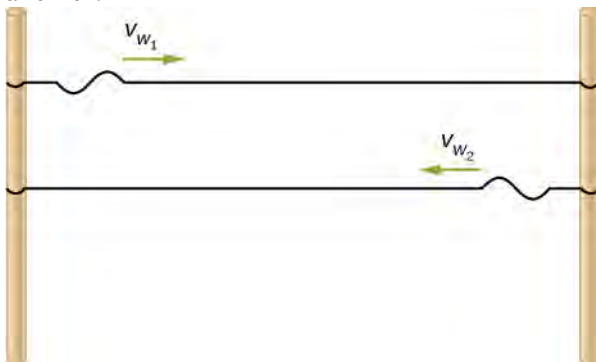
63. A string with a linear mass density of $\mu = 0.0060 \text{ kg/m}$ is tied to the ceiling. A 20-kg mass is tied to the free end of the string. The string is plucked, sending a pulse down the string. Estimate the speed of the pulse as it moves down the string.

64. A cord has a linear mass density of $\mu = 0.0075 \text{ kg/m}$ and a length of three meters. The cord is plucked and it takes 0.20 s for the pulse to reach the end of the string. What is the tension of the string?

65. A string is 3.00 m long with a mass of 5.00 g.

The string is held taut with a tension of 500.00 N applied to the string. A pulse is sent down the string. How long does it take the pulse to travel the 3.00 m of the string?

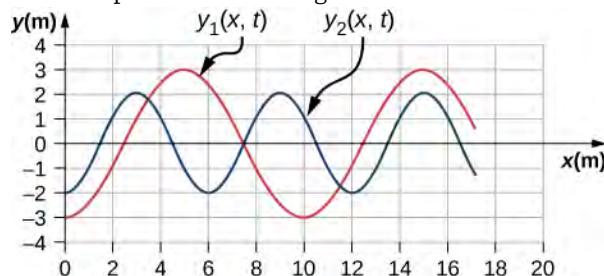
66. Two strings are attached to poles, however the first string is twice as long as the second. If both strings have the same tension and μ , what is the ratio of the speed of the pulse of the wave from the first string to the second string?
67. Two strings are attached to poles, however the first string is twice the linear mass density μ of the second. If both strings have the same tension, what is the ratio of the speed of the pulse of the wave from the first string to the second string?
68. Transverse waves travel through a string where the tension equals 7.00 N with a speed of 20.00 m/s. What tension would be required for a wave speed of 25.00 m/s?
69. Two strings are attached between two poles separated by a distance of 2.00 m as shown below, both under the same tension of 600.00 N. String 1 has a linear density of $\mu_1 = 0.0025$ kg/m and string 2 has a linear mass density of $\mu_2 = 0.0035$ kg/m. Transverse wave pulses are generated simultaneously at opposite ends of the strings. How much time passes before the pulses pass one another?



70. Two strings are attached between two poles separated by a distance of 2.00 meters as shown in the preceding figure, both strings have a linear density of $\mu_1 = 0.0025$ kg/m, the tension in string 1 is 600.00 N and the tension in string 2 is 700.00 N. Transverse wave pulses are generated simultaneously at opposite ends of the strings. How much time passes before the pulses pass one another?
71. The note E_4 is played on a piano and has a frequency of $f = 393.88$. If the linear mass density of this string of the piano is $\mu = 0.012$ kg/m and the string is under a tension of 1000.00 N, what is the speed of the

wave on the string and the wavelength of the wave?

72. Two transverse waves travel through a taut string. The speed of each wave is $v = 30.00$ m/s. A plot of the vertical position as a function of the horizontal position is shown below for the time $t = 0.00$ s. (a) What is the wavelength of each wave? (b) What is the frequency of each wave? (c) What is the maximum vertical speed of each string?



73. A sinusoidal wave travels down a taut, horizontal string with a linear mass density of $\mu = 0.060$ kg/m. The maximum vertical speed of the wave is $v_{y \max} = 0.30$ cm/s. The wave is modeled with the wave equation $y(x, t) = A \sin(6.00 \text{ m}^{-1}x - 24.00 \text{ s}^{-1}t)$. (a) What is the amplitude of the wave? (b) What is the tension in the string?
74. The speed of a transverse wave on a string is $v = 60.00$ m/s and the tension in the string is $F_T = 100.00$ N. What must the tension be to increase the speed of the wave to $v = 120.00$ m/s?

16.4 Energy and Power of a Wave

75. A string of length 5 m and a mass of 90 g is held under a tension of 100 N. A wave travels down the string that is modeled as $y(x, t) = 0.01 \text{ m} \sin(15.7 \text{ m}^{-1}x - 1170.12 \text{ s}^{-1}t)$. What is the power over one wavelength?
76. Ultrasound of intensity $1.50 \times 10^2 \text{ W/m}^2$ is produced by the rectangular head of a medical imaging device measuring 3.00 cm by 5.00 cm. What is its power output?
77. The low-frequency speaker of a stereo set has a surface area of $A = 0.05 \text{ m}^2$ and produces 1 W of acoustical power. (a) What is the intensity at the speaker? (b) If the speaker projects sound uniformly in all directions, at what distance from the speaker is the intensity 0.1 W/m^2 ?
78. To increase the intensity of a wave by a factor of 50, by what factor should the amplitude be increased?
79. A device called an insolation meter is used to measure the intensity of sunlight. It has an area

- of 100 cm^2 and registers 6.50 W . What is the intensity in W/m^2 ?
80. Energy from the Sun arrives at the top of Earth's atmosphere with an intensity of 1400 W/m^2 . How long does it take for $1.80 \times 10^9 \text{ J}$ to arrive on an area of 1.00 m^2 ?
 81. Suppose you have a device that extracts energy from ocean breakers in direct proportion to their intensity. If the device produces 10.0 kW of power on a day when the breakers are 1.20 m high, how much will it produce when they are 0.600 m high?
 82. A photovoltaic array of (solar cells) is 10.0% efficient in gathering solar energy and converting it to electricity. If the average intensity of sunlight on one day is 70.00 W/m^2 , what area should your array have to gather energy at the rate of 100 W ? (b) What is the maximum cost of the array if it must pay for itself in two years of operation averaging 10.0 hours per day? Assume that it earns money at the rate of 9.00 cents per kilowatt-hour.
 83. A microphone receiving a pure sound tone feeds an oscilloscope, producing a wave on its screen. If the sound intensity is originally $2.00 \times 10^{-5} \text{ W/m}^2$, but is turned up until the amplitude increases by 30.0% , what is the new intensity?
 84. A string with a mass of 0.30 kg has a length of 4.00 m . If the tension in the string is 50.00 N , and a sinusoidal wave with an amplitude of 2.00 cm is induced on the string, what must the frequency be for an average power of 100.00 W ?
 85. The power versus time for a point on a string ($\mu = 0.05 \text{ kg/m}$) in which a sinusoidal traveling wave is induced is shown in the preceding figure. The wave is modeled with the wave equation $y(x, t) = A \sin(20.93 \text{ m}^{-1}x - \omega t)$. What is the frequency and amplitude of the wave?
 86. A string is under tension F_{T1} . Energy is transmitted by a wave on the string at rate P_1 by a wave of frequency f_1 . What is the ratio of the new energy transmission rate P_2 to P_1 if the tension is doubled?
 87. A 250-Hz tuning fork is struck and the intensity at the source is I_1 at a distance of one meter from the source. (a) What is the intensity at a distance of 4.00 m from the source? (b) How far from the tuning fork is the intensity a tenth of the intensity at the source?
 88. A sound speaker is rated at a voltage of $P = 120.00 \text{ V}$ and a current of $I = 10.00 \text{ A}$. Electrical power consumption is $P = IV$. To test the speaker, a signal of a sine wave is applied to the speaker. Assuming that the sound wave moves as a spherical wave and that all of the energy applied to the speaker is converted to sound energy, how far from the speaker is the intensity equal to 3.82 W/m^2 ?
 89. The energy of a ripple on a pond is proportional to the amplitude squared. If the amplitude of the ripple is 0.1 cm at a distance from the source of 6.00 meters , what was the amplitude at a distance of 2.00 meters from the source?

16.5 Interference of Waves

90. Consider two sinusoidal waves traveling along a string, modeled as $y_1(x, t) = 0.3 \text{ m} \sin(4 \text{ m}^{-1}x + 3 \text{ s}^{-1}t)$ and $y_2(x, t) = 0.6 \text{ m} \sin(8 \text{ m}^{-1}x - 6 \text{ s}^{-1}t)$. What is the height of the resultant wave formed by the interference of the two waves at the position $x = 0.5 \text{ m}$ at time $t = 0.2 \text{ s}$?
91. Consider two sinusoidal sine waves traveling along a string, modeled as $y_1(x, t) = 0.3 \text{ m} \sin(4 \text{ m}^{-1}x + 3 \text{ s}^{-1}t + \frac{\pi}{3})$ and $y_2(x, t) = 0.6 \text{ m} \sin(8 \text{ m}^{-1}x - 6 \text{ s}^{-1}t)$. What is the height of the resultant wave formed by the interference of the two waves at the position $x = 1.0 \text{ m}$ at time $t = 3.0 \text{ s}$?
92. Consider two sinusoidal sine waves traveling along a string, modeled as $y_1(x, t) = 0.3 \text{ m} \sin(4 \text{ m}^{-1}x - 3 \text{ s}^{-1}t)$ and $y_2(x, t) = 0.3 \text{ m} \sin(4 \text{ m}^{-1}x + 3 \text{ s}^{-1}t)$. What is the wave function of the resulting wave? [Hint: Use the trig identity $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$]
93. Two sinusoidal waves are moving through a medium in the same direction, both having amplitudes of 3.00 cm , a wavelength of 5.20 m , and a period of 6.52 s , but one has a phase shift of an angle ϕ . What is the phase shift if the resultant wave has an amplitude of 5.00 cm ? [Hint: Use the trig identity $\sin u + \sin v = 2 \sin(\frac{u+v}{2}) \cos(\frac{u-v}{2})$]
94. Two sinusoidal waves are moving through a medium in the positive x -direction, both having amplitudes of 6.00 cm , a wavelength of 4.3 m , and a period of 6.00 s , but one has a phase shift of an angle $\phi = 0.50 \text{ rad}$. What is the height of the resultant wave at a time $t = 3.15 \text{ s}$ and a position $x = 0.45 \text{ m}$?
95. Two sinusoidal waves are moving through a

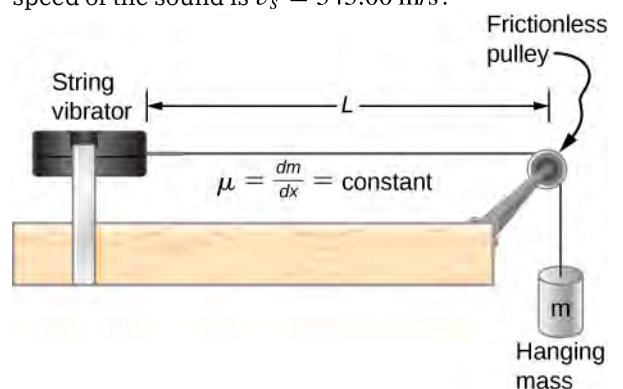
medium in the positive x -direction, both having amplitudes of 7.00 cm, a wave number of $k = 3.00 \text{ m}^{-1}$, an angular frequency of $\omega = 2.50 \text{ s}^{-1}$, and a period of 6.00 s, but one has a phase shift of an angle $\phi = \frac{\pi}{12}$ rad. What is the height of the resultant wave at a time $t = 2.00 \text{ s}$ and a position $x = 0.53 \text{ m}$?

96. Consider two waves $y_1(x, t)$ and $y_2(x, t)$ that are identical except for a phase shift propagating in the same medium. (a) What is the phase shift, in radians, if the amplitude of the resulting wave is 1.75 times the amplitude of the individual waves? (b) What is the phase shift in degrees? (c) What is the phase shift as a percentage of the individual wavelength?
97. Two sinusoidal waves, which are identical except for a phase shift, travel along in the same direction. The wave equation of the resultant wave is $y_R(x, t) = 0.70 \text{ m} \sin(3.00 \text{ m}^{-1}x - 6.28 \text{ s}^{-1}t + \pi/16 \text{ rad})$. What are the angular frequency, wave number, amplitude, and phase shift of the individual waves?
98. Two sinusoidal waves, which are identical except for a phase shift, travel along in the same direction. The wave equation of the resultant wave is $y_R(x, t) = 0.35 \text{ cm} \sin(6.28 \text{ m}^{-1}x - 1.57 \text{ s}^{-1}t + \frac{\pi}{4})$. What are the period, wavelength, amplitude, and phase shift of the individual waves?
99. Consider two wave functions, $y_1(x, t) = 4.00 \text{ m} \sin(\pi \text{ m}^{-1}x - \pi \text{ s}^{-1}t)$ and $y_2(x, t) = 4.00 \text{ m} \sin(\pi \text{ m}^{-1}x - \pi \text{ s}^{-1}t + \frac{\pi}{3})$. (a) Using a spreadsheet, plot the two wave functions and the wave that results from the superposition of the two wave functions as a function of position ($0.00 \leq x \leq 6.00 \text{ m}$) for the time $t = 0.00 \text{ s}$. (b) What are the wavelength and amplitude of the two original waves? (c) What are the wavelength and amplitude of the resulting wave?
100. Consider two wave functions, $y_2(x, t) = 2.00 \text{ m} \sin(\frac{\pi}{2} \text{ m}^{-1}x - \frac{\pi}{3} \text{ s}^{-1}t)$ and $y_2(x, t) = 2.00 \text{ m} \sin(\frac{\pi}{2} \text{ m}^{-1}x - \frac{\pi}{3} \text{ s}^{-1}t + \frac{\pi}{6})$. (a) Verify that $y_R = 2A \cos(\frac{\phi}{2}) \sin(kx - \omega t + \frac{\phi}{2})$ is the solution for the wave that results from a superposition of the two waves. Make a column for x , y_1 , y_2 , $y_1 + y_2$, and $y_R = 2A \cos(\frac{\phi}{2}) \sin(kx - \omega t + \frac{\phi}{2})$. Plot four waves as a function of position where the range of x is from 0 to 12 m.

101. Consider two wave functions that differ only by a phase shift, $y_1(x, t) = A \cos(kx - \omega t)$ and $y_2(x, t) = A \cos(kx - \omega t + \phi)$. Use the trigonometric identities $\cos u + \cos v = 2 \cos(\frac{u+v}{2}) \cos(\frac{u-v}{2})$ and $\cos(-\theta) = \cos(\theta)$ to find a wave equation for the wave resulting from the superposition of the two waves. Does the resulting wave function come as a surprise to you?

16.6 Standing Waves and Resonance

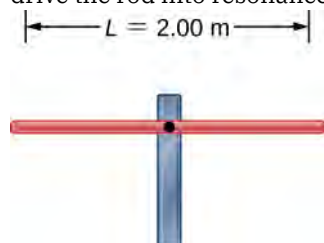
102. A wave traveling on a Slinky® that is stretched to 4 m takes 2.4 s to travel the length of the Slinky and back again. (a) What is the speed of the wave? (b) Using the same Slinky stretched to the same length, a standing wave is created which consists of three antinodes and four nodes. At what frequency must the Slinky be oscillating?
103. A 2-m long string is stretched between two supports with a tension that produces a wave speed equal to $v_w = 50.00 \text{ m/s}$. What are the wavelength and frequency of the first three modes that resonate on the string?
104. Consider the experimental setup shown below. The length of the string between the string vibrator and the pulley is $L = 1.00 \text{ m}$. The linear density of the string is $\mu = 0.006 \text{ kg/m}$. The string vibrator can oscillate at any frequency. The hanging mass is 2.00 kg. (a) What are the wavelength and frequency of $n = 6$ mode? (b) The string oscillates the air around the string. What is the wavelength of the sound if the speed of the sound is $v_s = 343.00 \text{ m/s}$?



105. A cable with a linear density of $\mu = 0.02 \text{ kg/m}$ is hung from telephone poles. The tension in the cable is 500.00 N. The distance between poles is 20 meters. The wind blows across the line, causing the cable resonate. A standing waves pattern is produced that has 4.5 wavelengths between the two poles. The speed of sound at the current temperature $T = 20^\circ\text{C}$

is 343.00 m/s. What are the frequency and wavelength of the hum?

- 106.** Consider a rod of length L , mounted in the center to a support. A node must exist where the rod is mounted on a support, as shown below. Draw the first two normal modes of the rod as it is driven into resonance. Label the wavelength and the frequency required to drive the rod into resonance.



- 107.** Consider two wave functions $y(x, t) = 0.30 \text{ cm} \sin(3 \text{ m}^{-1}x - 4 \text{ s}^{-1}t)$ and $y(x, t) = 0.30 \text{ cm} \sin(3 \text{ m}^{-1}x + 4 \text{ s}^{-1}t)$. Write a wave function for the resulting standing wave.
- 108.** A 2.40-m wire has a mass of 7.50 g and is under a tension of 160 N. The wire is held rigidly at both ends and set into oscillation. (a) What is the speed of waves on the wire? The string is driven into resonance by a frequency that produces a standing wave with a wavelength equal to 1.20 m. (b) What is the frequency used to drive the string into resonance?
- 109.** A string with a linear mass density of 0.0062 kg/m and a length of 3.00 m is set into the $n = 100$ mode of resonance. The tension in the string is 20.00 N. What is the wavelength and frequency of the wave?
- 110.** A string with a linear mass density of 0.0075 kg/m and a length of 6.00 m is set into the $n = 4$ mode of resonance by driving with a frequency of 100.00 Hz. What is the tension in the string?
- 111.** Two sinusoidal waves with identical wavelengths and amplitudes travel in opposite

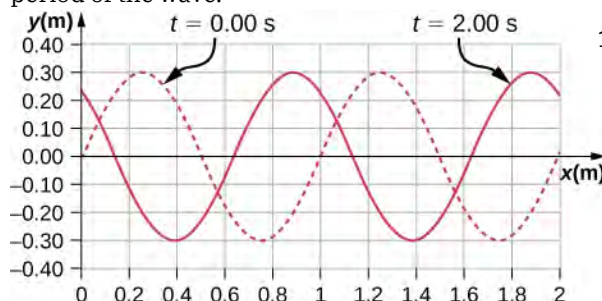
directions along a string producing a standing wave. The linear mass density of the string is $\mu = 0.075 \text{ kg/m}$ and the tension in the string is $F_T = 5.00 \text{ N}$. The time interval between instances of total destructive interference is $\Delta t = 0.13 \text{ s}$. What is the wavelength of the waves?

- 112.** A string, fixed on both ends, is 5.00 m long and has a mass of 0.15 kg. The tension in the string is 90 N. The string is vibrating to produce a standing wave at the fundamental frequency of the string. (a) What is the speed of the waves on the string? (b) What is the wavelength of the standing wave produced? (c) What is the period of the standing wave?
- 113.** A string is fixed at both ends. The mass of the string is 0.0090 kg and the length is 3.00 m. The string is under a tension of 200.00 N. The string is driven by a variable frequency source to produce standing waves on the string. Find the wavelengths and frequency of the first four modes of standing waves.
- 114.** The frequencies of two successive modes of standing waves on a string are 258.36 Hz and 301.42 Hz. What is the next frequency above 100.00 Hz that would produce a standing wave?
- 115.** A string is fixed at both ends to supports 3.50 m apart and has a linear mass density of $\mu = 0.005 \text{ kg/m}$. The string is under a tension of 90.00 N. A standing wave is produced on the string with six nodes and five antinodes. What are the wave speed, wavelength, frequency, and period of the standing wave?
- 116.** Sine waves are sent down a 1.5-m-long string fixed at both ends. The waves reflect back in the opposite direction. The amplitude of the wave is 4.00 cm. The propagation velocity of the waves is 175 m/s. The $n = 6$ resonance mode of the string is produced. Write an equation for the resulting standing wave.

Additional Problems

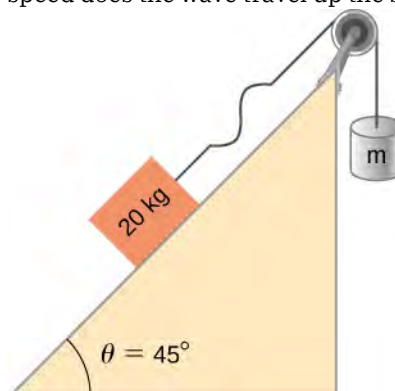
- 117.** Ultrasound equipment used in the medical profession uses sound waves of a frequency above the range of human hearing. If the frequency of the sound produced by the ultrasound machine is $f = 30 \text{ kHz}$, what is the wavelength of the ultrasound in bone, if the speed of sound in bone is $v = 3000 \text{ m/s}$?

- 118.** Shown below is the plot of a wave function that models a wave at time $t = 0.00$ s and $t = 2.00$ s. The dotted line is the wave function at time $t = 0.00$ s and the solid line is the function at time $t = 2.00$ s. Estimate the amplitude, wavelength, velocity, and period of the wave.



- 119.** The speed of light in air is approximately $v = 3.00 \times 10^8$ m/s and the speed of light in glass is $v = 2.00 \times 10^8$ m/s. A red laser with a wavelength of $\lambda = 633.00$ nm shines light incident on the glass, and some of the red light is transmitted to the glass. The frequency of the light is the same for the air and the glass. (a) What is the frequency of the light? (b) What is the wavelength of the light in the glass?
- 120.** A radio station broadcasts radio waves at a frequency of 101.7 MHz. The radio waves move through the air at approximately the speed of light in a vacuum. What is the wavelength of the radio waves?
- 121.** A sunbather stands waist deep in the ocean and observes that six crests of periodic surface waves pass each minute. The crests are 16.00 meters apart. What is the wavelength, frequency, period, and speed of the waves?
- 122.** A tuning fork vibrates producing sound at a frequency of 512 Hz. The speed of sound of sound in air is $v = 343.00$ m/s if the air is at a temperature of 20.00°C . What is the wavelength of the sound?
- 123.** A motorboat is traveling across a lake at a speed of $v_b = 15.00$ m/s. The boat bounces up and down every 0.50 s as it travels in the same direction as a wave. It bounces up and down every 0.30 s as it travels in a direction opposite the direction of the waves. What is the speed and wavelength of the wave?
- 124.** Use the linear wave equation to show that the wave speed of a wave modeled with the wave function $y(x, t) = 0.20 \text{ m} \sin(3.00 \text{ m}^{-1}x + 6.00 \text{ s}^{-1}t)$ is $v = 2.00$ m/s. What are the wavelength and the speed of the wave?
- 125.** Given the wave functions $y_1(x, t) = A \sin(kx - \omega t)$ and $y_2(x, t) = A \sin(kx - \omega t + \phi)$ with $\phi \neq \frac{\pi}{2}$, show that $y_1(x, t) + y_2(x, t)$ is a solution to the linear wave equation with a wave velocity of $v = \frac{\omega}{k}$.
- 126.** A transverse wave on a string is modeled with the wave function $y(x, t) = 0.10 \text{ m} \sin(0.15 \text{ m}^{-1}x + 1.50 \text{ s}^{-1}t + 0.20)$. (a) Find the wave velocity. (b) Find the position in the y -direction, the velocity perpendicular to the motion of the wave, and the acceleration perpendicular to the motion of the wave, of a small segment of the string centered at $x = 0.40$ m at time $t = 5.00$ s.
- 127.** A sinusoidal wave travels down a taut, horizontal string with a linear mass density of $\mu = 0.060$ kg/m. The magnitude of maximum vertical acceleration of the wave is $a_{y \text{ max}} = 0.90 \text{ cm/s}^2$ and the amplitude of the wave is 0.40 m. The string is under a tension of $F_T = 600.00$ N. The wave moves in the negative x -direction. Write an equation to model the wave.
- 128.** A transverse wave on a string ($\mu = 0.0030$ kg/m) is described with the equation $y(x, t) = 0.30 \text{ m} \sin\left(\frac{2\pi}{4.00 \text{ m}}(x - 16.00 \frac{\text{m}}{\text{s}}t)\right)$. What is the tension under which the string is held taut?
- 129.** A transverse wave on a horizontal string ($\mu = 0.0060$ kg/m) is described with the equation $y(x, t) = 0.30 \text{ m} \sin\left(\frac{2\pi}{4.00 \text{ m}}(x - v_w t)\right)$. The string is under a tension of 300.00 N. What are the wave speed, wave number, and angular frequency of the wave?
- 130.** A student holds an inexpensive sonic range finder and uses the range finder to find the distance to the wall. The sonic range finder emits a sound wave. The sound wave reflects off the wall and returns to the range finder. The round trip takes 0.012 s. The range finder was calibrated for use at room temperature $T = 20^\circ\text{C}$, but the temperature in the room is actually $T = 23^\circ\text{C}$. Assuming that the timing mechanism is perfect, what percentage of error can the student expect due to the calibration?

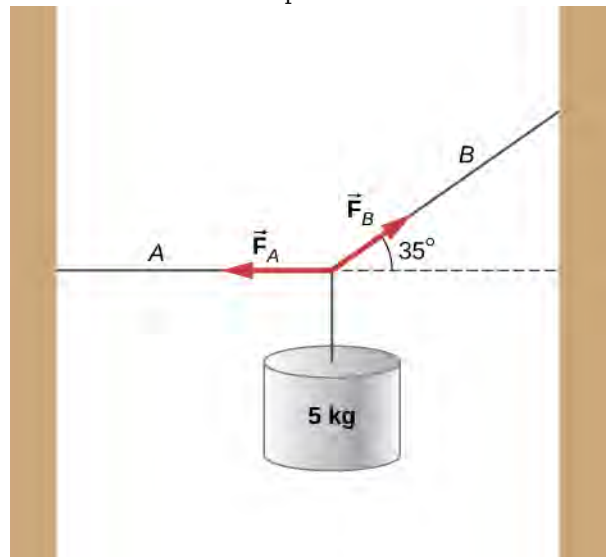
- 131.** A wave on a string is driven by a string vibrator, which oscillates at a frequency of 100.00 Hz and an amplitude of 1.00 cm. The string vibrator operates at a voltage of 12.00 V and a current of 0.20 A. The power consumed by the string vibrator is $P = IV$. Assume that the string vibrator is 90% efficient at converting electrical energy into the energy associated with the vibrations of the string. The string is 3.00 m long, and is under a tension of 60.00 N. What is the linear mass density of the string?
- 132.** A traveling wave on a string is modeled by the wave equation
 $y(x, t) = 3.00 \text{ cm} \sin(8.00 \text{ m}^{-1}x + 100.00 \text{ s}^{-1}t)$.
 The string is under a tension of 50.00 N and has a linear mass density of $\mu = 0.008 \text{ kg/m}$. What is the average power transferred by the wave on the string?
- 133.** A transverse wave on a string has a wavelength of 5.0 m, a period of 0.02 s, and an amplitude of 1.5 cm. The average power transferred by the wave is 5.00 W. What is the tension in the string?
- 134.** (a) What is the intensity of a laser beam used to burn away cancerous tissue that, when 90.0% absorbed, puts 500.0 J of energy into a circular spot 2.00 mm in diameter in 4.00 s? (b) Discuss how this intensity compares to the average intensity of sunlight (about 1 kW/m^2) and the implications that would have if the laser beam entered your eye. Note how your answer depends on the time duration of the exposure.
- 135.** Consider two periodic wave functions,
 $y_1(x, t) = A \sin(kx - \omega t)$ and
 $y_2(x, t) = A \sin(kx - \omega t + \phi)$. (a) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of $2A$? (b) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of zero?
- 136.** Consider two periodic wave functions,
 $y_1(x, t) = A \sin(kx - \omega t)$ and
 $y_2(x, t) = A \cos(kx - \omega t + \phi)$. (a) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of $2A$? (b) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of zero?
- 137.** A trough with dimensions 10.00 meters by 0.10 meters by 0.10 meters is partially filled with water. Small-amplitude surface water waves are produced from both ends of the trough by paddles oscillating in simple harmonic motion. The height of the water waves are modeled with two sinusoidal wave equations,
 $y_1(x, t) = 0.3 \text{ m} \sin(4 \text{ m}^{-1}x - 3 \text{ s}^{-1}t)$ and
 $y_2(x, t) = 0.3 \text{ m} \cos(4 \text{ m}^{-1}x + 3 \text{ s}^{-1}t - \frac{\pi}{2})$.
 What is the wave function of the resulting wave after the waves reach one another and before they reach the end of the trough (i.e., assume that there are only two waves in the trough and ignore reflections)? Use a spreadsheet to check your results. (*Hint:* Use the trig identities $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$ and $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$)
- 138.** A seismograph records the S- and P-waves from an earthquake 20.00 s apart. If they traveled the same path at constant wave speeds of $v_S = 4.00 \text{ km/s}$ and $v_P = 7.50 \text{ km/s}$, how far away is the epicenter of the earthquake?
- 139.** Consider what is shown below. A 20.00-kg mass rests on a frictionless ramp inclined at 45° . A string with a linear mass density of $\mu = 0.025 \text{ kg/m}$ is attached to the 20.00-kg mass. The string passes over a frictionless pulley of negligible mass and is attached to a hanging mass (m). The system is in static equilibrium. A wave is induced on the string and travels up the ramp. (a) What is the mass of the hanging mass (m)? (b) At what wave speed does the wave travel up the string?



- 140.** Consider the superposition of three wave functions
 $y(x, t) = 3.00 \text{ cm} \sin(2 \text{ m}^{-1}x - 3 \text{ s}^{-1}t)$,
 $y(x, t) = 3.00 \text{ cm} \sin(6 \text{ m}^{-1}x + 3 \text{ s}^{-1}t)$, and
 $y(x, t) = 3.00 \text{ cm} \sin(2 \text{ m}^{-1}x - 4 \text{ s}^{-1}t)$. What is the height of the resulting wave at position $x = 3.00 \text{ m}$ at time $t = 10.0 \text{ s}$?
- 141.** A string has a mass of 150 g and a length of 3.4 m . One end of the string is fixed to a lab stand and the other is attached to a spring with a spring constant of $k_s = 100 \text{ N/m}$. The free end of the spring is attached to another lab pole. The tension in the string is maintained by the spring. The lab poles are separated by a distance that stretches the spring 2.00 cm . The string is plucked and a pulse travels along the string. What is the propagation speed of the pulse?

Challenge Problems

- 144.** A copper wire has a radius of $200 \mu\text{m}$ and a length of 5.0 m . The wire is placed under a tension of 3000 N and the wire stretches by a small amount. The wire is plucked and a pulse travels down the wire. What is the propagation speed of the pulse? (Assume the temperature does not change:
 $(\rho = 8.96 \frac{\text{g}}{\text{cm}^3}, Y = 1.1 \times 10^{11} \frac{\text{N}}{\text{m}})$.)
- 145.** A pulse moving along the x axis can be modeled as the wave function
 $y(x, t) = 4.00 \text{ me}^{-\left(\frac{x+(2.00 \text{ m/s})t}{1.00 \text{ m}}\right)^2}$. (a) What are the direction and propagation speed of the pulse? (b) How far has the wave moved in 3.00 s ? (c) Plot the pulse using a spreadsheet at time $t = 0.00 \text{ s}$ and $t = 3.00 \text{ s}$ to verify your answer in part (b).
- 142.** A standing wave is produced on a string under a tension of 70.0 N by two sinusoidal transverse waves that are identical, but moving in opposite directions. The string is fixed at $x = 0.00 \text{ m}$ and $x = 10.00 \text{ m}$. Nodes appear at $x = 0.00 \text{ m}$, 2.00 m , 4.00 m , 6.00 m , 8.00 m , and 10.00 m . The amplitude of the standing wave is 3.00 cm . It takes 0.10 s for the antinodes to make one complete oscillation. (a) What are the wave functions of the two sine waves that produce the standing wave? (b) What are the maximum velocity and acceleration of the string, perpendicular to the direction of motion of the transverse waves, at the antinodes?
- 143.** A string with a length of 4 m is held under a constant tension. The string has a linear mass density of $\mu = 0.006 \text{ kg/m}$. Two resonant frequencies of the string are 400 Hz and 480 Hz . There are no resonant frequencies between the two frequencies. (a) What are the wavelengths of the two resonant modes? (b) What is the tension in the string?
- 146.** A string with a linear mass density of $\mu = 0.0085 \text{ kg/m}$ is fixed at both ends. A 5.0-kg mass is hung from the string, as shown below. If a pulse is sent along section A, what is the wave speed in section A and the wave speed in section B?



- 147.** Consider two wave functions

$$y_1(x, t) = A \sin(kx - \omega t) \text{ and}$$

$y_2(x, t) = A \sin(kx + \omega t + \phi)$. What is the wave function resulting from the interference of the two wave? (*Hint:*

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \text{ and}$$

$$\phi = \frac{\phi}{2} + \frac{\phi}{2}.)$$

- 148.** The wave function that models a standing wave is given as

$$y_R(x, t) = 6.00 \text{ cm} \sin(3.00 \text{ m}^{-1}x + 1.20 \text{ rad})$$

$\cos(6.00 \text{ s}^{-1}t + 1.20 \text{ rad})$. What are two wave functions that interfere to form this wave function? Plot the two wave functions and the sum of the sum of the two wave functions at $t = 1.00 \text{ s}$ to verify your answer.

- 149.** Consider two wave functions

$$y_1(x, t) = A \sin(kx - \omega t) \text{ and}$$

$y_2(x, t) = A \sin(kx + \omega t + \phi)$. The resultant wave form when you add the two functions is

$$y_R = 2A \sin\left(kx + \frac{\phi}{2}\right) \cos\left(\omega t + \frac{\phi}{2}\right).$$

Consider the case where $A = 0.03 \text{ m}^{-1}$, $k = 1.26 \text{ m}^{-1}$, $\omega = \pi \text{ s}^{-1}$, and $\phi = \frac{\pi}{10}$. (a)

Where are the first three nodes of the standing wave function starting at zero and moving in the positive x direction? (b) Using a spreadsheet, plot the two wave functions and the resulting function at time $t = 1.00 \text{ s}$ to verify your answer.

CHAPTER 17

Sound



Figure 17.1 Hearing is an important human sense that can detect frequencies of sound, ranging between 20 Hz and 20 kHz. However, other species have very different ranges of hearing. Bats, for example, emit clicks in ultrasound, using frequencies beyond 20 kHz. They can detect nearby insects by hearing the echo of these ultrasonic clicks. Ultrasound is important in several human applications, including probing the interior structures of human bodies, Earth, and the Sun. Ultrasound is also useful in industry for nondestructive testing. (credit: modification of work by Derek Keats/flickr)

Chapter Outline

[17.1 Sound Waves](#)

[17.2 Speed of Sound](#)

[17.3 Sound Intensity](#)

[17.4 Normal Modes of a Standing Sound Wave](#)

[17.5 Sources of Musical Sound](#)

[17.6 Beats](#)

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[17.8 Shock Waves](#)

INTRODUCTION Sound is an example of a mechanical wave, specifically, a pressure wave: Sound waves travel

through the air and other media as oscillations of molecules. Normal human hearing encompasses an impressive range of frequencies from 20 Hz to 20 kHz. Sounds below 20 Hz are called infrasound, whereas those above 20 kHz are called ultrasound. Some animals, like the bat shown in [Figure 17.1](#), can hear sounds in the ultrasonic range.

Many of the concepts covered in [Waves](#) also have applications in the study of sound. For example, when a sound wave encounters an interface between two media with different wave speeds, reflection and transmission of the wave occur.

Ultrasound has many uses in science, engineering, and medicine. Ultrasound is used for nondestructive testing in engineering, such as testing the thickness of coating on metal. In medicine, sound waves are far less destructive than X-rays and can be used to image the fetus in a mother's womb without danger to the fetus or the mother. Later in this chapter, we discuss the Doppler effect, which can be used to determine the velocity of blood in the arteries or wind speed in weather systems.

17.1 Sound Waves

Learning Objectives

By the end of this section, you will be able to:

- Explain the difference between sound and hearing
- Describe sound as a wave
- List the equations used to model sound waves
- Describe compression and rarefactions as they relate to sound

The physical phenomenon of **sound** is a disturbance of matter that is transmitted from its source outward.

Hearing is the perception of sound, just as seeing is the perception of visible light. On the atomic scale, sound is a disturbance of atoms that is far more ordered than their thermal motions. In many instances, sound is a periodic wave, and the atoms undergo simple harmonic motion. Thus, sound waves can induce oscillations and resonance effects ([Figure 17.2](#)).



Figure 17.2 This glass has been shattered by a high-intensity sound wave of the same frequency as the resonant frequency of the glass.
(credit: “||read||”/Flickr)

INTERACTIVE

This [video \(https://openstax.org/l/21waveswineglas\)](https://openstax.org/l/21waveswineglas) shows waves on the surface of a wine glass, being driven by sound waves from a speaker. As the frequency of the sound wave approaches the resonant frequency of the wine glass, the amplitude and frequency of the waves on the wine glass increase. When the resonant frequency is reached, the glass shatters.

A speaker produces a sound wave by oscillating a cone, causing vibrations of air molecules. In [Figure 17.3](#), a speaker vibrates at a constant frequency and amplitude, producing vibrations in the surrounding air molecules. As the speaker oscillates back and forth, it transfers energy to the air, mostly as thermal energy. But a small part of the speaker's energy goes into compressing and expanding the surrounding air, creating slightly higher and lower local pressures. These compressions (high-pressure regions) and rarefactions (low-pressure regions) move out as longitudinal pressure waves having the same frequency as the speaker—they are the disturbance that is a sound wave. (Sound waves in air and most fluids are longitudinal, because fluids have almost no shear strength. In solids, sound waves can be both transverse and longitudinal.)

[Figure 17.3\(a\)](#) shows the compressions and rarefactions, and also shows a graph of gauge pressure versus distance from a speaker. As the speaker moves in the positive x -direction, it pushes air molecules, displacing them from their equilibrium positions. As the speaker moves in the negative x -direction, the air molecules move back toward their equilibrium positions due to a restoring force. The air molecules oscillate in simple harmonic motion about their equilibrium positions, as shown in part (b). Note that sound waves in air are longitudinal, and in the figure, the wave propagates in the positive x -direction and the molecules oscillate parallel to the direction in which the wave propagates.

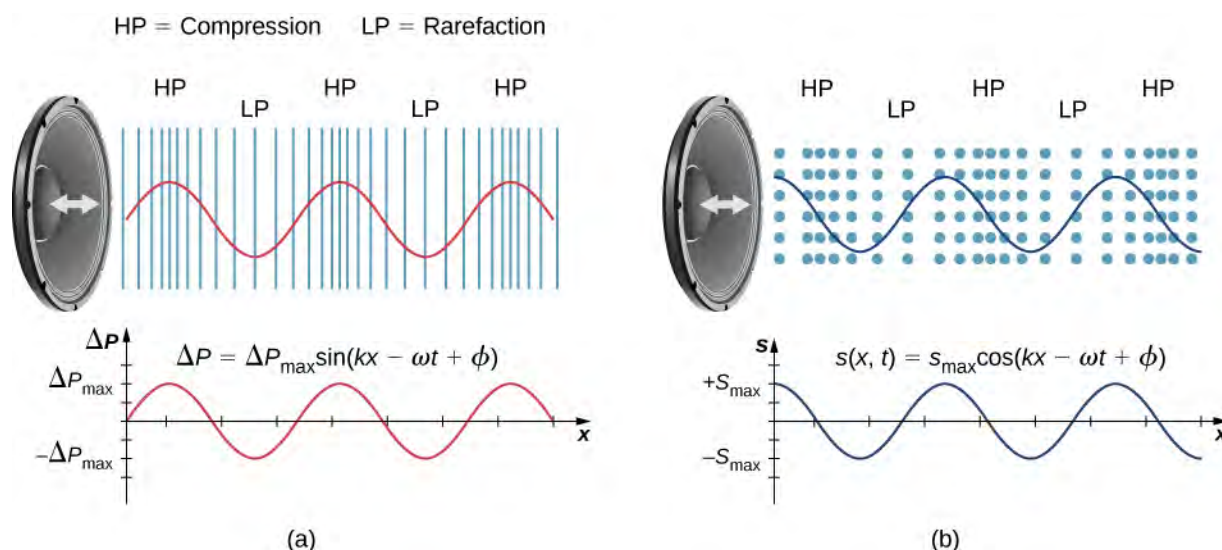


Figure 17.3 (a) A vibrating cone of a speaker, moving in the positive x -direction, compresses the air in front of it and expands the air behind it. As the speaker oscillates, it creates another compression and rarefaction as those on the right move away from the speaker. After many vibrations, a series of compressions and rarefactions moves out from the speaker as a sound wave. The red graph shows the gauge pressure of the air versus the distance from the speaker. Pressures vary only slightly from atmospheric pressure for ordinary sounds. Note that gauge pressure is modeled with a sine function, where the crests of the function line up with the compressions and the troughs line up with the rarefactions. (b) Sound waves can also be modeled using the displacement of the air molecules. The blue graph shows the displacement of the air molecules versus the position from the speaker and is modeled with a cosine function. Notice that the displacement is zero for the molecules in their equilibrium position and are centered at the compressions and rarefactions. Compressions are formed when molecules on either side of the equilibrium molecules are displaced toward the equilibrium position. Rarefactions are formed when the molecules are displaced away from the equilibrium position.

Models Describing Sound

Sound can be modeled as a pressure wave by considering the change in pressure from average pressure,

$$\Delta P = \Delta P_{\max} \sin(kx \mp \omega t + \phi). \quad 17.1$$

This equation is similar to the periodic wave equations seen in [Waves](#), where ΔP is the change in pressure, $\Delta P_{\max}$ is the maximum change in pressure, $k = \frac{2\pi}{\lambda}$ is the wave number, $\omega = \frac{2\pi}{T} = 2\pi f$ is the angular frequency, and ϕ is the initial phase. The wave speed can be determined from $v = \frac{\omega}{k} = \frac{\lambda}{T}$. Sound waves can also be modeled in terms of the displacement of the air molecules. The displacement of the air molecules can be modeled using a cosine function:

$$s(x, t) = s_{\max} \cos(kx \mp \omega t + \phi). \quad 17.2$$

In this equation, s is the displacement and $s_{\max}$ is the maximum displacement.

Not shown in the figure is the amplitude of a sound wave as it decreases with distance from its source, because the energy of the wave is spread over a larger and larger area. The intensity decreases as it moves away from the speaker, as discussed in [Waves](#). The energy is also absorbed by objects and converted into thermal energy by the viscosity of the air. In addition, during each compression, a little heat transfers to the air; during each rarefaction, even less heat transfers from the air, and these heat transfers reduce the organized disturbance into random thermal motions. Whether the heat transfer from compression to rarefaction is significant depends on how far apart they are—that is, it depends on wavelength. Wavelength, frequency, amplitude, and speed of propagation are important characteristics for sound, as they are for all waves.

17.2 Speed of Sound

Learning Objectives

By the end of this section, you will be able to:

- Explain the relationship between wavelength and frequency of sound
- Determine the speed of sound in different media
- Derive the equation for the speed of sound in air
- Determine the speed of sound in air for a given temperature

Sound, like all waves, travels at a certain speed and has the properties of frequency and wavelength. You can observe direct evidence of the speed of sound while watching a fireworks display ([Figure 17.4](#)). You see the flash of an explosion well before you hear its sound and possibly feel the pressure wave, implying both that sound travels at a finite speed and that it is much slower than light.



Figure 17.4 When a firework shell explodes, we perceive the light energy before the sound energy because sound travels more slowly than light does.

The difference between the speed of light and the speed of sound can also be experienced during an electrical storm. The flash of lighting is often seen before the clap of thunder. You may have heard that if you count the number of seconds between the flash and the sound, you can estimate the distance to the source. Every five seconds converts to about one mile. The velocity of any wave is related to its frequency and wavelength by

$$v = f\lambda,$$

17.3

where v is the speed of the wave, f is its frequency, and λ is its wavelength. Recall from [Waves](#) that the wavelength is the length of the wave as measured between sequential identical points. For example, for a surface water wave or sinusoidal wave on a string, the wavelength can be measured between any two convenient sequential points with the same height and slope, such as between two sequential crests or two sequential troughs. Similarly, the wavelength of a sound wave is the distance between sequential identical parts of a wave—for example, between sequential compressions ([Figure 17.5](#)). The frequency is the same as that of the source and is the number of waves that pass a point per unit time.

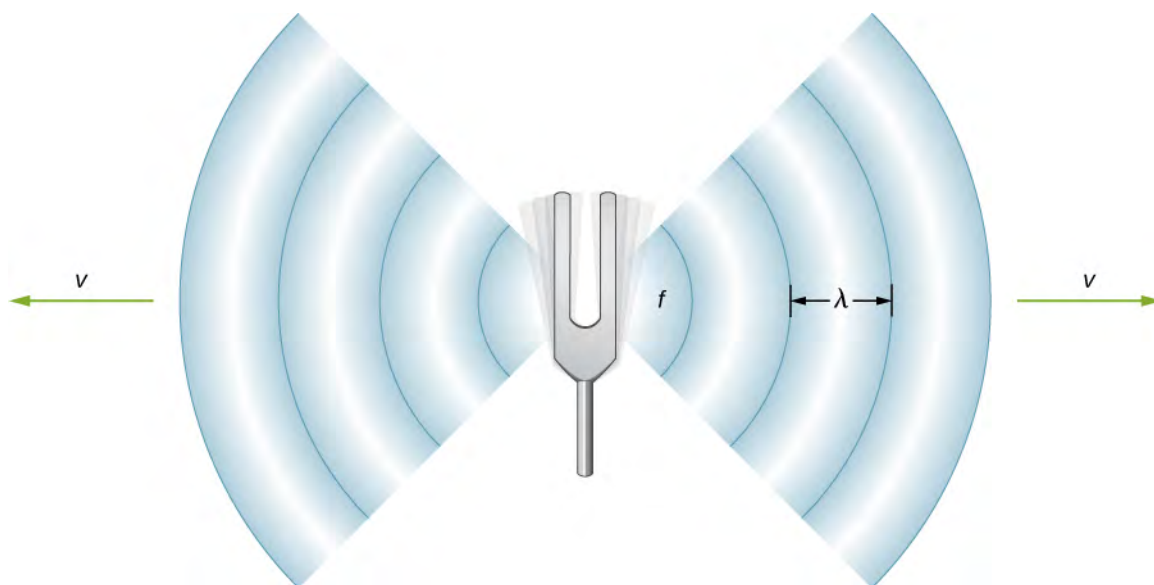


Figure 17.5 A sound wave emanates from a source, such as a tuning fork, vibrating at a frequency f . It propagates at speed v and has a wavelength λ .

Speed of Sound in Various Media

[Table 17.1](#) shows that the speed of sound varies greatly in different media. The speed of sound in a medium depends on how quickly vibrational energy can be transferred through the medium. For this reason, the derivation of the speed of sound in a medium depends on the medium and on the state of the medium. In general, the equation for the speed of a mechanical wave in a medium depends on the square root of the restoring force, or the elastic property, divided by the inertial property,

$$v = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}.$$

Also, sound waves satisfy the wave equation derived in [Waves](#),

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}.$$

Recall from [Waves](#) that the speed of a wave on a string is equal to $v = \sqrt{\frac{F_T}{\mu}}$, where the restoring force is the tension in the string F_T and the linear density μ is the inertial property. In a fluid, the speed of sound depends on the bulk modulus and the density,

$$v = \sqrt{\frac{B}{\rho}}. \quad 17.4$$

The speed of sound in a solid depends on the Young's modulus of the medium and the density,

$$v = \sqrt{\frac{Y}{\rho}}. \quad 17.5$$

In an ideal gas (see [The Kinetic Theory of Gases](#)), the equation for the speed of sound is

$$v = \sqrt{\frac{\gamma R T_K}{M}}, \quad 17.6$$

where γ is the adiabatic index, $R = 8.31 \text{ J/mol} \cdot \text{K}$ is the gas constant, T_K is the absolute temperature in

kelvins, and M is the molar mass. In general, the more rigid (or less compressible) the medium, the faster the speed of sound. This observation is analogous to the fact that the frequency of simple harmonic motion is directly proportional to the stiffness of the oscillating object as measured by k , the spring constant. The greater the density of a medium, the slower the speed of sound. This observation is analogous to the fact that the frequency of a simple harmonic motion is inversely proportional to m , the mass of the oscillating object. The speed of sound in air is low, because air is easily compressible. Because liquids and solids are relatively rigid and very difficult to compress, the speed of sound in such media is generally greater than in gases.

Medium	v (m/s)
<i>Gases at 0°C</i>	
Air	331
Carbon dioxide	259
Oxygen	316
Helium	965
Hydrogen	1290
<i>Liquids at 20°C</i>	
Ethanol	1160
Mercury	1450
Water, fresh	1480
Sea Water	1540
Human tissue	1540
<i>Solids (longitudinal or bulk)</i>	
Vulcanized rubber	54
Polyethylene	920
Marble	3810
Glass, Pyrex	5640
Lead	1960
Aluminum	5120
Steel	5960

Table 17.1 Speed of Sound in Various Media

Because the speed of sound depends on the density of the material, and the density depends on the

temperature, there is a relationship between the temperature in a given medium and the speed of sound in the medium. For air at sea level, the speed of sound is given by

$$v = 331 \frac{\text{m}}{\text{s}} \sqrt{1 + \frac{T_C}{273^\circ\text{C}}} = 331 \frac{\text{m}}{\text{s}} \sqrt{\frac{T_K}{273 \text{ K}}} \quad 17.7$$

where the temperature in the first equation (denoted as T_C) is in degrees Celsius and the temperature in the second equation (denoted as T_K) is in kelvins. The speed of sound in gases is related to the average speed of particles in the gas, $v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}}$, where k_B is the Boltzmann constant ($1.38 \times 10^{-23} \text{ J/K}$) and m is the mass of each (identical) particle in the gas. Note that v refers to the speed of the coherent propagation of a disturbance (the wave), whereas v_{rms} describes the speeds of particles in random directions. Thus, it is reasonable that the speed of sound in air and other gases should depend on the square root of temperature. While not negligible, this is not a strong dependence. At 0°C , the speed of sound is 331 m/s , whereas at 20.0°C , it is 343 m/s , less than a 4% increase. [Figure 17.6](#) shows how a bat uses the speed of sound to sense distances.

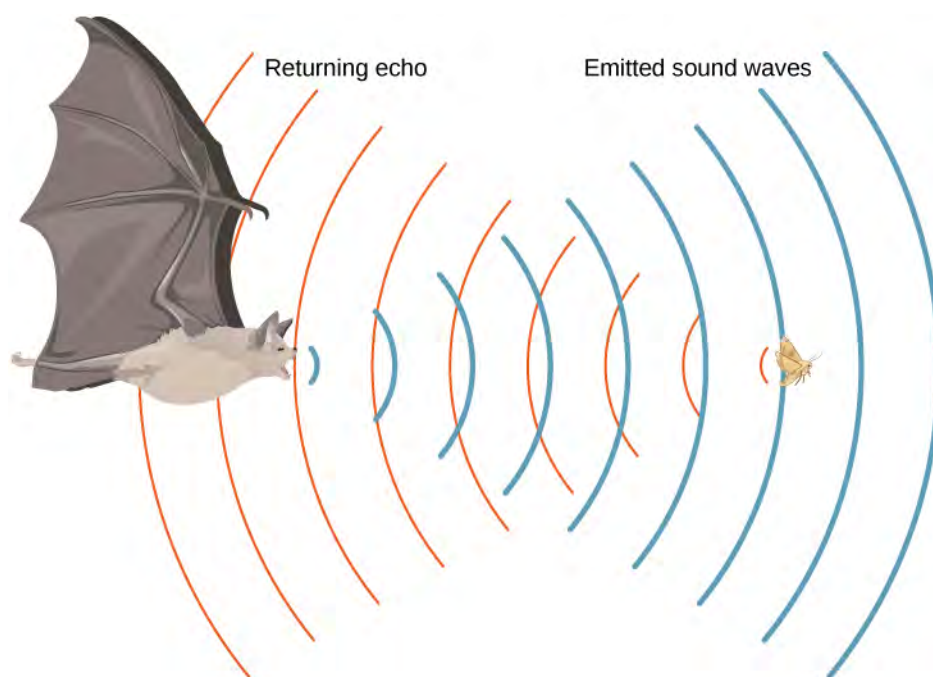


Figure 17.6 A bat uses sound echoes to find its way about and to catch prey. The time for the echo to return is directly proportional to the distance.

Derivation of the Speed of Sound in Air

As stated earlier, the speed of sound in a medium depends on the medium and the state of the medium. The derivation of the equation for the speed of sound in air starts with the mass flow rate and continuity equation discussed in [Fluid Mechanics](#).

Consider fluid flow through a pipe with cross-sectional area A ([Figure 17.7](#)). The mass in a small volume of length x of the pipe is equal to the density times the volume, or $m = \rho V = \rho Ax$. The mass flow rate is

$$\frac{dm}{dt} = \frac{d}{dt}(\rho V) = \frac{d}{dt}(\rho Ax) = \rho A \frac{dx}{dt} = \rho A v.$$

The continuity equation from [Fluid Mechanics](#) states that the mass flow rate into a volume has to equal the mass flow rate out of the volume, $\rho_{\text{in}} A_{\text{in}} v_{\text{in}} = \rho_{\text{out}} A_{\text{out}} v_{\text{out}}$.

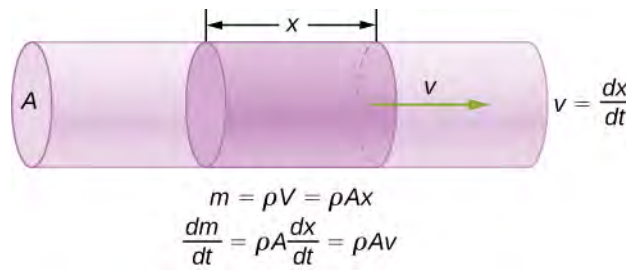


Figure 17.7 The mass of a fluid in a volume is equal to the density times the volume, $m = \rho V = \rho A x$. The mass flow rate is the time derivative of the mass.

Now consider a sound wave moving through a parcel of air. A parcel of air is a small volume of air with imaginary boundaries ([Figure 17.8](#)). The density, temperature, and velocity on one side of the volume of the fluid are given as ρ, T, v , and on the other side are $\rho + d\rho, T + dT, v + dv$.

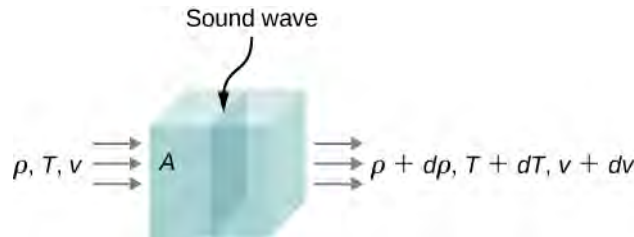


Figure 17.8 A sound wave moves through a volume of fluid. The density, temperature, and velocity of the fluid change from one side to the other.

The continuity equation states that the mass flow rate entering the volume is equal to the mass flow rate leaving the volume, so

$$\rho A v = (\rho + d\rho) A (v + dv).$$

This equation can be simplified, noting that the area cancels and considering that the multiplication of two infinitesimals is approximately equal to zero: $d\rho (dv) \approx 0$,

$$\begin{aligned} \rho v &= (\rho + d\rho)(v + dv) \\ \rho v &= \rho v + \rho(dv) + (d\rho)v + (d\rho)(dv) \\ 0 &= \rho(dv) + (d\rho)v \\ \rho dv &= -vd\rho. \end{aligned}$$

The net force on the volume of fluid ([Figure 17.9](#)) equals the sum of the forces on the left face and the right face:

$$\begin{aligned} F_{\text{net}} &= p dy dz - (p + dp) dy dz \\ &= p dy dz - p dy dz - dp dy dz \\ &= -dp dy dz \\ ma &= -dp dy dz. \end{aligned}$$

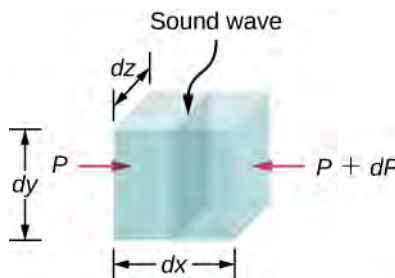


Figure 17.9 A sound wave moves through a volume of fluid. The force on each face can be found by the pressure times the area.

The acceleration is the force divided by the mass and the mass is equal to the density times the volume,

$m = \rho V = \rho \, dx \, dy \, dz$. We have

$$\begin{aligned} ma &= -dp \, dy \, dz \\ a &= -\frac{dp \, dy \, dz}{m} = -\frac{dp \, dy \, dz}{\rho \, dx \, dy \, dz} = -\frac{dp}{(\rho \, dx)} \\ \frac{dv}{dt} &= -\frac{dp}{(\rho \, dx)} \\ dv &= -\frac{dp}{(\rho \, dx)} dt = -\frac{dp}{\rho} \frac{1}{v} \\ \rho v \, dv &= -dp. \end{aligned}$$

From the continuity equation $\rho \, dv = -v \, d\rho$, we obtain

$$\begin{aligned} \rho v \, dv &= -dp \\ (-v \, d\rho) v &= -dp \\ v &= \sqrt{\frac{dp}{d\rho}}. \end{aligned}$$

Consider a sound wave moving through air. During the process of compression and expansion of the gas, no heat is added or removed from the system. A process where heat is not added or removed from the system is known as an adiabatic system. Adiabatic processes are covered in detail in [The First Law of Thermodynamics](#), but for now it is sufficient to say that for an adiabatic process, $pV^\gamma = \text{constant}$, where p is the pressure, V is the volume, and gamma (γ) is a constant that depends on the gas. For air, $\gamma = 1.40$. The density equals the number of moles times the molar mass divided by the volume, so the volume is equal to $V = \frac{nM}{\rho}$. The number of moles and the molar mass are constant and can be absorbed into the constant $p\left(\frac{1}{\rho}\right)^\gamma = \text{constant}$. Taking the natural logarithm of both sides yields $\ln p - \gamma \ln \rho = \text{constant}$. Differentiating with respect to the density, the equation becomes

$$\begin{aligned} \ln p - \gamma \ln \rho &= \text{constant} \\ \frac{d}{d\rho}(\ln p - \gamma \ln \rho) &= \frac{d}{d\rho}(\text{constant}) \\ \frac{1}{p} \frac{dp}{d\rho} - \frac{\gamma}{\rho} &= 0 \\ \frac{dp}{d\rho} &= \frac{\gamma p}{\rho}. \end{aligned}$$

If the air can be considered an ideal gas, we can use the ideal gas law:

$$\begin{aligned} pV &= nRT = \frac{m}{M} RT \\ p &= \frac{m}{V} \frac{RT}{M} = \rho \frac{RT}{M}. \end{aligned}$$

Here M is the molar mass of air:

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho} = \frac{\gamma \left(\rho \frac{RT}{M}\right)}{\rho} = \frac{\gamma RT}{M}.$$

Since the speed of sound is equal to $v = \sqrt{\frac{dp}{d\rho}}$, the speed is equal to

$$v = \sqrt{\frac{\gamma RT}{M}}.$$

Note that the velocity is faster at higher temperatures and slower for heavier gases. For air, $\gamma = 1.4$, $M = 0.02897 \frac{\text{kg}}{\text{mol}}$, and $R = 8.31 \frac{\text{J}}{\text{mol} \cdot \text{K}}$. If the temperature is $T_C = 20^\circ\text{C}$ ($T = 293 \text{ K}$), the speed of sound is $v = 343 \text{ m/s}$.

The equation for the speed of sound in air $v = \sqrt{\frac{\gamma RT}{M}}$ can be simplified to give the equation for the speed of sound in air as a function of absolute temperature:

$$\begin{aligned}
 v &= \sqrt{\frac{\gamma RT}{M}} \\
 &= \sqrt{\frac{\gamma RT}{M} \left(\frac{273 \text{ K}}{273 \text{ K}} \right)} = \sqrt{\frac{(273 \text{ K})\gamma R}{M}} \sqrt{\frac{T}{273 \text{ K}}} \\
 &\approx 331 \frac{\text{m}}{\text{s}} \sqrt{\frac{T}{273 \text{ K}}}.
 \end{aligned}$$

One of the more important properties of sound is that its speed is nearly independent of the frequency. This independence is certainly true in open air for sounds in the audible range. If this independence were not true, you would certainly notice it for music played by a marching band in a football stadium, for example. Suppose that high-frequency sounds traveled faster—then the farther you were from the band, the more the sound from the low-pitch instruments would lag that from the high-pitch ones. But the music from all instruments arrives in cadence independent of distance, so all frequencies must travel at nearly the same speed. Recall that

$$v = f\lambda.$$

In a given medium under fixed conditions, v is constant, so there is a relationship between f and λ ; the higher the frequency, the smaller the wavelength ([Figure 17.10](#)).

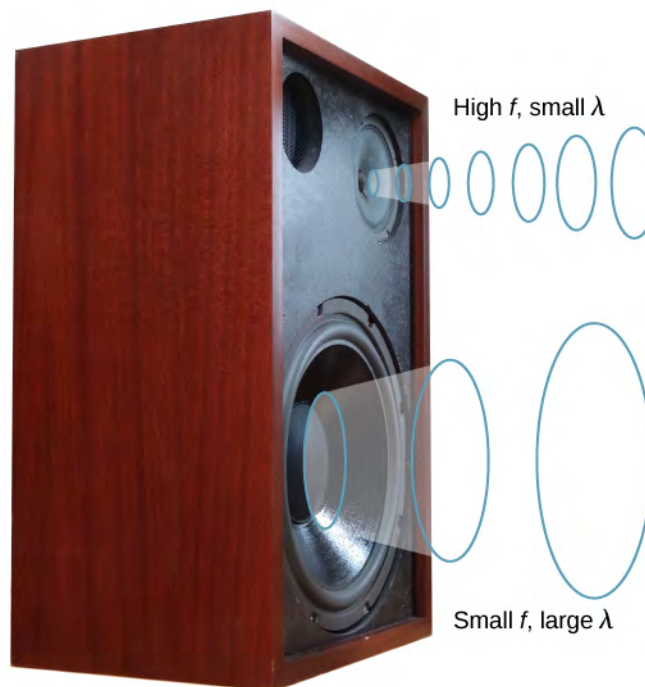


Figure 17.10 Because they travel at the same speed in a given medium, low-frequency sounds must have a greater wavelength than high-frequency sounds. Here, the lower-frequency sounds are emitted by the large speaker, called a woofer, whereas the higher-frequency sounds are emitted by the small speaker, called a tweeter. (credit: modification of work by Jane Whitney)



EXAMPLE 17.1

Calculating Wavelengths

Calculate the wavelengths of sounds at the extremes of the audible range, 20 and 20,000 Hz, in 30.0°C air. (Assume that the frequency values are accurate to two significant figures.)

Strategy

To find wavelength from frequency, we can use $v = f\lambda$.

Solution

1. Identify knowns. The value for v is given by

$$v = (331 \text{ m/s}) \sqrt{\frac{T}{273 \text{ K}}}.$$

2. Convert the temperature into kelvins and then enter the temperature into the equation

$$v = (331 \text{ m/s}) \sqrt{\frac{303 \text{ K}}{273 \text{ K}}} = 348.7 \text{ m/s}.$$

3. Solve the relationship between speed and wavelength for λ :

$$\lambda = \frac{v}{f}.$$

4. Enter the speed and the minimum frequency to give the maximum wavelength:

$$\lambda_{\text{max}} = \frac{348.7 \text{ m/s}}{20 \text{ Hz}} = 17 \text{ m}.$$

5. Enter the speed and the maximum frequency to give the minimum wavelength:

$$\lambda_{\text{min}} = \frac{348.7 \text{ m/s}}{20,000 \text{ Hz}} = 0.017 \text{ m} = 1.7 \text{ cm}.$$

Significance

Because the product of f multiplied by λ equals a constant, the smaller f is, the larger λ must be, and vice versa.

The speed of sound can change when sound travels from one medium to another, but the frequency usually remains the same. This is similar to the frequency of a wave on a string being equal to the frequency of the force oscillating the string. If v changes and f remains the same, then the wavelength λ must change. That is, because $v = f\lambda$, the higher the speed of a sound, the greater its wavelength for a given frequency.

CHECK YOUR UNDERSTANDING 17.1

Imagine you observe two firework shells explode. You hear the explosion of one as soon as you see it. However, you see the other shell for several milliseconds before you hear the explosion. Explain why this is so.

Although sound waves in a fluid are longitudinal, sound waves in a solid travel both as longitudinal waves and transverse waves. Seismic waves, which are essentially sound waves in Earth's crust produced by earthquakes, are an interesting example of how the speed of sound depends on the rigidity of the medium. Earthquakes produce both longitudinal and transverse waves, and these travel at different speeds. The bulk modulus of granite is greater than its shear modulus. For that reason, the speed of longitudinal or pressure waves (P-waves) in earthquakes in granite is significantly higher than the speed of transverse or shear waves (S-waves). Both types of earthquake waves travel slower in less rigid material, such as sediments. P-waves have speeds of 4 to 7 km/s, and S-waves range in speed from 2 to 5 km/s, both being faster in more rigid material. The P-wave gets progressively farther ahead of the S-wave as they travel through Earth's crust. The time between the P- and S-waves is routinely used to determine the distance to their source, the epicenter of the earthquake. Because S-waves do not pass through the liquid core, two shadow regions are produced ([Figure 17.11](#)).

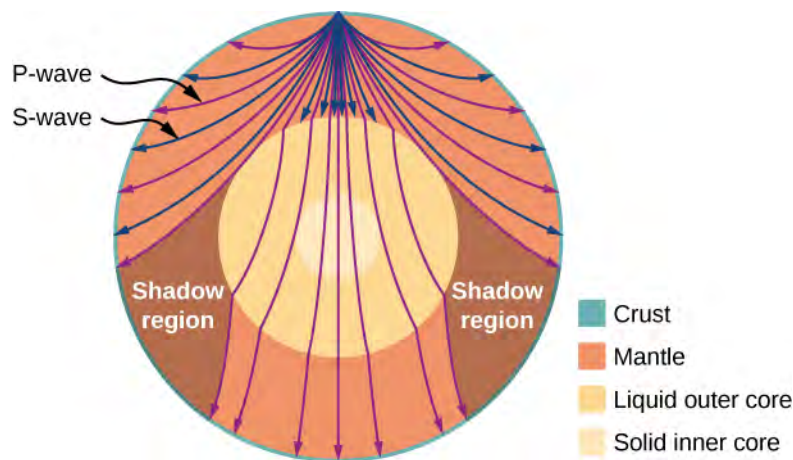


Figure 17.11 Earthquakes produce both longitudinal waves (P-waves) and transverse waves (S-waves), and these travel at different speeds. Both waves travel at different speeds in the different regions of Earth, but in general, P-waves travel faster than S-waves. S-waves cannot be supported by the liquid core, producing shadow regions.

As sound waves move away from a speaker, or away from the epicenter of an earthquake, their power per unit area decreases. This is why the sound is very loud near a speaker and becomes less loud as you move away from the speaker. This also explains why there can be an extreme amount of damage at the epicenter of an earthquake but only tremors are felt in areas far from the epicenter. The power per unit area is known as the intensity, and in the next section, we will discuss how the intensity depends on the distance from the source.

17.3 Sound Intensity

Learning Objectives

By the end of this section, you will be able to:

- Define the term intensity
- Explain the concept of sound intensity level
- Describe how the human ear translates sound

In a quiet forest, you can sometimes hear a single leaf fall to the ground. But when a passing motorist has his stereo turned up, you cannot even hear what the person next to you in your car is saying ([Figure 17.12](#)). We are all very familiar with the loudness of sounds and are aware that loudness is related to how energetically the source is vibrating. High noise exposure is hazardous to hearing, which is why it is important for people working in industrial settings to wear ear protection. The relevant physical quantity is sound intensity, a concept that is valid for all sounds whether or not they are in the audible range.



Figure 17.12 Noise on crowded roadways, like this one in Delhi, makes it hard to hear others unless they shout. (credit: “Lingaraj G J”/Flickr)

In [Waves](#), we defined intensity as the power per unit area carried by a wave. Power is the rate at which energy is transferred by the wave. In equation form, intensity I is

$$I = \frac{P}{A}, \quad 17.8$$

where P is the power through an area A . The SI unit for I is W/m^2 . If we assume that the sound wave is spherical, and that no energy is lost to thermal processes, the energy of the sound wave is spread over a larger area as distance increases, so the intensity decreases. The area of a sphere is $A = 4\pi r^2$. As the wave spreads out from r_1 to r_2 , the energy also spreads out over a larger area:

$$\begin{aligned} P_1 &= P_2 \\ I_1 4\pi r_1^2 &= I_2 4\pi r_2^2; \end{aligned}$$

$$I_2 = I_1 \left(\frac{r_1}{r_2} \right)^2. \quad 17.9$$

The intensity decreases as the wave moves out from the source. In an inverse square relationship, such as the intensity, when you double the distance, the intensity decreases to one quarter,

$$I_2 = I_1 \left(\frac{r_1}{r_2} \right)^2 = I_1 \left(\frac{r_1}{2r_1} \right)^2 = \frac{1}{4} I_1.$$

Generally, when considering the intensity of a sound wave, we take the intensity to be the time-averaged value of the power, denoted by $\langle P \rangle$, divided by the area,

$$I = \frac{\langle P \rangle}{A}. \quad 17.10$$

The intensity of a sound wave is proportional to the change in the pressure squared and inversely proportional to the density and the speed. Consider a parcel of a medium initially undisturbed and then influenced by a

sound wave at time t , as shown in [Figure 17.13](#).

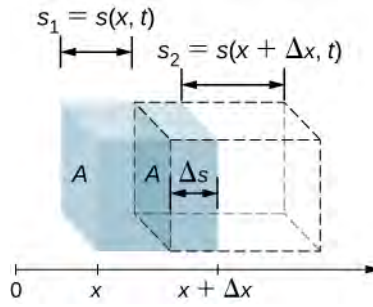


Figure 17.13 An undisturbed parcel of a medium with a volume $V = A\Delta x$ shown in blue. A sound wave moves through the medium at time t , and the parcel is displaced and expands, as shown by dotted lines. The change in volume is $\Delta V = A\Delta s = A(s_2 - s_1)$, where s_1 is the displacement of the leading edge of the parcel and s_2 is the displacement of the trailing edge of the parcel. In the figure, $s_2 > s_1$ and the parcel expands, but the parcel can either expand or compress ($s_2 < s_1$), depending on which part of the sound wave (compression or rarefaction) is moving through the parcel.

As the sound wave moves through the parcel, the parcel is displaced and may expand or contract. If $s_2 > s_1$, the volume has increased and the pressure decreases. If $s_2 < s_1$, the volume has decreased and the pressure increases. The change in the volume is

$$\Delta V = A\Delta s = A(s_2 - s_1) = A(s(x + \Delta x, t) - s(x, t)).$$

The fractional change in the volume is the change in volume divided by the original volume:

$$\frac{dV}{V} = \lim_{\Delta x \rightarrow 0} \frac{A[s(x + \Delta x, t) - s(x, t)]}{A\Delta x} = \frac{\partial s(x, t)}{\partial x}.$$

The fractional change in volume is related to the pressure fluctuation by the bulk modulus $\beta = -\frac{\Delta p(x, t)}{dV/V}$. Recall that the minus sign is required because the volume is *inversely* related to the pressure. (We use lowercase p for pressure to distinguish it from power, denoted by P .) The change in pressure is therefore

$\Delta p(x, t) = -\beta \frac{dV}{V} = -\beta \frac{\partial s(x, t)}{\partial x}$. If the sound wave is sinusoidal, then the displacement as shown in [Equation 17.2](#) is $s(x, t) = s_{\max} \cos(kx - \omega t + \phi)$ and the pressure is found to be

$$\Delta p(x, t) = -\beta \frac{dV}{V} = -\beta \frac{\partial s(x, t)}{\partial x} = \beta k s_{\max} \sin(kx - \omega t + \phi) = \Delta p_{\max} \sin(kx - \omega t + \phi).$$

The intensity of the sound wave is the power per unit area, and the power is the force times the velocity, $I = \frac{P}{A} = \frac{Fv}{A} = pv$. Here, the velocity is the velocity of the oscillations of the medium, and not the velocity of the sound wave. The velocity of the medium is the time rate of change in the displacement:

$$v(x, t) = \frac{\partial}{\partial t} s(x, t) = \frac{\partial}{\partial t} (s_{\max} \cos(kx - \omega t + \phi)) = s_{\max} \omega \sin(kx - \omega t + \phi).$$

Thus, the intensity becomes

$$\begin{aligned} I &= \Delta p(x, t) v(x, t) \\ &= \beta k s_{\max} \sin(kx - \omega t + \phi) [s_{\max} \omega \sin(kx - \omega t + \phi)] \\ &= \beta k \omega s_{\max}^2 \sin^2(kx - \omega t + \phi). \end{aligned}$$

To find the time-averaged intensity over one period $T = \frac{2\pi}{\omega}$ for a position x , we integrate over the period,

$I = \frac{\beta k \omega s_{\max}^2}{2}$. Using $\Delta p_{\max} = \beta k s_{\max}$, $v = \sqrt{\frac{\beta}{\rho}}$, and $v = \frac{\omega}{k}$, we obtain

$$I = \frac{\beta k \omega s_{\max}^2}{2} = \frac{\beta^2 k^2 \omega s_{\max}^2}{2\beta k} = \frac{\omega (\Delta p_{\max})^2}{2(\rho v^2)k} = \frac{v(\Delta p_{\max})^2}{2(\rho v^2)} = \frac{(\Delta p_{\max})^2}{2\rho v}.$$

That is, the intensity of a sound wave is related to its amplitude squared by

$$I = \frac{(\Delta p_{\max})^2}{2\rho v}. \quad 17.11$$

Here, $\Delta p_{\max}$ is the pressure variation or pressure amplitude in units of pascals (Pa) or N/m^2 . The energy (as kinetic energy $\frac{1}{2}mv^2$) of an oscillating element of air due to a traveling sound wave is proportional to its amplitude squared. In this equation, ρ is the density of the material in which the sound wave travels, in units of kg/m^3 , and v is the speed of sound in the medium, in units of m/s . The pressure variation is proportional to the amplitude of the oscillation, so I varies as $(\Delta p)^2$. This relationship is consistent with the fact that the sound wave is produced by some vibration; the greater its pressure amplitude, the more the air is compressed in the sound it creates.

Human Hearing and Sound Intensity Levels

As stated earlier in this chapter, hearing is the perception of sound. The hearing mechanism involves some interesting physics. The sound wave that impinges upon our ear is a pressure wave. The ear is a **transducer** that converts sound waves into electrical nerve impulses in a manner much more sophisticated than, but analogous to, a microphone. [Figure 17.14](#) shows the anatomy of the ear.

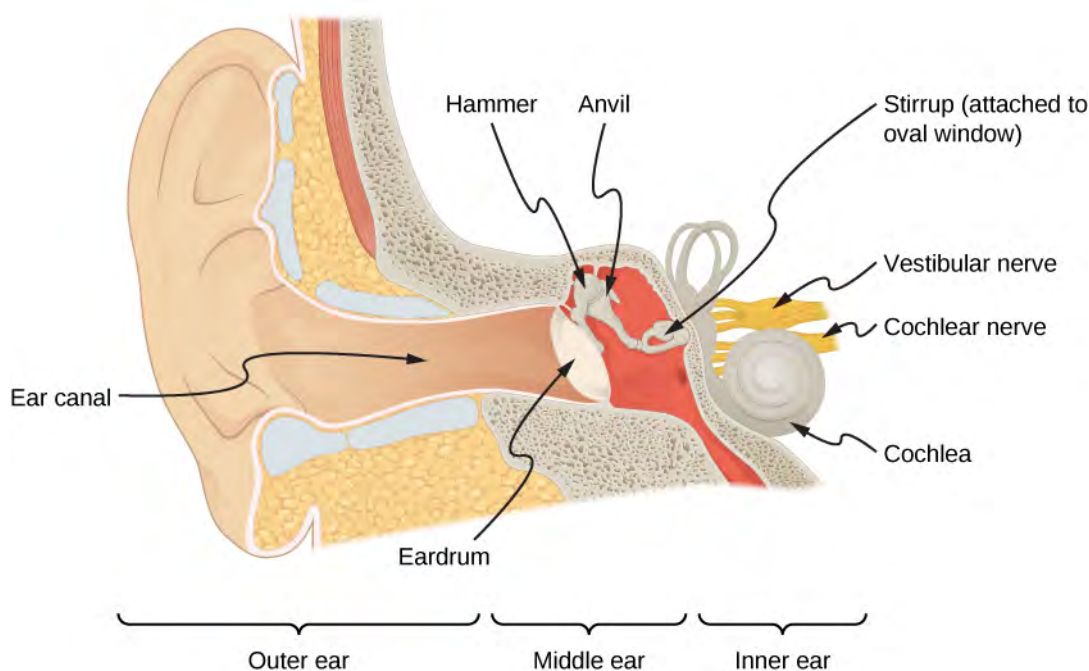


Figure 17.14 The anatomy of the human ear.

The outer ear, or ear canal, carries sound to the recessed, protected eardrum. The air column in the ear canal resonates and is partially responsible for the sensitivity of the ear to sounds in the 2000–5000-Hz range. The middle ear converts sound into mechanical vibrations and applies these vibrations to the cochlea.

INTERACTIVE

Watch this [video \(https://openstax.org/l/21humanear\)](https://openstax.org/l/21humanear) for a more detailed discussion of the workings of the human ear.

The range of intensities that the human ear can hear depends on the frequency of the sound, but, in general, the range is quite large. The minimum threshold intensity that can be heard is $I_0 = 10^{-12} \text{ W/m}^2$. Pain is experienced at intensities of $I_{\text{pain}} = 1 \text{ W/m}^2$. Measurements of sound intensity (in units of W/m^2) are very cumbersome due to this large range in values. For this reason, as well as for other reasons, the concept of sound intensity level was proposed.

The **sound intensity level** β of a sound, measured in decibels, having an intensity I in watts per meter squared, is defined as

$$\beta(\text{dB}) = 10 \log_{10} \left(\frac{I}{I_0} \right), \quad 17.12$$

where $I_0 = 10^{-12} \text{ W/m}^2$ is a reference intensity, corresponding to the threshold intensity of sound that a person with normal hearing can perceive at a frequency of 1.00 kHz. It is more common to consider sound intensity levels in dB than in W/m^2 . How human ears perceive sound can be more accurately described by the logarithm of the intensity rather than directly by the intensity. Because β is defined in terms of a ratio, it is a unitless quantity, telling you the *level* of the sound relative to a fixed standard (10^{-12} W/m^2). The units of decibels (dB) are used to indicate this ratio is multiplied by 10 in its definition. The bel, upon which the decibel is based, is named for Alexander Graham Bell, the inventor of the telephone.

The decibel level of a sound having the threshold intensity of 10^{-12} W/m^2 is $\beta = 0 \text{ dB}$, because $\log_{10} 1 = 0$. [Table 17.2](#) gives levels in decibels and intensities in watts per meter squared for some familiar sounds. The ear is sensitive to as little as a trillionth of a watt per meter squared—even more impressive when you realize that the area of the eardrum is only about 1 cm^2 , so that only 10^{-16} W falls on it at the threshold of hearing. Air molecules in a sound wave of this intensity vibrate over a distance of less than one molecular diameter, and the gauge pressures involved are less than 10^{-9} atm .

Sound intensity level β (dB)	Intensity I (W/m^2)	Example/effect
0	1×10^{-12}	Threshold of hearing at 1000 Hz
10	1×10^{-11}	Rustle of leaves
20	1×10^{-10}	Whisper at 1-m distance
30	1×10^{-9}	Quiet home
40	1×10^{-8}	Average home
50	1×10^{-7}	Average office, soft music
60	1×10^{-6}	Normal conversation
70	1×10^{-5}	Noisy office, busy traffic
80	1×10^{-4}	Loud radio, classroom lecture
90	1×10^{-3}	Inside a heavy truck; damage from prolonged exposure ¹
100	1×10^{-2}	Noisy factory, siren at 30 m; damage from 8 h per day exposure
110	1×10^{-1}	Damage from 30 min per day exposure
120	1	Loud rock concert; pneumatic chipper at 2 m; threshold of pain

Sound intensity level β (dB)	Intensity I (W/m ²)	Example/effect
140	1×10^2	Jet airplane at 30 m; severe pain, damage in seconds
160	1×10^4	Bursting of eardrums

Table 17.2 Sound Intensity Levels and Intensities ^[1] Several government agencies and health-related professional associations recommend that 85 dB not be exceeded for 8-hour daily exposures in the absence of hearing protection.

An observation readily verified by examining [Table 17.2](#) or by using [Equation 17.12](#) is that each factor of 10 in intensity corresponds to 10 dB. For example, a 90-dB sound compared with a 60-dB sound is 30 dB greater, or three factors of 10 (that is, 10^3 times) as intense. Another example is that if one sound is 10^7 as intense as another, it is 70 dB higher ([Table 17.3](#)).

I_2/I_1	$\beta_2 - \beta_1$
2.0	3.0 dB
5.0	7.0 dB
10.0	10.0 dB
100.0	20.0 dB
1000.0	30.0 dB

Table 17.3 Ratios of Intensities and Corresponding Differences in Sound Intensity Levels



EXAMPLE 17.2

Calculating Sound Intensity Levels

Calculate the sound intensity level in decibels for a sound wave traveling in air at 0°C and having a pressure amplitude of 0.656 Pa.

Strategy

We are given Δp , so we can calculate I using the equation $I = \frac{(\Delta p)^2}{2\rho v_w}$. Using I , we can calculate β straight from its definition in $\beta(\text{dB}) = 10 \log_{10} \left(\frac{I}{I_0} \right)$.

Solution

- Identify knowns:
Sound travels at 331 m/s in air at 0°C.
Air has a density of 1.29 kg/m³ at atmospheric pressure and 0°C.
- Enter these values and the pressure amplitude into $I = \frac{(\Delta p)^2}{2\rho v}$.

¹ Several government agencies and health-related professional associations recommend that 85 dB not be exceeded for 8-hour daily exposures in the absence of hearing protection.

$$I = \frac{(\Delta p)^2}{2\rho v} = \frac{(0.656 \text{ Pa})^2}{2(1.29 \text{ kg/m}^3)(331 \text{ m/s})} = 5.04 \times 10^{-4} \text{ W/m}^2.$$

3. Enter the value for I and the known value for I_0 into $\beta(\text{dB}) = 10 \log_{10}(I/I_0)$. Calculate to find the sound intensity level in decibels:

$$10 \log_{10}(5.04 \times 10^8) = 10(8.70) \text{ dB} = 87 \text{ dB}.$$

Significance

This 87-dB sound has an intensity five times as great as an 80-dB sound. So a factor of five in intensity corresponds to a difference of 7 dB in sound intensity level. This value is true for any intensities differing by a factor of five.



EXAMPLE 17.3

Changing Intensity Levels of a Sound

Show that if one sound is twice as intense as another, it has a sound level about 3 dB higher.

Strategy

We are given that the ratio of two intensities is 2 to 1, and are then asked to find the difference in their sound levels in decibels. We can solve this problem by using of the properties of logarithms.

Solution

1. Identify knowns:

The ratio of the two intensities is 2 to 1, or

$$\frac{I_2}{I_1} = 2.00.$$

We wish to show that the difference in sound levels is about 3 dB. That is, we want to show:

$$\beta_2 - \beta_1 = 3 \text{ dB}.$$

Note that

$$\log_{10} b - \log_{10} a = \log_{10} \left(\frac{b}{a} \right).$$

2. Use the definition of β to obtain

$$\beta_2 - \beta_1 = 10 \log_{10} \left(\frac{I_2}{I_1} \right) = 10 \log_{10} 2.00 = 10(0.301) \text{ dB}.$$

Thus,

$$\beta_2 - \beta_1 = 3.01 \text{ dB}.$$

Significance

This means that the two sound intensity levels differ by 3.01 dB, or about 3 dB, as advertised. Note that because only the ratio I_2/I_1 is given (and not the actual intensities), this result is true for any intensities that differ by a factor of two. For example, a 56.0-dB sound is twice as intense as a 53.0-dB sound, a 97.0-dB sound is half as intense as a 100-dB sound, and so on.



CHECK YOUR UNDERSTANDING 17.2

Identify common sounds at the levels of 10 dB, 50 dB, and 100 dB.

Another decibel scale is also in use, called the **sound pressure level**, based on the ratio of the pressure amplitude to a reference pressure. This scale is used particularly in applications where sound travels in water. It is beyond the scope of this text to treat this scale because it is not commonly used for sounds in air, but it is

important to note that very different decibel levels may be encountered when sound pressure levels are quoted.

Hearing and Pitch

The human ear has a tremendous range and sensitivity. It can give us a wealth of simple information—such as pitch, loudness, and direction.

The perception of frequency is called **pitch**. Typically, humans have excellent relative pitch and can discriminate between two sounds if their frequencies differ by 0.3% or more. For example, 500.0 and 501.5 Hz are noticeably different. Musical **notes** are sounds of a particular frequency that can be produced by most instruments and in Western music have particular names, such as A-sharp, C, or E-flat.

The perception of intensity is called **loudness**. At a given frequency, it is possible to discern differences of about 1 dB, and a change of 3 dB is easily noticed. But loudness is not related to intensity alone. Frequency has a major effect on how loud a sound seems. Sounds near the high- and low-frequency extremes of the hearing range seem even less loud, because the ear is less sensitive at those frequencies. When a violin plays middle C, there is no mistaking it for a piano playing the same note. The reason is that each instrument produces a distinctive set of frequencies and intensities. We call our perception of these combinations of frequencies and intensities tone quality or, more commonly, the **timbre** of the sound. Timbre is the shape of the wave that arises from the many reflections, resonances, and superposition in an instrument.

A unit called a **phon** is used to express loudness numerically. Phons differ from decibels because the phon is a unit of loudness perception, whereas the decibel is a unit of physical intensity. [Figure 17.15](#) shows the relationship of loudness to intensity (or intensity level) and frequency for persons with normal hearing. The curved lines are equal-loudness curves. Each curve is labeled with its loudness in phons. Any sound along a given curve is perceived as equally loud by the average person. The curves were determined by having large numbers of people compare the loudness of sounds at different frequencies and sound intensity levels. At a frequency of 1000 Hz, phons are taken to be numerically equal to decibels.

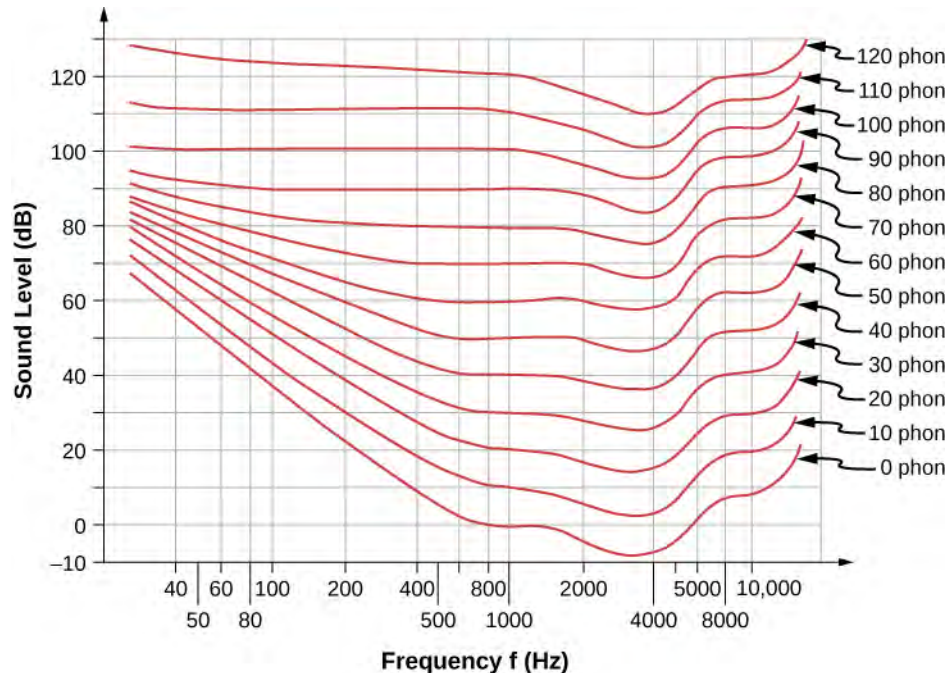


Figure 17.15 The relationship of loudness in phons to intensity level (in decibels) and intensity (in watts per meter squared) for persons with normal hearing. The curved lines are equal-loudness curves—all sounds on a given curve are perceived as equally loud. Phons and decibels are defined to be the same at 1000 Hz.



EXAMPLE 17.4

Measuring Loudness

(a) What is the loudness in phons of a 100-Hz sound that has an intensity level of 80 dB? (b) What is the intensity level in decibels of a 4000-Hz sound having a loudness of 70 phons? (c) At what intensity level will an 8000-Hz sound have the same loudness as a 200-Hz sound at 60 dB?

Strategy

The graph in [Figure 17.15](#) should be referenced to solve this example. To find the loudness of a given sound, you must know its frequency and intensity level, locate that point on the square grid, and then interpolate between loudness curves to get the loudness in phons. Once that point is located, the intensity level can be determined from the vertical axis.

Solution

1. Identify knowns: The square grid of the graph relating phons and decibels is a plot of intensity level versus frequency—both physical quantities: 100 Hz at 80 dB lies halfway between the curves marked 70 and 80 phons.
Find the loudness: 75 phons.
2. Identify knowns: Values are given to be 4000 Hz at 70 phons.
Follow the 70-phon curve until it reaches 4000 Hz. At that point, it is below the 70 dB line at about 67 dB.
Find the intensity level: 67 dB.
3. Locate the point for a 200 Hz and 60 dB sound.
Find the loudness: This point lies just slightly above the 50-phon curve, and so its loudness is 51 phons.
Look for the 51-phon level is at 8000 Hz: 63 dB.

Significance

These answers, like all information extracted from [Figure 17.15](#), have uncertainties of several phons or several decibels, partly due to difficulties in interpolation, but mostly related to uncertainties in the equal-loudness curves.



CHECK YOUR UNDERSTANDING 17.3

Describe how amplitude is related to the loudness of a sound.

In this section, we discussed the characteristics of sound and how we hear, but how are the sounds we hear produced? Interesting sources of sound are musical instruments and the human voice, and we will discuss these sources. But before we can understand how musical instruments produce sound, we need to look at the basic mechanisms behind these instruments. The theories behind the mechanisms used by musical instruments involve interference, superposition, and standing waves, which we discuss in the next section.

17.4 Normal Modes of a Standing Sound Wave

Learning Objectives

By the end of this section, you will be able to:

- Explain the mechanism behind sound-reducing headphones
- Describe resonance in a tube closed at one end and open at the other end
- Describe resonance in a tube open at both ends

Interference is the hallmark of waves, all of which exhibit constructive and destructive interference exactly analogous to that seen for water waves. In fact, one way to prove something “is a wave” is to observe interference effects. Since sound is a wave, we expect it to exhibit interference.

Interference of Sound Waves

In [Waves](#), we discussed the interference of wave functions that differ only in a phase shift. We found that the wave function resulting from the superposition of $y_1(x, t) = A \sin(kx - \omega t + \phi)$ and $y_2(x, t) = A \sin(kx - \omega t)$ is

$$y(x, t) = \left[2A \cos\left(\frac{\phi}{2}\right) \right] \sin\left(kx - \omega t + \frac{\phi}{2}\right).$$

One way for two identical waves that are initially in phase to become out of phase with one another is to have the waves travel different distances; that is, they have different path lengths. Sound waves provide an excellent example of a phase shift due to a path difference. As we have discussed, sound waves can basically be modeled as longitudinal waves, where the molecules of the medium oscillate around an equilibrium position, or as pressure waves.

When the waves leave the speakers, they move out as spherical waves ([Figure 17.16](#)). The waves interfere; constructive interference is produced by the combination of two crests or two troughs, as shown. Destructive interference is produced by the combination of a trough and a crest.

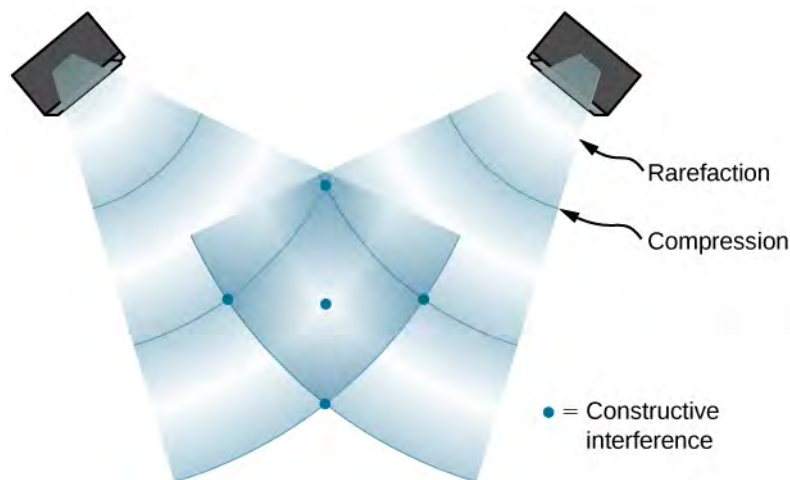


Figure 17.16 When sound waves are produced by a speaker, they travel at the speed of sound and move out as spherical waves. Here, two speakers produce the same steady tone (frequency). The result is points of high-intensity sound (highlighted), which result from two crests (compression) or two troughs (rarefaction) overlapping. Destructive interference results from a crest and trough overlapping. The points where there is constructive interference in the figure occur because the two waves are in phase at those points. Points of destructive interference ([Figure 17.17](#)) are the result of the two waves being out of phase.

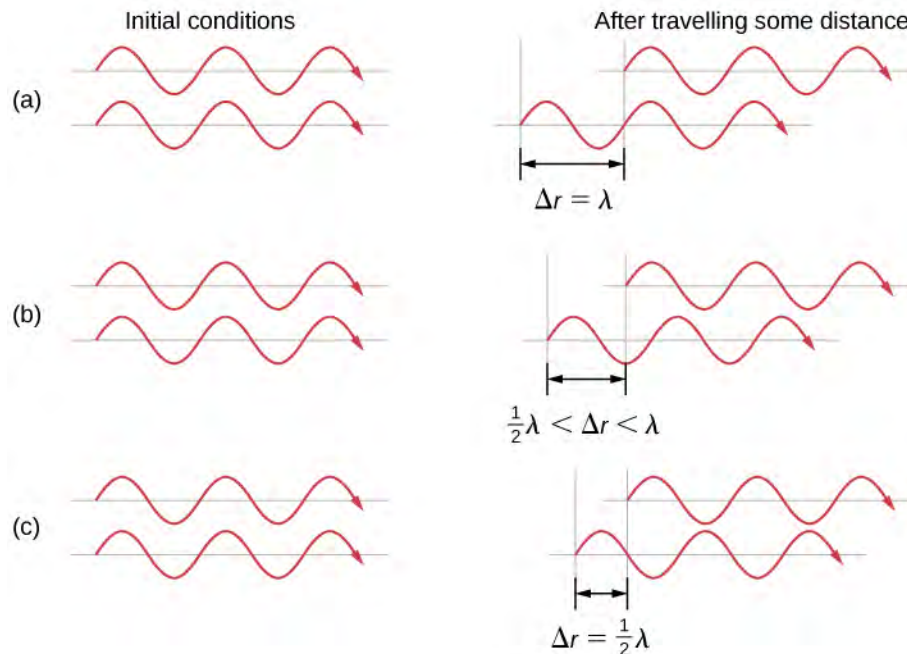
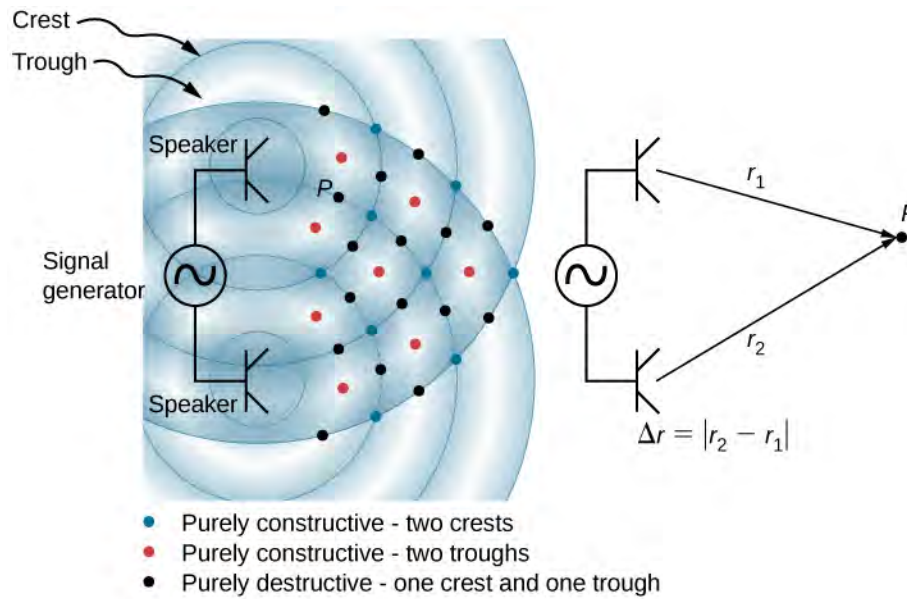


Figure 17.17 Two speakers being driven by a single signal generator. The sound waves produced by the speakers are in phase and are of a single frequency. The sound waves interfere with each other. When two crests or two troughs coincide, there is constructive interference, marked by the red and blue dots. When a trough and a crest coincide, destructive interference occurs, marked by black dots. The phase difference is due to the path lengths traveled by the individual waves. Two identical waves travel two different path lengths to a point P . (a) The difference in the path lengths is one wavelength, resulting in total constructive interference and a resulting amplitude equal to twice the original amplitude. (b) The difference in the path lengths is less than one wavelength but greater than one half a wavelength, resulting in an amplitude greater than zero and less than twice the original amplitude. (c) The difference in the path lengths is one half of a wavelength, resulting in total destructive interference and a resulting amplitude of zero.

The phase difference at each point is due to the different path lengths traveled by each wave. When the difference in the path lengths is an integer multiple of a wavelength,

$$\Delta r = |r_2 - r_1| = n\lambda, \text{ where } n = 0, 1, 2, 3, \dots,$$

the waves are in phase and there is constructive interference. When the difference in path lengths is an odd

multiple of a half wavelength,

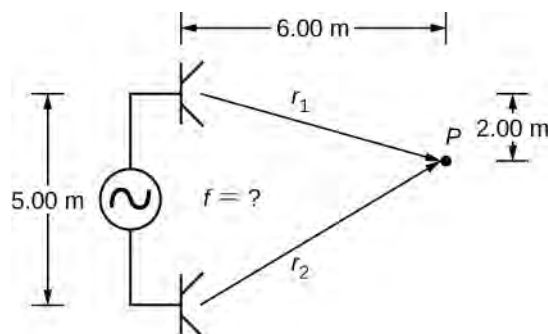
$$\Delta r = |r_2 - r_1| = n \frac{\lambda}{2}, \text{ where } n = 1, 3, 5, \dots,$$

the waves are $180^\circ (\pi \text{ rad})$ out of phase and the result is destructive interference. These points can be located with a sound-level intensity meter.

EXAMPLE 17.5

Interference of Sound Waves

Two speakers are separated by 5.00 m and are being driven by a signal generator at an unknown frequency. A student with a sound-level meter walks out 6.00 m and down 2.00 m, and finds the first minimum intensity, as shown below. What is the frequency supplied by the signal generator? Assume the wave speed of sound is $v = 343.00 \text{ m/s}$.

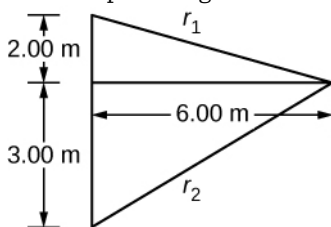


Strategy

The wave velocity is equal to $v = \frac{\lambda}{T} = \lambda f$. The frequency is then $f = \frac{v}{\lambda}$. A minimum intensity indicates destructive interference and the first such point occurs where there is path difference of $\Delta r = \lambda/2$, which can be found from the geometry.

Solution

- Find the path length to the minimum point from each speaker.



$$r_1 = \sqrt{(6.00 \text{ m})^2 + (2.00 \text{ m})^2} = 6.32 \text{ m}, \quad r_2 = \sqrt{(6.00 \text{ m})^2 + (3.00 \text{ m})^2} = 6.71 \text{ m}$$

- Use the difference in the path length to find the wavelength.

$$\Delta r = |r_2 - r_1| = |6.71 \text{ m} - 6.32 \text{ m}| = 0.39 \text{ m}$$

$$\lambda = 2\Delta r = 2(0.39 \text{ m}) = 0.78 \text{ m}$$

- Find the frequency.

$$f = \frac{v}{\lambda} = \frac{343.00 \text{ m/s}}{0.78 \text{ m}} = 439.74 \text{ Hz}$$

Significance

If point P were a point of maximum intensity, then the path length would be an integer multiple of the wavelength.

✓ CHECK YOUR UNDERSTANDING 17.4

If you walk around two speakers playing music, how come you do not notice places where the music is very loud or very soft, that is, where there is constructive and destructive interference?

The concept of a phase shift due to a difference in path length is very important. You will use this concept again in [Interference](#) and [Photons and Matter Waves](#), where we discuss how Thomas Young used this method in his famous double-slit experiment to provide evidence that light has wavelike properties.

Noise Reduction through Destructive Interference

[Figure 17.18](#) shows a clever use of sound interference to cancel noise. Larger-scale applications of active noise reduction by destructive interference have been proposed for entire passenger compartments in commercial aircraft. To obtain destructive interference, a fast electronic analysis is performed, and a second sound is introduced 180° out of phase with the original sound, with its maxima and minima exactly reversed from the incoming noise. Sound waves in fluids are pressure waves and are consistent with Pascal's principle; that is, pressures from two different sources add and subtract like simple numbers. Therefore, positive and negative gauge pressures add to a much smaller pressure, producing a lower-intensity sound. Although completely destructive interference is possible only under the simplest conditions, it is possible to reduce noise levels by 30 dB or more using this technique.

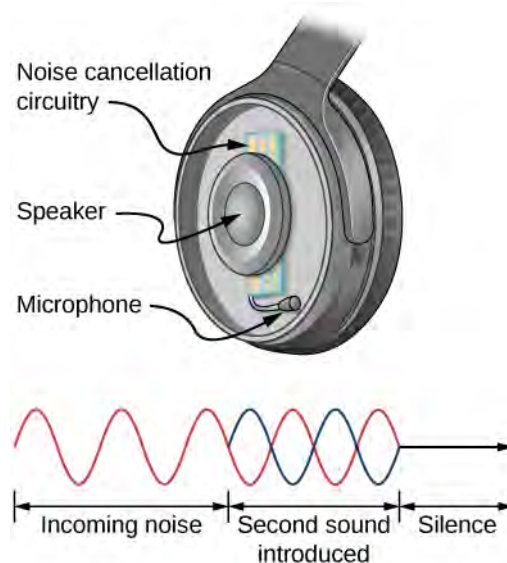


Figure 17.18 Headphones designed to cancel noise with destructive interference create a sound wave exactly opposite to the incoming sound. These headphones can be more effective than the simple passive attenuation used in most ear protection. Such headphones were used on the record-setting, around-the-world nonstop flight of the *Voyager* aircraft in 1986 to protect the pilots' hearing from engine noise.

✓ CHECK YOUR UNDERSTANDING 17.5

Describe how noise-canceling headphones differ from standard headphones used to block outside sounds.

Where else can we observe sound interference? All sound resonances, such as in musical instruments, are due to constructive and destructive interference. Only the resonant frequencies interfere constructively to form standing waves, whereas others interfere destructively and are absent.

Resonance in a Tube Closed at one End

As we discussed in [Waves](#), *standing waves* are formed by two waves moving in opposite directions. When two identical sinusoidal waves move in opposite directions, the waves may be modeled as

$$y_1(x, t) = A \sin(kx - \omega t) \text{ and } y_2(x, t) = A \sin(kx + \omega t).$$

When these two waves interfere, the resultant wave is a standing wave:

$$y_R(x, t) = [2A \sin(kx)] \cos(\omega t).$$

Resonance can be produced due to the boundary conditions imposed on a wave. In [Waves](#), we showed that resonance could be produced in a string under tension that had symmetrical boundary conditions, specifically, a node at each end. We defined a node as a fixed point where the string did not move. We found that the symmetrical boundary conditions resulted in some frequencies resonating and producing standing waves, while other frequencies interfere destructively. Sound waves can resonate in a hollow tube, and the frequencies of the sound waves that resonate depend on the boundary conditions.

Suppose we have a tube that is closed at one end and open at the other. If we hold a vibrating tuning fork near the open end of the tube, an incident sound wave travels through the tube and reflects off the closed end. The reflected sound has the same frequency and wavelength as the incident sound wave, but is traveling in the opposite direction. At the closed end of the tube, the molecules of air have very little freedom to oscillate, and a node arises. At the open end, the molecules are free to move, and at the right frequency, an antinode occurs. Unlike the symmetrical boundary conditions for the standing waves on the string, the boundary conditions for a tube open at one end and closed at the other end are anti-symmetrical: a node at the closed end and an antinode at the open end.

If the tuning fork has just the right frequency, the air column in the tube resonates loudly, but at most frequencies it vibrates very little. This observation just means that the air column has only certain natural frequencies. Consider the lowest frequency that will cause the tube to resonate, producing a loud sound. There will be a node at the closed end and an antinode at the open end, as shown in [Figure 17.19](#).

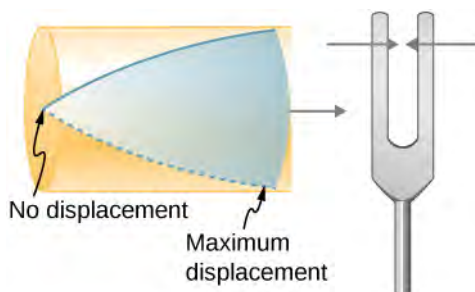


Figure 17.19 Resonance of air in a tube closed at one end, caused by a tuning fork that vibrates at the lowest frequency that can produce resonance (the fundamental frequency). A node exists at the closed end and an antinode at the open end.

The standing wave formed in the tube has an antinode at the open end and a node at the closed end. The distance from a node to an antinode is one-fourth of a wavelength, and this equals the length of the tube; thus, $\lambda_1 = 4L$. This same resonance can be produced by a vibration introduced at or near the closed end of the tube ([Figure 17.20](#)). It is best to consider this a natural vibration of the air column, independently of how it is induced.

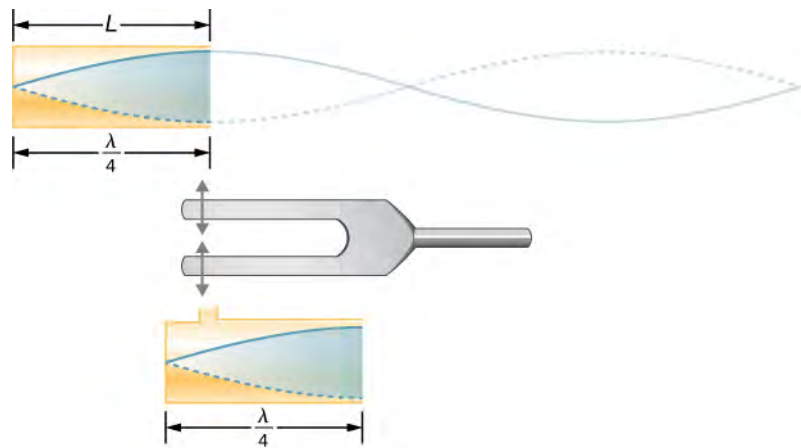


Figure 17.20 The same standing wave is created in the tube by a vibration introduced near its closed end.

Given that maximum air displacements are possible at the open end and none at the closed end, other shorter wavelengths can resonate in the tube, such as the one shown in [Figure 17.21](#). Here the standing wave has three-fourths of its wavelength in the tube, or $\frac{3}{4}\lambda_3 = L$, so that $\lambda_3 = \frac{4}{3}L$. Continuing this process reveals a whole series of shorter-wavelength and higher-frequency sounds that resonate in the tube. We use specific terms for the resonances in any system. The lowest resonant frequency is called the **fundamental**, while all higher resonant frequencies are called **overtones**. The resonant frequencies that are integral multiples of the fundamental are collectively called **harmonics**. The fundamental is the first harmonic, the second harmonic is twice the frequency of the first harmonic, and so on. Some of these harmonics may not exist for a given scenario. [Figure 17.22](#) shows the fundamental and the first three overtones (or the first, third, fifth, and seventh harmonics) in a tube closed at one end.

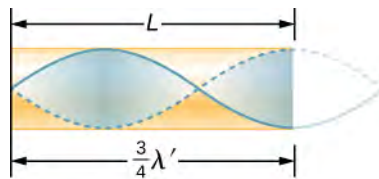


Figure 17.21 Another resonance for a tube closed at one end. This standing wave has maximum air displacement at the open end and none at the closed end. The wavelength is shorter, with three-fourths λ' equaling the length of the tube, so that $\lambda' = 4L/3$. This higher-frequency vibration is the first overtone.

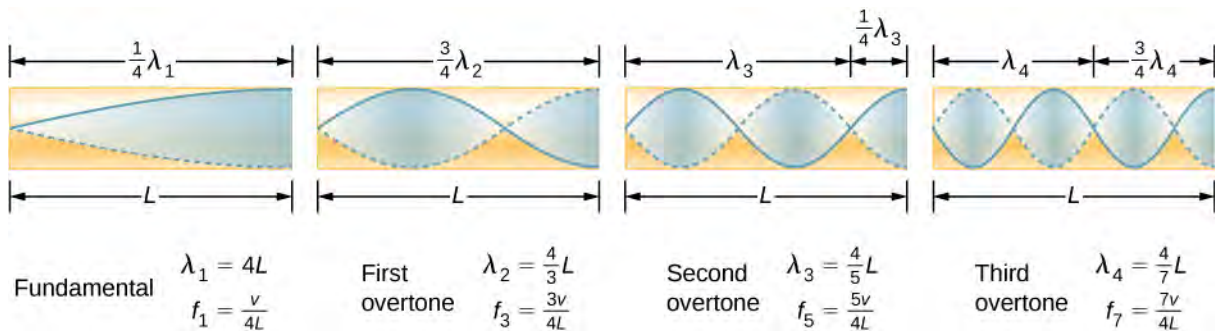


Figure 17.22 The fundamental and three lowest overtones for a tube closed at one end. All have maximum air displacements at the open end and none at the closed end.

The relationship for the resonant wavelengths of a tube closed at one end is

$$\lambda_n = \frac{4}{n}L \quad n = 1, 3, 5, \dots$$

17.13

Now let us look for a pattern in the resonant frequencies for a simple tube that is closed at one end. The

fundamental has $\lambda = 4L$, and frequency is related to wavelength and the speed of sound as given by $v = f\lambda$.

Solving for f in this equation gives

$$f = \frac{v}{\lambda} = \frac{v}{4L},$$

where v is the speed of sound in air. Similarly, the first overtone has $\lambda = 4L/3$ (see [Figure 17.22](#)), so that

$$f_3 = 3\frac{v}{4L} = 3f_1.$$

Because $f_3 = 3f_1$, we call the first overtone the third harmonic. Continuing this process, we see a pattern that can be generalized in a single expression. The resonant frequencies of a tube closed at one end are

$$f_n = n\frac{v}{4L}, \quad n = 1, 3, 5, \dots, \quad 17.14$$

where f_1 is the fundamental, f_3 is the first overtone, and so on. It is interesting that the resonant frequencies depend on the speed of sound and, hence, on temperature. This dependence poses a noticeable problem for organs in old unheated cathedrals, and it is also the reason why musicians commonly bring their wind instruments to room temperature before playing them.

Resonance in a Tube Open at Both Ends

Another source of standing waves is a tube that is open at both ends. In this case, the boundary conditions are symmetrical: an antinode at each end. The resonances of tubes open at both ends can be analyzed in a very similar fashion to those for tubes closed at one end. The air columns in tubes open at both ends have maximum air displacements at both ends ([Figure 17.23](#)). Standing waves form as shown.

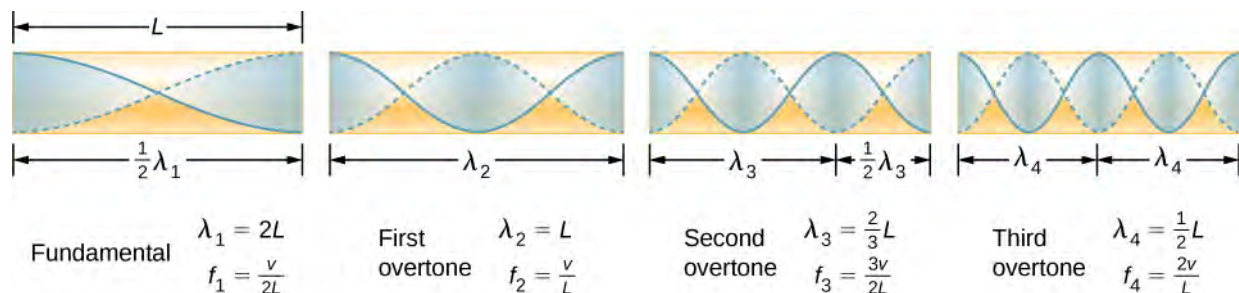


Figure 17.23 The resonant frequencies of a tube open at both ends, including the fundamental and the first three overtones. In all cases, the maximum air displacements occur at both ends of the tube, giving it different natural frequencies than a tube closed at one end.

The relationship for the resonant wavelengths of a tube open at both ends is

$$\lambda_n = \frac{2}{n}L, \quad n = 1, 2, 3, \dots \quad 17.15$$

Based on the fact that a tube open at both ends has maximum air displacements at both ends, and using [Figure 17.23](#) as a guide, we can see that the resonant frequencies of a tube open at both ends are

$$f_n = n\frac{v}{2L}, \quad n = 1, 2, 3, \dots, \quad 17.16$$

where f_1 is the fundamental, f_2 is the first overtone, f_3 is the second overtone, and so on. Note that a tube open at both ends has a fundamental frequency twice what it would have if closed at one end. It also has a different spectrum of overtones than a tube closed at one end.

Note that a tube open at both ends has symmetrical boundary conditions, similar to the string fixed at both

ends discussed in [Waves](#). The relationships for the wavelengths and frequencies of a stringed instrument are the same as given in [Equation 17.15](#) and [Equation 17.16](#). The speed of the wave on the string (from [Waves](#)) is $v = \sqrt{\frac{F_T}{\mu}}$. The air around the string vibrates at the same frequency as the string, producing sound of the same frequency. The sound wave moves at the speed of sound and the wavelength can be found using $v = \lambda f$.

✓ CHECK YOUR UNDERSTANDING 17.6

How is it possible to use a standing wave's node and antinode to determine the length of a closed-end tube?

🔗 INTERACTIVE

This [video \(https://openstax.org/l/21soundwaves\)](https://openstax.org/l/21soundwaves) lets you visualize sound waves.

✓ CHECK YOUR UNDERSTANDING 17.7

You observe two musical instruments that you cannot identify. One plays high-pitched sounds and the other plays low-pitched sounds. How could you determine which is which without hearing either of them play?

17.5 Sources of Musical Sound

Learning Objectives

By the end of this section, you will be able to:

- Describe the resonant frequencies in instruments that can be modeled as a tube with symmetrical boundary conditions
- Describe the resonant frequencies in instruments that can be modeled as a tube with anti-symmetrical boundary conditions

Some musical instruments, such as woodwinds, brass, and pipe organs, can be modeled as tubes with symmetrical boundary conditions, that is, either open at both ends or closed at both ends ([Figure 17.24](#)). Other instruments can be modeled as tubes with anti-symmetrical boundary conditions, such as a tube with one end open and the other end closed ([Figure 17.25](#)).

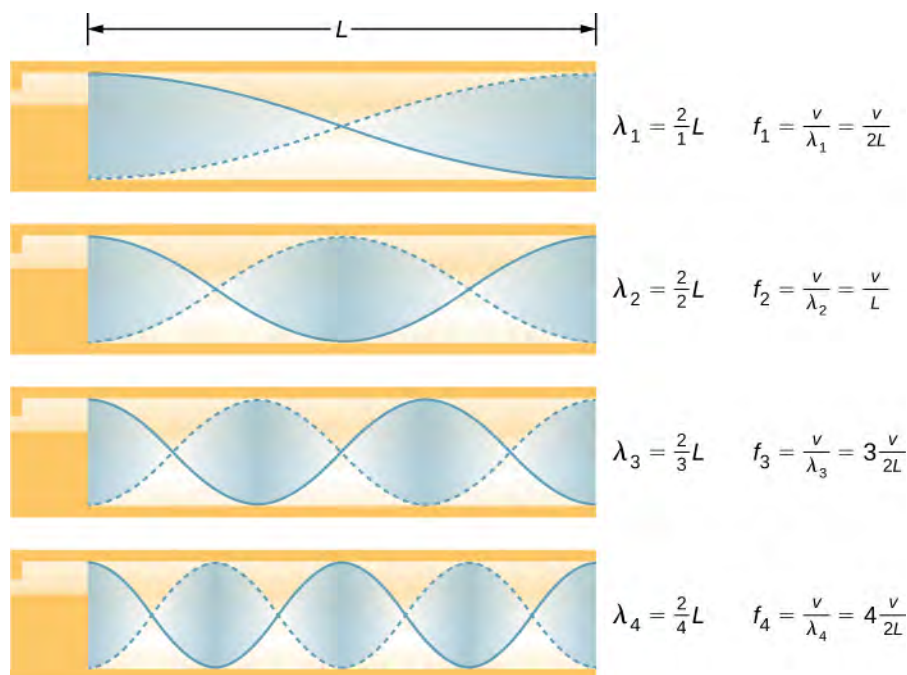


Figure 17.24 Some musical instruments can be modeled as a pipe open at both ends.

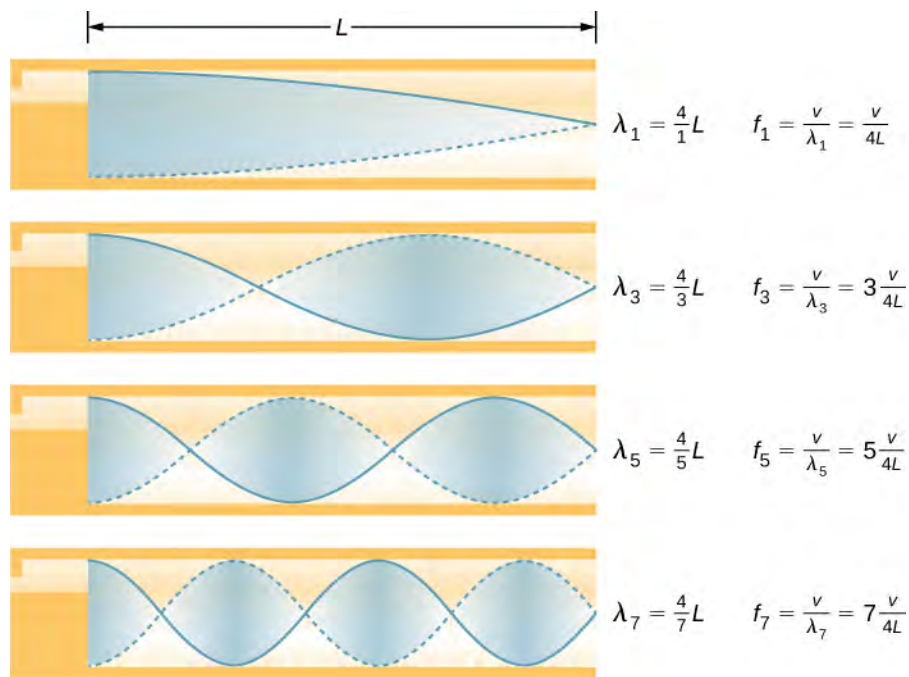


Figure 17.25 Some musical instruments can be modeled as a pipe closed at one end.

Resonant frequencies are produced by longitudinal waves that travel down the tubes and interfere with the reflected waves traveling in the opposite direction. A pipe organ is manufactured with various tubes of fixed lengths to produce different frequencies. The waves are the result of compressed air allowed to expand in the tubes. Even in open tubes, some reflection occurs due to the constraints of the sides of the tubes and the atmospheric pressure outside the open tube.

The antinodes do not occur at the opening of the tube, but rather depend on the radius of the tube. The waves do not fully expand until they are outside the open end of a tube, and for a thin-walled tube, an *end correction* should be added. This end correction is approximately 0.6 times the radius of the tube and should be added to the length of the tube.

Players of instruments such as the flute or oboe vary the length of the tube by opening and closing finger holes. On a trombone, you change the tube length by using a sliding tube. Bugles have a fixed length and can produce only a limited range of frequencies.

The fundamental and overtones can be present simultaneously in a variety of combinations. For example, middle C on a trumpet sounds distinctively different from middle C on a clarinet, although both instruments are modified versions of a tube closed at one end. The fundamental frequency is the same (and usually the most intense), but the overtones and their mix of intensities are different and subject to shading by the musician. This mix is what gives various musical instruments (and human voices) their distinctive characteristics, whether they have air columns, strings, sounding boxes, or drumheads. In fact, much of our speech is determined by shaping the cavity formed by the throat and mouth, and positioning the tongue to adjust the fundamental and combination of overtones. For example, simple resonant cavities can be made to resonate with the sound of the vowels ([Figure 17.26](#)). In boys at puberty, the larynx grows and the shape of the resonant cavity changes, giving rise to the difference in predominant frequencies in speech between men and women.

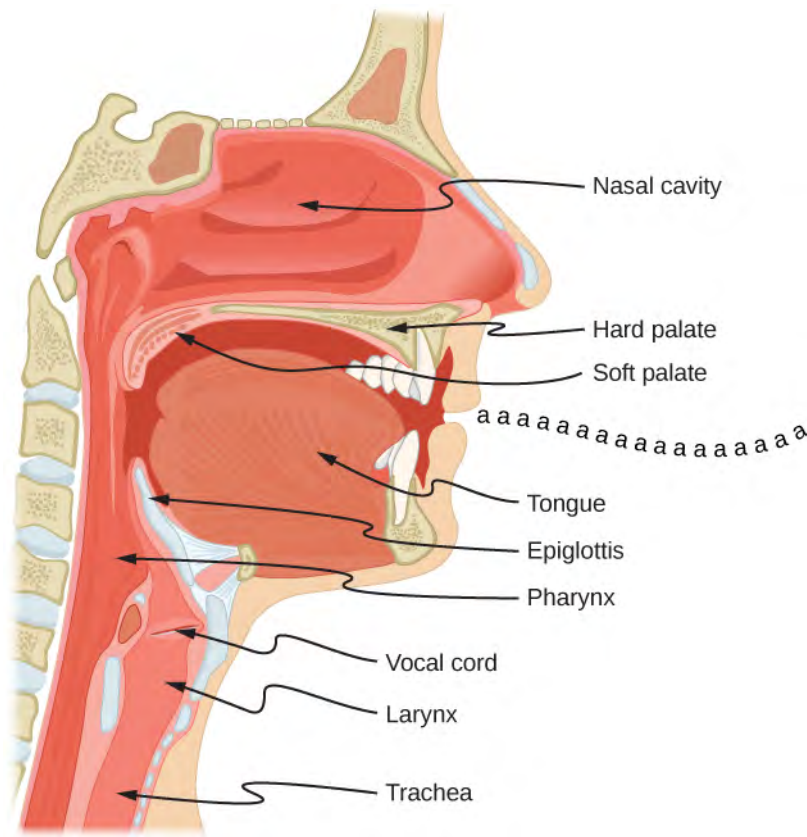


Figure 17.26 The throat and mouth form an air column closed at one end that resonates in response to vibrations in the voice box. The spectrum of overtones and their intensities vary with mouth shaping and tongue position to form different sounds. The voice box can be replaced with a mechanical vibrator, and understandable speech is still possible. Variations in basic shapes make different voices recognizable.



EXAMPLE 17.6

Finding the Length of a Tube with a 128-Hz Fundamental

- (a) What length should a tube closed at one end have on a day when the air temperature is 22.0°C if its fundamental frequency is to be 128 Hz (C below middle C)?
- (b) What is the frequency of its fourth overtone?

Strategy

The length L can be found from the relationship $f_n = n \frac{v}{4L}$, but we first need to find the speed of sound v .

Solution

- a. Identify knowns: The fundamental frequency is 128 Hz, and the air temperature is 22.0°C .

Use $f_n = n \frac{v}{4L}$ to find the fundamental frequency ($n = 1$),

$$f_1 = \frac{v}{4L}.$$

Solve this equation for length,

$$L = \frac{v}{4f_1}.$$

Find the speed of sound using $v = (331 \text{ m/s})\sqrt{\frac{T}{273 \text{ K}}}$,

$$v = (331 \text{ m/s}) \sqrt{\frac{295 \text{ K}}{273 \text{ K}}} = 344 \text{ m/s}.$$

Enter the values of the speed of sound and frequency into the expression for L .

$$L = \frac{v}{4f_1} = \frac{344 \text{ m/s}}{4(128 \text{ Hz})} = 0.672 \text{ m}$$

- b. Identify knowns: The first overtone has $n = 3$, the second overtone has $n = 5$, the third overtone has $n = 7$, and the fourth overtone has $n = 9$.

Enter the value for the fourth overtone into $f_n = n \frac{v}{4L}$,

$$f_9 = 9 \frac{v}{4L} = 9f_1 = 1.15 \text{ kHz}.$$

Significance

Many wind instruments are modified tubes that have finger holes, valves, and other devices for changing the length of the resonating air column and hence, the frequency of the note played. Horns producing very low frequencies require tubes so long that they are coiled into loops. An example is the tuba. Whether an overtone occurs in a simple tube or a musical instrument depends on how it is stimulated to vibrate and the details of its shape. The trombone, for example, does not produce its fundamental frequency and only makes overtones.

If you have two tubes with the same fundamental frequency, but one is open at both ends and the other is closed at one end, they would sound different when played because they have different overtones. Middle C, for example, would sound richer played on an open tube, because it has even multiples of the fundamental as well as odd. A closed tube has only odd multiples.

Resonance

Resonance occurs in many different systems, including strings, air columns, and atoms. As we discussed in earlier chapters, resonance is the driven or forced oscillation of a system at its natural frequency. At resonance, energy is transferred rapidly to the oscillating system, and the amplitude of its oscillations grows until the system can no longer be described by Hooke's law. An example of this is the distorted sound intentionally produced in certain types of rock music.

Wind instruments use resonance in air columns to amplify tones made by lips or vibrating reeds. Other instruments also use air resonance in clever ways to amplify sound. [Figure 17.27](#) shows a violin and a guitar, both of which have sounding boxes but with different shapes, resulting in different overtone structures. The vibrating string creates a sound that resonates in the sounding box, greatly amplifying the sound and creating overtones that give the instrument its characteristic timbre. The more complex the shape of the sounding box, the greater its ability to resonate over a wide range of frequencies. The marimba, like the one shown in [Figure 17.28](#), uses pots or gourds below the wooden slats to amplify their tones. The resonance of the pot can be adjusted by adding water.



(a)



(b)

Figure 17.27 String instruments such as (a) violins and (b) guitars use resonance in their sounding boxes to amplify and enrich the sound created by their vibrating strings. The bridge and supports couple the string vibrations to the sounding boxes and air within. (credit a: modification of work by Feliciano Guimarães; credit b: modification of work by Steve Snodgrass)



Figure 17.28 This marimba uses gourds as resonance chambers to amplify its sound. (credit: "APC Events"/Flickr)

We have emphasized sound applications in our discussions of resonance and standing waves, but these ideas apply to any system that has wave characteristics. Vibrating strings, for example, are actually resonating and have fundamentals and overtones similar to those for air columns. More subtle are the resonances in atoms due to the wave character of their electrons. Their orbitals can be viewed as standing waves, which have a fundamental (ground state) and overtones (excited states). It is fascinating that wave characteristics apply to such a wide range of physical systems.

17.6 Beats

Learning Objectives

By the end of this section, you will be able to:

- Determine the beat frequency produced by two sound waves that differ in frequency
- Describe how beats are produced by musical instruments

The study of music provides many examples of the superposition of waves and the constructive and destructive interference that occurs. Very few examples of music being performed consist of a single source playing a single frequency for an extended period of time. You will probably agree that a single frequency of sound for an extended period might be boring to the point of irritation, similar to the unwanted drone of an aircraft engine or a loud fan. Music is pleasant and interesting due to mixing the changing frequencies of various instruments and voices.

An interesting phenomenon that occurs due to the constructive and destructive interference of two or more frequencies of sound is the phenomenon of **beats**. If two sounds differ in frequencies, the sound waves can be modeled as

$$y_1 = A \cos(k_1 x - 2\pi f_1 t) \text{ and } y_2 = A \cos(k_2 x - 2\pi f_2 t).$$

Using the trigonometric identity $\cos u + \cos v = 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$ and considering the point in space as $x = 0.0 \text{ m}$, we find the resulting sound at a point in space, from the superposition of the two sound waves, is equal to [Figure 17.29](#):

$$y(t) = 2A \cos(2\pi f_{\text{avg}} t) \cos\left(2\pi \left(\frac{|f_2 - f_1|}{2}\right) t\right),$$

where the **beat frequency** is

$$f_{\text{beat}} = |f_2 - f_1|.$$

17.17

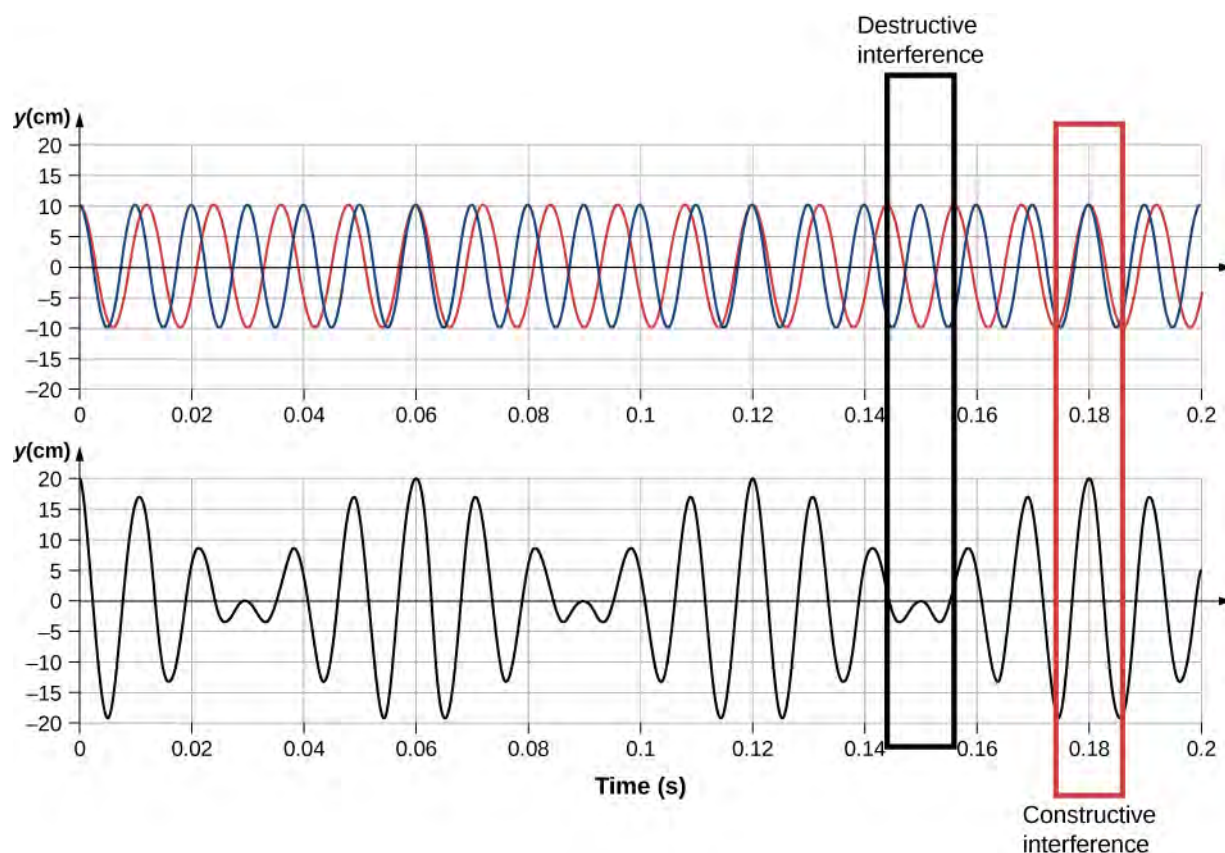


Figure 17.29 Beats produced by the constructive and destructive interference of two sound waves that differ in frequency.

These beats can be used by piano tuners to tune a piano. A tuning fork is struck and a note is played on the piano. As the piano tuner tunes the string, the beats have a lower frequency as the frequency of the note played approaches the frequency of the tuning fork.

**EXAMPLE 17.7****Find the Beat Frequency Between Two Tuning Forks**

What is the beat frequency produced when a tuning fork of a frequency of 256 Hz and a tuning fork of a frequency of 512 Hz are struck simultaneously?

Strategy

The beat frequency is the difference of the two frequencies.

Solution

We use $f_{\text{beat}} = |f_2 - f_1|$:

$$|f_2 - f_1| = (512 - 256) \text{ Hz} = 256 \text{ Hz}.$$

Significance

The beat frequency is the absolute value of the difference between the two frequencies. A negative frequency would not make sense.

**CHECK YOUR UNDERSTANDING 17.8**

What would happen if more than two frequencies interacted? Consider three frequencies.

The study of the superposition of various waves has many interesting applications beyond the study of sound. In later chapters, we will discuss the wave properties of particles. The particles can be modeled as a “wave packet” that results from the superposition of various waves, where the particle moves at the “group velocity” of the wave packet.

17.7 The Doppler Effect

Learning Objectives

By the end of this section, you will be able to:

- Explain the change in observed frequency as a moving source of sound approaches or departs from a stationary observer
- Explain the change in observed frequency as an observer moves toward or away from a stationary source of sound

The characteristic sound of a motorcycle buzzing by is an example of the **Doppler effect**. Specifically, if you are standing on a street corner and observe an ambulance with a siren sounding passing at a constant speed, you notice two characteristic changes in the sound of the siren. First, the sound increases in loudness as the ambulance approaches and decreases in loudness as it moves away, which is expected. But in addition, the high-pitched siren shifts dramatically to a lower-pitched sound. As the ambulance passes, the frequency of the sound heard by a stationary observer changes from a constant high frequency to a constant lower frequency, even though the siren is producing a constant source frequency. The closer the ambulance brushes by, the more abrupt the shift. Also, the faster the ambulance moves, the greater the shift. We also hear this characteristic shift in frequency for passing cars, airplanes, and trains.

The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer. Although less familiar, this effect is easily noticed for a stationary source and moving observer. For example, if you ride a train past a stationary warning horn, you will hear the horn’s frequency shift from high to low as you pass by. The actual change in frequency due to relative motion of source and observer is called a **Doppler shift**. The Doppler effect and Doppler shift are named for the Austrian physicist and mathematician Christian Johann Doppler (1803–1853), who did experiments with both moving sources and moving observers. Doppler, for example, had musicians play on a moving open train car and also play standing next to the train tracks as a train passed by. Their music was observed both on and off the train, and changes in frequency were measured.

What causes the Doppler shift? [Figure 17.30](#) illustrates sound waves emitted by stationary and moving sources in a stationary air mass. Each disturbance spreads out spherically from the point at which the sound is emitted. If the source is stationary, then all of the spheres representing the air compressions in the sound wave are centered on the same point, and the stationary observers on either side hear the same wavelength and frequency as emitted by the source (case a). If the source is moving, the situation is different. Each compression of the air moves out in a sphere from the point at which it was emitted, but the point of emission moves. This moving emission point causes the air compressions to be closer together on one side and farther apart on the other. Thus, the wavelength is shorter in the direction the source is moving (on the right in case b), and longer in the opposite direction (on the left in case b). Finally, if the observers move, as in case (c), the frequency at which they receive the compressions changes. The observer moving toward the source receives them at a higher frequency, and the person moving away from the source receives them at a lower frequency.

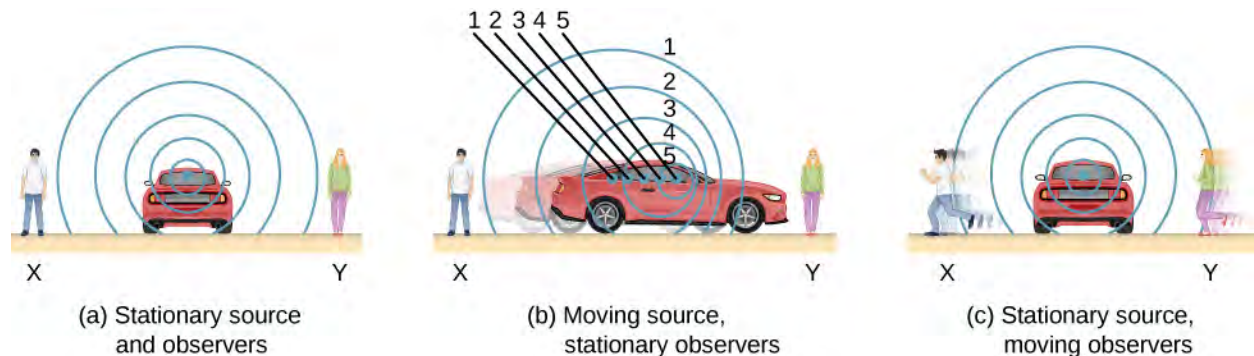


Figure 17.30 Sounds emitted by a source spread out in spherical waves. (a) When the source, observers, and air are stationary, the wavelength and frequency are the same in all directions and to all observers. (b) Sounds emitted by a source moving to the right spread out from the points at which they were emitted. The wavelength is reduced, and consequently, the frequency is increased in the direction of motion, so that the observer on the right hears a higher-pitched sound. The opposite is true for the observer on the left, where the wavelength is increased and the frequency is reduced. (c) The same effect is produced when the observers move relative to the source. Motion toward the source increases frequency as the observer on the right passes through more wave crests than she would if stationary. Motion away from the source decreases frequency as the observer on the left passes through fewer wave crests than he would if stationary.

We know that wavelength and frequency are related by $v = f\lambda$, where v is the fixed speed of sound. The sound moves in a medium and has the same speed v in that medium whether the source is moving or not. Thus, f multiplied by λ is a constant. Because the observer on the right in case (b) receives a shorter wavelength, the frequency she receives must be higher. Similarly, the observer on the left receives a longer wavelength, and hence he hears a lower frequency. The same thing happens in case (c). A higher frequency is received by the observer moving toward the source, and a lower frequency is received by an observer moving away from the source. In general, then, relative motion of source and observer toward one another increases the received frequency. Relative motion apart decreases frequency. The greater the relative speed, the greater the effect.

The Doppler effect occurs not only for sound, but for any wave when there is relative motion between the observer and the source. Doppler shifts occur in the frequency of sound, light, and water waves, for example. Doppler shifts can be used to determine velocity, such as when ultrasound is reflected from blood in a medical diagnostic. The relative velocities of stars and galaxies is determined by the shift in the frequencies of light received from them and has implied much about the origins of the universe. Modern physics has been profoundly affected by observations of Doppler shifts.

Derivation of the Observed Frequency due to the Doppler Shift

Consider two stationary observers X and Y in [Figure 17.31](#), located on either side of a stationary source. Each observer hears the same frequency, and that frequency is the frequency produced by the stationary source.

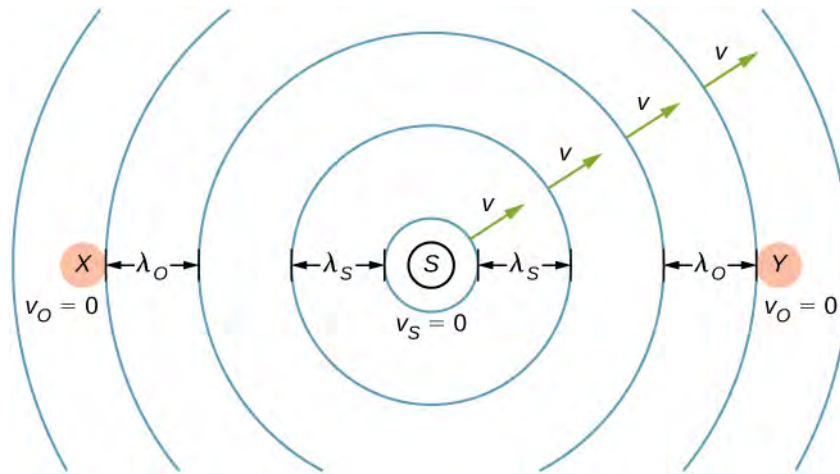


Figure 17.31 A stationary source sends out sound waves at a constant frequency f_s , with a constant wavelength λ_s , at the speed of sound v . Two stationary observers X and Y, on either side of the source, observe a frequency $f_o = f_s$, with a wavelength $\lambda_o = \lambda_s$.

Now consider a stationary observer X with a source moving away from the observer with a constant speed $v_s < v$ (Figure 17.32). At time $t = 0$, the source sends out a sound wave, indicated in black. This wave moves out at the speed of sound v . The position of the sound wave at each time interval of period T_s is shown as dotted lines. After one period, the source has moved $\Delta x = v_s T_s$ and emits a second sound wave, which moves out at the speed of sound. The source continues to move and produce sound waves, as indicated by the circles numbered 3 and 4. Notice that as the waves move out, they remained centered at their respective point of origin.

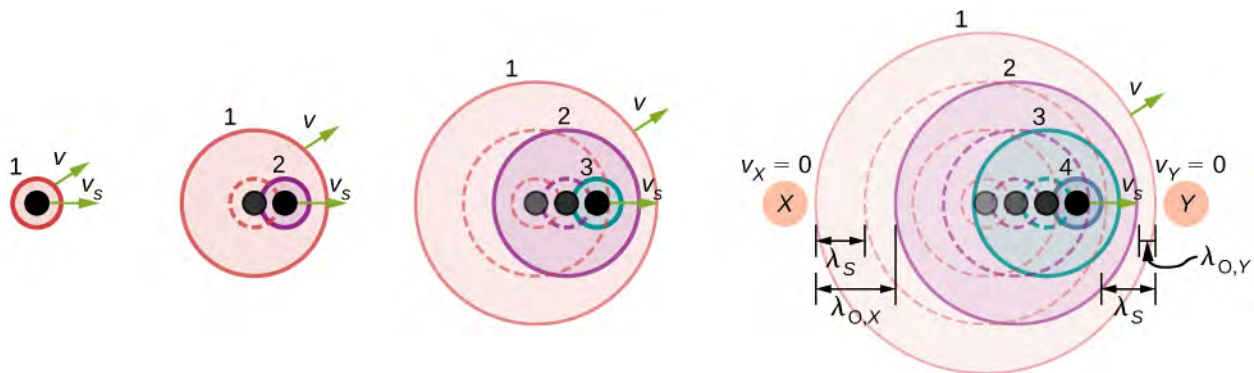


Figure 17.32 A source moving at a constant speed v_s away from an observer X. The moving source sends out sound waves at a constant frequency f_s , with a constant wavelength λ_s , at the speed of sound v . Snapshots of the source at an interval of T_s are shown as the source moves away from the stationary observer X. The solid lines represent the position of the sound waves after four periods from the initial time. The dotted lines are used to show the positions of the waves at each time period. The observer hears a wavelength of $\lambda_o = \lambda_s + \Delta x = \lambda_s + v_s T_s$.

Using the fact that the wavelength is equal to the speed times the period, and the period is the inverse of the frequency, we can derive the observed frequency:

$$\begin{aligned}\lambda_o &= \lambda_s + \Delta x \\ vT_o &= vT_s + v_s T_s \\ \frac{v}{f_o} &= \frac{v}{f_s} = \frac{v_s}{f_s} = \frac{v + v_s}{f_s} \\ f_o &= f_s \left(\frac{v}{v + v_s} \right).\end{aligned}$$

As the source moves away from the observer, the observed frequency is lower than the source frequency.

Now consider a source moving at a constant velocity v_s , moving toward a stationary observer Y, also shown in Figure 17.32. The wavelength is observed by Y as $\lambda_o = \lambda_s - \Delta x = \lambda_s - v_s T_s$. Once again, using the fact that

the wavelength is equal to the speed times the period, and the period is the inverse of the frequency, we can derive the observed frequency:

$$\begin{aligned}\lambda_o &= \lambda_s - \Delta x \\ vT_o &= vT_s - v_s T_s \\ \frac{v}{f_o} &= \frac{v}{f_s} - \frac{v_s}{f_s} = \frac{v-v_s}{f_s} \\ f_o &= f_s \left(\frac{v}{v-v_s} \right).\end{aligned}$$

When a source is moving and the observer is stationary, the observed frequency is

$$f_o = f_s \left(\frac{v}{v \mp v_s} \right), \quad 17.18$$

where f_o is the frequency observed by the stationary observer, f_s is the frequency produced by the moving source, v is the speed of sound, v_s is the constant speed of the source, and the top sign is for the source approaching the observer and the bottom sign is for the source departing from the observer.

What happens if the observer is moving and the source is stationary? If the observer moves toward the stationary source, the observed frequency is higher than the source frequency. If the observer is moving away from the stationary source, the observed frequency is lower than the source frequency. Consider observer X in [Figure 17.33](#) as the observer moves toward a stationary source with a speed v_o . The source emits a tone with a constant frequency f_s and constant period T_s . The observer hears the first wave emitted by the source. If the observer were stationary, the time for one wavelength of sound to pass should be equal to the period of the source T_s . Since the observer is moving toward the source, the time for one wavelength to pass is less than T_s and is equal to the observed period $T_o = T_s - \Delta t$. At time $t = 0$, the observer starts at the beginning of a wavelength and moves toward the second wavelength as the wavelength moves out from the source. The wavelength is equal to the distance the observer traveled plus the distance the sound wave traveled until it is met by the observer:

$$\begin{aligned}\lambda_s &= vT_o + v_o T_o \\ vT_s &= (v + v_o) T_o \\ v \left(\frac{1}{f_s} \right) &= (v + v_o) \left(\frac{1}{f_o} \right) \\ f_o &= f_s \left(\frac{v+v_o}{v} \right).\end{aligned}$$

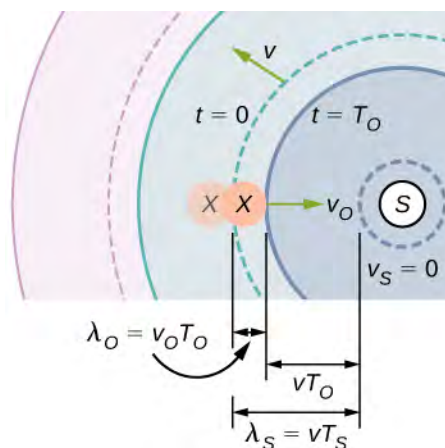


Figure 17.33 A stationary source emits a sound wave with a constant frequency f_s , with a constant wavelength λ_s moving at the speed of sound v . Observer X moves toward the source with a constant speed v_o , and the figure shows the initial and final position of observer X. Observer X observes a frequency higher than the source frequency. The dotted lines show the position of the waves at $t = 0$. The solid lines show the position of the waves at $t = T_o$.

If the observer is moving away from the source ([Figure 17.34](#)), the observed frequency can be found:

$$\begin{aligned}
 \lambda_s &= vT_o - v_oT_o \\
 vT_s &= (v - v_o)T_o \\
 v\left(\frac{1}{f_s}\right) &= (v - v_o)\left(\frac{1}{f_o}\right) \\
 f_o &= f_s \left(\frac{v - v_o}{v}\right).
 \end{aligned}$$

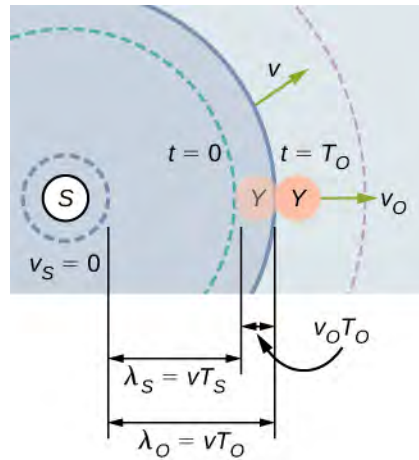


Figure 17.34 A stationary source emits a sound wave with a constant frequency f_s , with a constant wavelength λ_s moving at the speed of sound v . Observer Y moves away from the source with a constant speed v_o , and the figure shows initial and final position of the observer Y . Observer Y observes a frequency lower than the source frequency. The dotted lines show the position of the waves at $t = 0$. The solid lines show the position of the waves at $t = T_o$.

The equations for an observer moving toward or away from a stationary source can be combined into one equation:

$$f_o = f_s \left(\frac{v \pm v_o}{v} \right), \quad 17.19$$

where f_o is the observed frequency, f_s is the source frequency, v is the speed of sound, v_o is the speed of the observer, the top sign is for the observer approaching the source and the bottom sign is for the observer departing from the source.

[Equation 17.18](#) and [Equation 17.19](#) can be summarized in one equation (the top sign is for approaching) and is further illustrated in [Table 17.4](#):

$$f_o = f_s \left(\frac{v \pm v_o}{v \mp v_s} \right), \quad 17.20$$

Doppler shift $f_o = f_s \left(\frac{v \pm v_o}{v \mp v_s} \right)$	Stationary observer	Observer moving towards source	Observer moving away from source
Stationary source	$f_o = f_s$	$f_o = f_s \left(\frac{v + v_o}{v} \right)$	$f_o = f_s \left(\frac{v - v_o}{v} \right)$
Source moving towards observer	$f_o = f_s \left(\frac{v}{v - v_s} \right)$	$f_o = f_s \left(\frac{v + v_o}{v - v_s} \right)$	$f_o = f_s \left(\frac{v - v_o}{v - v_s} \right)$
Source moving away from observer	$f_o = f_s \left(\frac{v}{v + v_s} \right)$	$f_o = f_s \left(\frac{v + v_o}{v + v_s} \right)$	$f_o = f_s \left(\frac{v - v_o}{v + v_s} \right)$

Table 17.4

where f_o is the observed frequency, f_s is the source frequency, v is the speed of sound, v_o is the speed of the observer, v_s is the speed of the source, the top sign is for approaching and the bottom sign is for departing.

INTERACTIVE

The Doppler effect involves motion and a [video \(https://openstax.org/l/21doppler\)](https://openstax.org/l/21doppler) will help visualize the effects of a moving observer or source. This video shows a moving source and a stationary observer, and a moving observer and a stationary source. It also discusses the Doppler effect and its application to light.



EXAMPLE 17.8

Calculating a Doppler Shift

Suppose a train that has a 150-Hz horn is moving at 35.0 m/s in still air on a day when the speed of sound is 340 m/s.

- What frequencies are observed by a stationary person at the side of the tracks as the train approaches and after it passes?
- What frequency is observed by the train's engineer traveling on the train?

Strategy

To find the observed frequency in (a), we must use $f_{\text{obs}} = f_s \left(\frac{v}{v \pm v_s} \right)$ because the source is moving. The minus sign is used for the approaching train, and the plus sign for the receding train. In (b), there are two Doppler shifts—one for a moving source and the other for a moving observer.

Solution

- Enter known values into $f_o = f_s \left(\frac{v}{v - v_s} \right)$:

$$f_o = f_s \left(\frac{v}{v - v_s} \right) = (150 \text{ Hz}) \left(\frac{340 \text{ m/s}}{340 \text{ m/s} - 35.0 \text{ m/s}} \right).$$

Calculate the frequency observed by a stationary person as the train approaches:

$$f_o = (150 \text{ Hz})(1.11) = 167 \text{ Hz}.$$

Use the same equation with the plus sign to find the frequency heard by a stationary person as the train recedes:

$$f_o = f_s \left(\frac{v}{v + v_s} \right) = (150 \text{ Hz}) \left(\frac{340 \text{ m/s}}{340 \text{ m/s} + 35.0 \text{ m/s}} \right).$$

Calculate the second frequency:

$$f_o = (150 \text{ Hz})(0.907) = 136 \text{ Hz}.$$

- Identify knowns:
 - It seems reasonable that the engineer would receive the same frequency as emitted by the horn, because the relative velocity between them is zero.
 - Relative to the medium (air), the speeds are $v_s = v_o = 35.0 \text{ m/s}$.
 - The first Doppler shift is for the moving observer; the second is for the moving source.

Use the following equation:

$$f_o = \left[f_s \left(\frac{v \pm v_o}{v} \right) \right] \left(\frac{v}{v \mp v_s} \right).$$

The quantity in the square brackets is the Doppler-shifted frequency due to a moving observer. The factor on the right is the effect of the moving source.

Because the train engineer is moving in the direction toward the horn, we must use the plus sign for v_{obs} ; however, because the horn is also moving in the direction away from the engineer, we also use the plus

sign for v_s . But the train is carrying both the engineer and the horn at the same velocity, so $v_s = v_o$. As a result, everything but f_s cancels, yielding

$$f_o = f_s.$$

Significance

For the case where the source and the observer are not moving together, the numbers calculated are valid when the source (in this case, the train) is far enough away that the motion is nearly along the line joining source and observer. In both cases, the shift is significant and easily noticed. Note that the shift is 17.0 Hz for motion toward and 14.0 Hz for motion away. The shifts are not symmetric.

For the engineer riding in the train, we may expect that there is no change in frequency because the source and observer move together. This matches your experience. For example, there is no Doppler shift in the frequency of conversations between driver and passenger on a motorcycle. People talking when a wind moves the air between them also observe no Doppler shift in their conversation. The crucial point is that source and observer are not moving relative to each other.

✓ CHECK YOUR UNDERSTANDING 17.9

Describe a situation in your life when you might rely on the Doppler shift to help you either while driving a car or walking near traffic.

The Doppler effect and the Doppler shift have many important applications in science and engineering. For example, the Doppler shift in ultrasound can be used to measure blood velocity, and police use the Doppler shift in radar (a microwave) to measure car velocities. In meteorology, the Doppler shift is used to track the motion of storm clouds; such “Doppler Radar” can give the velocity and direction of rain or snow in weather fronts. In astronomy, we can examine the light emitted from distant galaxies and determine their speed relative to ours. As galaxies move away from us, their light is shifted to a lower frequency, and so to a longer wavelength—the so-called red shift. Such information from galaxies far, far away has allowed us to estimate the age of the universe (from the Big Bang) as about 14 billion years.

17.8 Shock Waves

Learning Objectives

By the end of this section, you will be able to:

- Explain the mechanism behind sonic booms
- Describe the difference between sonic booms and shock waves
- Describe a bow wake

When discussing the Doppler effect of a moving source and a stationary observer, the only cases we considered were cases where the source was moving at speeds that were less than the speed of sound. Recall that the observed frequency for a moving source approaching a stationary observer is $f_o = f_s \left(\frac{v}{v - v_s} \right)$. As the source approaches the speed of sound, the observed frequency increases. According to the equation, if the source moves at the speed of sound, the denominator is equal to zero, implying the observed frequency is infinite. If the source moves at speeds greater than the speed of sound, the observed frequency is negative.

What could this mean? What happens when a source approaches the speed of sound? It was once argued by some scientists that such a large pressure wave would result from the constructive interference of the sound waves, that it would be impossible for a plane to exceed the speed of sound because the pressures would be great enough to destroy the airplane. But now planes routinely fly faster than the speed of sound. On July 28, 1976, Captain Eldon W. Joersz and Major George T. Morgan flew a Lockheed SR-71 Blackbird #61-7958 at 3529.60 km/h (2193.20 mi/h), which is Mach 2.85. The Mach number is the speed of the source divided by the speed of sound:

$$M = \frac{v_s}{v}.$$

17.21

You will see that interesting phenomena occur when a source approaches and exceeds the speed of sound.

Doppler Effect and High Velocity

What happens to the sound produced by a moving source, such as a jet airplane, that approaches or even exceeds the speed of sound? The answer to this question applies not only to sound but to all other waves as well. Suppose a jet plane is coming nearly straight at you, emitting a sound of frequency f_s . The greater the plane's speed v_s , the greater the Doppler shift and the greater the value observed for f_o (Figure 17.35).

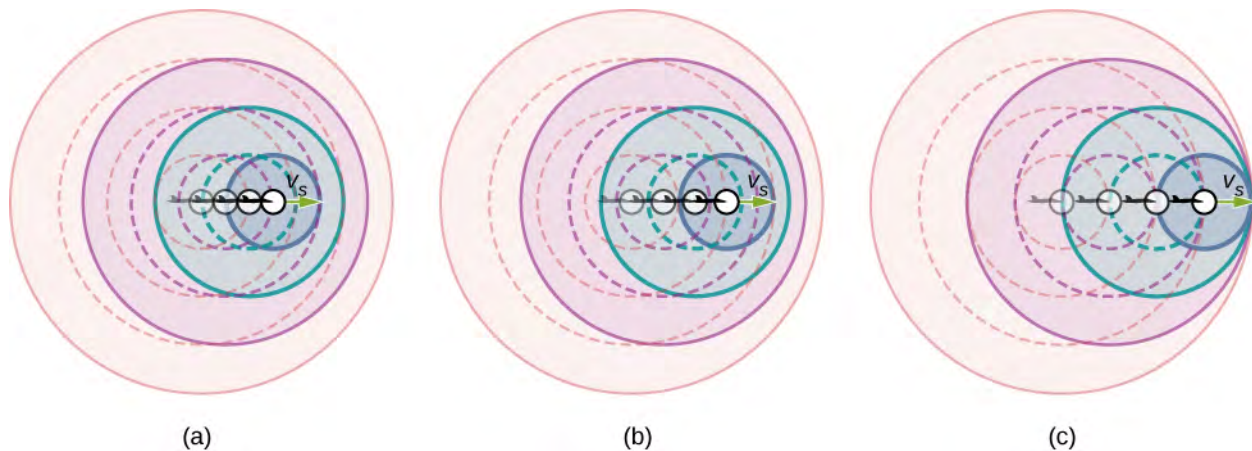


Figure 17.35 Because of the Doppler shift, as a moving source approaches a stationary observer, the observed frequency is higher than the source frequency. The faster the source is moving, the higher the observed frequency. In this figure, the source in (b) is moving faster than the source in (a). Shown are four time steps, the first three shown as dotted lines. (c) If a source moves at the speed of sound, each successive wave interfere with the previous one and the observer observes them all at the same instant.

Now, as v_s approaches the speed of sound, f_o approaches infinity, because the denominator in $f_o = f_s \left(\frac{v}{v \mp v_s} \right)$ approaches zero. At the speed of sound, this result means that in front of the source, each successive wave interferes with the previous one because the source moves forward at the speed of sound. The observer gets them all at the same instant, so the frequency is infinite [part (c) of the figure].

Shock Waves and Sonic Booms

If the source exceeds the speed of sound, no sound is received by the observer until the source has passed, so that the sounds from the approaching source are mixed with those from it when receding. This mixing appears messy, but something interesting happens—a shock wave is created (Figure 17.36).

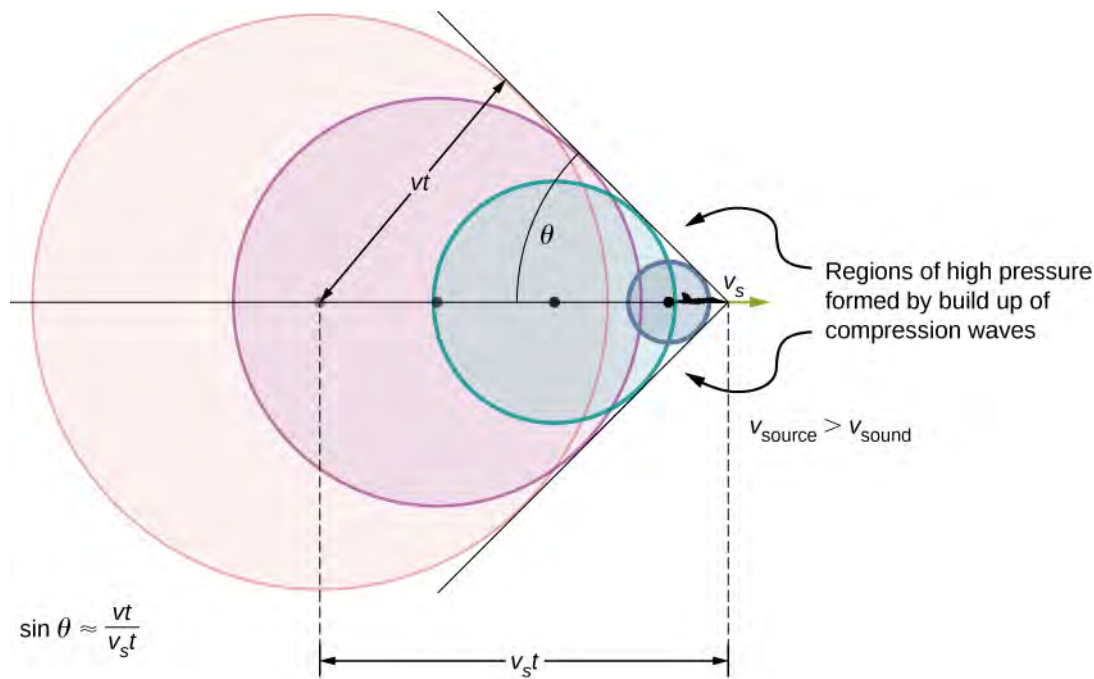


Figure 17.36 Sound waves from a source that moves faster than the speed of sound spread spherically from the point where they are emitted, but the source moves ahead of each wave. Constructive interference along the lines shown (actually a cone in three dimensions) creates a shock wave called a sonic boom. The faster the speed of the source, the smaller the angle θ .

Constructive interference along the lines shown (a cone in three dimensions) from similar sound waves arriving there simultaneously. This superposition forms a disturbance called a **shock wave**, a constructive interference of sound created by an object moving faster than sound. Inside the cone, the interference is mostly destructive, so the sound intensity there is much less than on the shock wave. The angle of the shock wave can be found from the geometry. In time t the source has moved $v_s t$ and the sound wave has moved a distance vt and the angle can be found using $\sin \theta = \frac{vt}{v_s t} = \frac{v}{v_s}$. Note that the Mach number is defined as $\frac{v_s}{v}$ so the sine of the angle equals the inverse of the Mach number,

$$\sin \theta = \frac{v}{v_s} = \frac{1}{M}. \quad 17.22$$

You may have heard of the common term ‘**sonic boom**.’ A common misconception is that the sonic boom occurs as the plane breaks the sound barrier; that is, accelerates to a speed higher than the speed of sound. Actually, the sonic boom occurs as the shock wave sweeps along the ground.

An aircraft creates two shock waves, one from its nose and one from its tail ([Figure 17.37](#)). During television coverage of space shuttle landings, two distinct booms could often be heard. These were separated by exactly the time it would take the shuttle to pass by a point. Observers on the ground often do not see the aircraft creating the sonic boom, because it has passed by before the shock wave reaches them, as seen in the figure. If the aircraft flies close by at low altitude, pressures in the sonic boom can be destructive and break windows as well as rattle nerves. Because of how destructive sonic booms can be, supersonic flights are banned over populated areas.

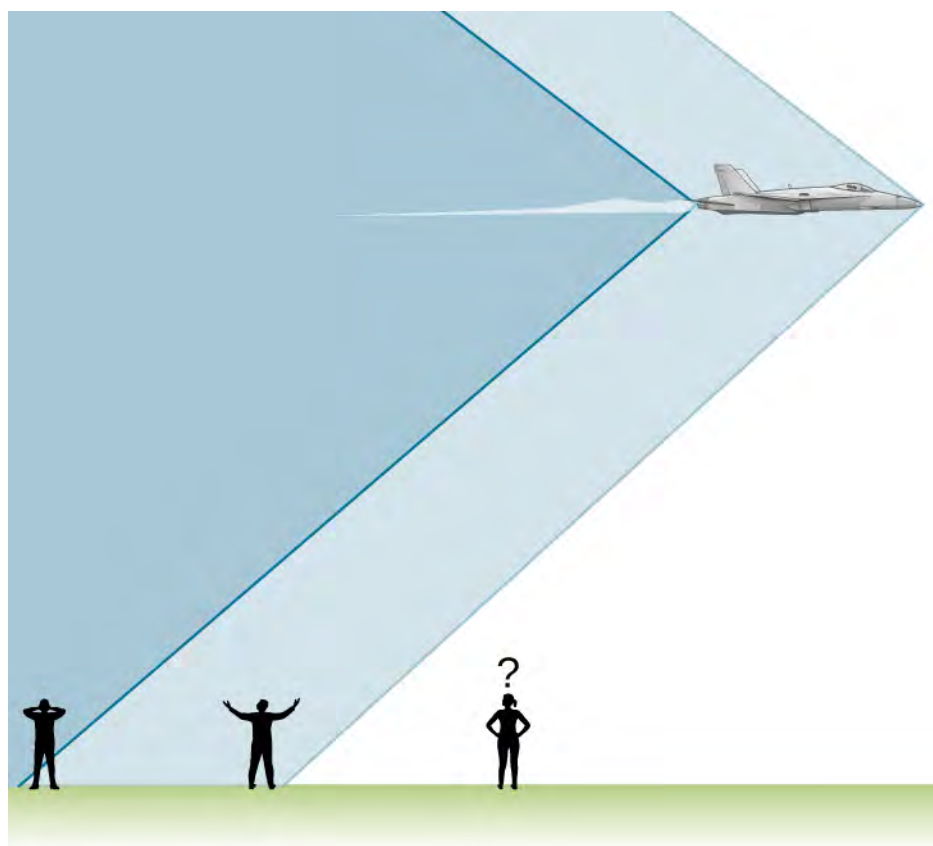


Figure 17.37 Two sonic booms experienced by observers, created by the nose and tail of an aircraft as the shock wave sweeps along the ground, are observed on the ground after the plane has passed by.

Shock waves are one example of a broader phenomenon called bow wakes. A **bow wake**, such as the one in [Figure 17.38](#), is created when the wave source moves faster than the wave propagation speed. Water waves spread out in circles from the point where created, and the bow wake is the familiar V-shaped wake, trailing the source. A more exotic bow wake is created when a subatomic particle travels through a medium faster than the speed of light travels in that medium. (In a vacuum, the maximum speed of light is $c = 3.00 \times 10^8$ m/s; in the medium of water, the speed of light is closer to $0.75c$.) If the particle creates light in its passage, that light spreads on a cone with an angle indicative of the speed of the particle, as illustrated in [Figure 17.39](#). Such a bow wake is called Cerenkov radiation and is commonly observed in particle physics.

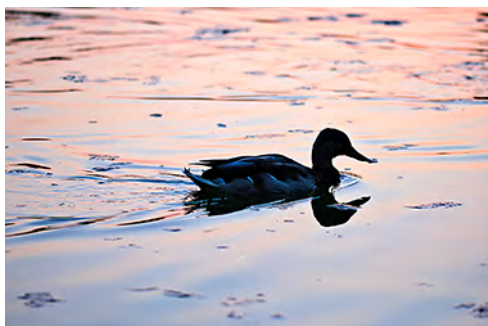


Figure 17.38 Bow wake created by a duck. Constructive interference produces the rather structured wake, whereas relatively little wave action occurs inside the wake, where interference is mostly destructive. (credit: Horia Varlan)



Figure 17.39 The blue glow in this research reactor pool is Cerenkov radiation caused by subatomic particles traveling faster than the speed of light in water. (credit: Idaho National Laboratory)

CHAPTER REVIEW

Key Terms

beat frequency frequency of beats produced by sound waves that differ in frequency

beats constructive and destructive interference of two or more frequencies of sound

bow wake v-shaped disturbance created when the wave source moves faster than the wave propagation speed

Doppler effect alteration in the observed frequency of a sound due to motion of either the source or the observer

Doppler shift actual change in frequency due to relative motion of source and observer

fundamental the lowest-frequency resonance

harmonics the term used to refer collectively to the fundamental and its overtones

hearing perception of sound

loudness perception of sound intensity

notes basic unit of music with specific names, combined to generate tunes

overtones all resonant frequencies higher than the fundamental

phon numerical unit of loudness

pitch perception of the frequency of a sound

shock wave wave front that is produced when a sound source moves faster than the speed of sound

sonic boom loud noise that occurs as a shock wave as it sweeps along the ground

sound traveling pressure wave that may be periodic; the wave can be modeled as a pressure wave or as an oscillation of molecules

sound intensity level unitless quantity telling you the level of the sound relative to a fixed standard

sound pressure level ratio of the pressure amplitude to a reference pressure

timbre number and relative intensity of multiple sound frequencies

transducer device that converts energy of a signal into measurable energy form, for example, a microphone converts sound waves into an electrical signal

Key Equations

Pressure of a sound wave

$$\Delta P = \Delta P_{\max} \sin(kx \mp \omega t + \phi)$$

Displacement of the oscillating molecules of a sound wave

$$s(x, t) = s_{\max} \cos(kx \mp \omega t + \phi)$$

Velocity of a wave

$$v = f\lambda$$

Speed of sound in a fluid

$$v = \sqrt{\frac{\beta}{\rho}}$$

Speed of sound in a solid

$$v = \sqrt{\frac{Y}{\rho}}$$

Speed of sound in an ideal gas

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Speed of sound in air as a function of temperature

$$v = 331 \frac{\text{m}}{\text{s}} \sqrt{\frac{T_K}{273 \text{ K}}} = 331 \frac{\text{m}}{\text{s}} \sqrt{1 + \frac{T_C}{273^\circ\text{C}}}$$

Decrease in intensity as a spherical wave expands

$$I_2 = I_1 \left(\frac{r_1}{r_2} \right)^2$$

Intensity averaged over a period

$$I = \frac{\langle P \rangle}{A}$$

Intensity of sound	$I = \frac{(\Delta p_{\max})^2}{2\rho v}$
Sound intensity level	$\beta(\text{dB}) = 10 \log_{10} \left(\frac{I}{I_0} \right)$
Resonant wavelengths of a tube closed at one end	$\lambda_n = \frac{4}{n} L, \quad n = 1, 3, 5, \dots$
Resonant frequencies of a tube closed at one end	$f_n = n \frac{v}{4L}, \quad n = 1, 3, 5, \dots$
Resonant wavelengths of a tube open at both ends	$\lambda_n = \frac{2}{n} L, \quad n = 1, 2, 3, \dots$
Resonant frequencies of a tube open at both ends	$f_n = n \frac{v}{2L}, \quad n = 1, 2, 3, \dots$
Beat frequency produced by two waves that differ in frequency	$f_{\text{beat}} = f_2 - f_1 $
Observed frequency for a stationary observer and a moving source	$f_o = f_s \left(\frac{v}{v \mp v_s} \right)$
Observed frequency for a moving observer and a stationary source	$f_o = f_s \left(\frac{v \pm v_o}{v} \right)$
Doppler shift for the observed frequency	$f_o = f_s \left(\frac{v \pm v_o}{v \mp v_s} \right)$
Mach number	$M = \frac{v_s}{v}$
Sine of angle formed by shock wave	$\sin \theta = \frac{v}{v_s} = \frac{1}{M}$

Summary

17.1 Sound Waves

- Sound is a disturbance of matter (a pressure wave) that is transmitted from its source outward. Hearing is the perception of sound.
- Sound can be modeled in terms of pressure or in terms of displacement of molecules.
- The human ear is sensitive to frequencies between 20 Hz and 20 kHz.

17.2 Speed of Sound

- The speed of sound depends on the medium and the state of the medium.
- In a fluid, because the absence of shear forces, sound waves are longitudinal. A solid can support both longitudinal and transverse sound waves.
- In air, the speed of sound is related to air temperature T by

$$v = 331 \frac{\text{m}}{\text{s}} \sqrt{\frac{T_K}{273 \text{ K}}} = 331 \frac{\text{m}}{\text{s}} \sqrt{1 + \frac{T_C}{273^\circ\text{C}}}.$$

- v is the same for all frequencies and wavelengths of sound in air.

17.3 Sound Intensity

- Intensity $I = P/A$ is the same for a sound wave as was defined for all waves, where P is the power crossing area A . The SI unit for I is watts per meter squared. The intensity of a sound wave is also related to the pressure amplitude Δp :

$$I = \frac{(\Delta p)^2}{2\rho v},$$

where ρ is the density of the medium in which the sound wave travels and v_w is the speed of sound in the medium.

- Sound intensity level in units of decibels (dB) is

$$\beta(\text{dB}) = 10 \log_{10} \left(\frac{I}{I_0} \right),$$

where $I_0 = 10^{-12} \text{ W/m}^2$ is the threshold intensity of hearing.

- The perception of frequency is pitch. The perception of intensity is loudness and loudness has units of phons.

17.4 Normal Modes of a Standing Sound Wave

- Unwanted sound can be reduced using destructive interference.
- Sound has the same properties of interference and resonance as defined for all waves.
- In air columns, the lowest-frequency resonance is called the fundamental, whereas all higher resonant frequencies are called overtones. Collectively, they are called harmonics.

17.5 Sources of Musical Sound

- Some musical instruments can be modeled as pipes that have symmetrical boundary conditions: open at both ends or closed at both ends. Other musical instruments can be modeled as pipes that have anti-symmetrical boundary conditions: closed at one end and open at the other.
- Some instruments, such as the pipe organ, have several tubes with different lengths. Instruments such as the flute vary the length of the tube by closing the holes along the tube. The trombone varies the length of the tube using a sliding bar.
- String instruments produce sound using a vibrating string with nodes at each end. The air

around the string oscillates at the frequency of the string. The relationship for the frequencies for the string is the same as for the symmetrical boundary conditions of the pipe, with the length of the pipe replaced by the length of the string and the velocity replaced by $v = \sqrt{\frac{F_T}{\mu}}$.

17.6 Beats

- When two sound waves that differ in frequency interfere, beats are created with a beat frequency that is equal to the absolute value of the difference in the frequencies.

17.7 The Doppler Effect

- The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer.
- The actual change in frequency is called the Doppler shift.

17.8 Shock Waves

- The Mach number is the velocity of a source divided by the speed of sound, $M = \frac{v_s}{v}$.
- When a sound source moves faster than the speed of sound, a shock wave is produced as the sound waves interfere.
- A sonic boom is the intense sound that occurs as the shock wave moves along the ground.
- The angle the shock wave produces can be found as $\sin \theta = \frac{v}{v_s} = \frac{1}{M}$.
- A bow wake is produced when an object moves faster than the speed of a mechanical wave in the medium, such as a boat moving through the water.

Conceptual Questions

17.1 Sound Waves

1. What is the difference between sound and hearing?
2. You will learn that light is an electromagnetic wave that can travel through a vacuum. Can sound waves travel through a vacuum?
3. Sound waves can be modeled as a change in pressure. Why is the change in pressure used and not the actual pressure?

17.2 Speed of Sound

4. How do sound vibrations of atoms differ from thermal motion?
5. When sound passes from one medium to another

where its propagation speed is different, does its frequency or wavelength change? Explain your answer briefly.

6. A popular party trick is to inhale helium and speak in a high-frequency, funny voice. Explain this phenomenon.
7. You may have used a sonic range finder in lab to measure the distance of an object using a clicking sound from a sound transducer. What is the principle used in this device?
8. The sonic range finder discussed in the preceding question often needs to be calibrated. During the calibration, the software asks for the room temperature. Why do you suppose the room temperature is required?

17.3 Sound Intensity

9. Six members of a synchronized swim team wear earplugs to protect themselves against water pressure at depths, but they can still hear the music and perform the combinations in the water perfectly. One day, they were asked to leave the pool so the dive team could practice a few dives, and they tried to practice on a mat, but seemed to have a lot more difficulty. Why might this be?
10. A community is concerned about a plan to bring train service to their downtown from the town's outskirts. The current sound intensity level, even though the rail yard is blocks away, is 70 dB downtown. The mayor assures the public that there will be a difference of only 30 dB in sound in the downtown area. Should the townspeople be concerned? Why?

17.4 Normal Modes of a Standing Sound Wave

11. You are given two wind instruments of identical length. One is open at both ends, whereas the other is closed at one end. Which is able to produce the lowest frequency?
12. What is the difference between an overtone and a harmonic? Are all harmonics overtones? Are all overtones harmonics?
13. Two identical columns, open at both ends, are in separate rooms. In room *A*, the temperature is $T = 20^\circ\text{C}$ and in room *B*, the temperature is $T = 25^\circ\text{C}$. A speaker is attached to the end of each tube, causing the tubes to resonate at the fundamental frequency. Is the frequency the same for both tubes? Which has the higher frequency?

17.5 Sources of Musical Sound

14. How does an unamplified guitar produce sounds so much more intense than those of a plucked string held taut by a simple stick?
15. Consider three pipes of the same length (L). Pipe *A* is open at both ends, pipe *B* is closed at both ends, and pipe *C* has one open end and one closed end. If the velocity of sound is the same in each of the three tubes, in which of the tubes could the lowest fundamental frequency be produced? In which of the tubes could the highest fundamental frequency be produced?
16. Pipe *A* has a length L and is open at both ends. Pipe *B* has a length $L/2$ and has one open end

and one closed end. Assume the speed of sound to be the same in both tubes. Which of the harmonics in each tube would be equal?

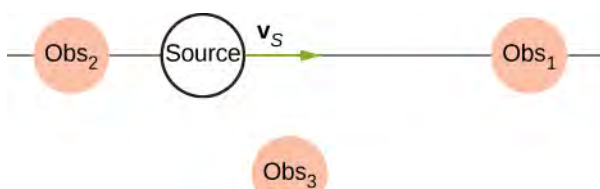
17. A string is tied between two lab posts a distance L apart. The tension in the string and the linear mass density is such that the speed of a wave on the string is $v = 343\text{ m/s}$. A tube with symmetric boundary conditions has a length L and the speed of sound in the tube is $v = 343\text{ m/s}$. What could be said about the frequencies of the harmonics in the string and the tube? What if the velocity in the string were $v = 686\text{ m/s}$?

17.6 Beats

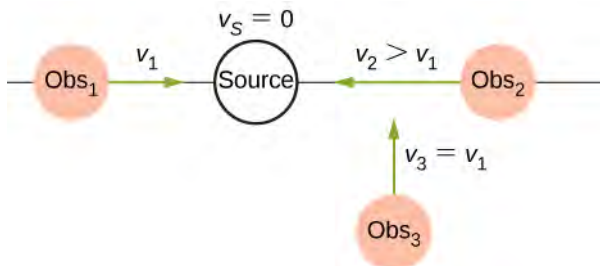
18. Two speakers are attached to variable-frequency signal generator. Speaker *A* produces a constant-frequency sound wave of 1.00 kHz, and speaker *B* produces a tone of 1.10 kHz. The beat frequency is 0.10 kHz. If the frequency of each speaker is doubled, what is the beat frequency produced?
19. The label has been scratched off a tuning fork and you need to know its frequency. From its size, you suspect that it is somewhere around 250 Hz. You find a 250-Hz tuning fork and a 270-Hz tuning fork. When you strike the 250-Hz fork and the fork of unknown frequency, a beat frequency of 5 Hz is produced. When you strike the unknown with the 270-Hz fork, the beat frequency is 15 Hz. What is the unknown frequency? Could you have deduced the frequency using just the 250-Hz fork?
20. Referring to the preceding question, if you had only the 250-Hz fork, could you come up with a solution to the problem of finding the unknown frequency?
21. A "showy" custom-built car has two brass horns that are supposed to produce the same frequency but actually emit 263.8 and 264.5 Hz. What beat frequency is produced?

17.7 The Doppler Effect

22. Is the Doppler shift real or just a sensory illusion?
23. Three stationary observers observe the Doppler shift from a source moving at a constant velocity. The observers are stationed as shown below. Which observer will observe the highest frequency? Which observer will observe the lowest frequency? What can be said about the frequency observed by observer 3?



24. Shown below is a stationary source and moving observers. Describe the frequencies observed by the observers for this configuration.



25. Prior to 1980, conventional radar was used by

Problems

17.1 Sound Waves

29. Consider a sound wave modeled with the equations $s(x, t) = 4.00 \text{ nm} \cos(3.66 \text{ m}^{-1}x - 1256 \text{ s}^{-1}t)$. What is the maximum displacement, the wavelength, the frequency, and the speed of the sound wave?
30. Consider a sound wave moving through the air modeled with the equations $s(x, t) = 6.00 \text{ nm} \cos(54.93 \text{ m}^{-1}x - 18.84 \times 10^3 \text{ s}^{-1}t)$. What is the shortest time required for an air molecule to move between 3.00 nm and -3.00 nm ?
31. Consider a diagnostic ultrasound of frequency 5.00 MHz that is used to examine an irregularity in soft tissue. (a) What is the wavelength in air of such a sound wave if the speed of sound is 343 m/s ? (b) If the speed of sound in tissue is 1800 m/s , what is the wavelength of this wave in tissue?
32. A sound wave is modeled as $\Delta P = 1.80 \text{ Pa} \sin(55.41 \text{ m}^{-1}x - 18,840 \text{ s}^{-1}t)$. What is the maximum change in pressure, the wavelength, the frequency, and the speed of the sound wave?
33. A sound wave is modeled with the wave function $\Delta P = 1.20 \text{ Pa} \sin(kx - 6.28 \times 10^4 \text{ s}^{-1}t)$ and the sound wave travels in air at a speed of $v = 343.00 \text{ m/s}$. (a) What is the wave number of the sound wave? (b) What is the value for ΔP (3.00 m , 20.00 s)?
34. The displacement of the air molecules in sound wave is

modeled with the wave function $s(x, t) = 5.00 \text{ nm} \cos(91.54 \text{ m}^{-1}x - 3.14 \times 10^4 \text{ s}^{-1}t)$. (a) What is the wave speed of the sound wave? (b) What is the maximum speed of the air molecules as they oscillate in simple harmonic motion? (c) What is the magnitude of the maximum acceleration of the air molecules as they oscillate in simple harmonic motion?

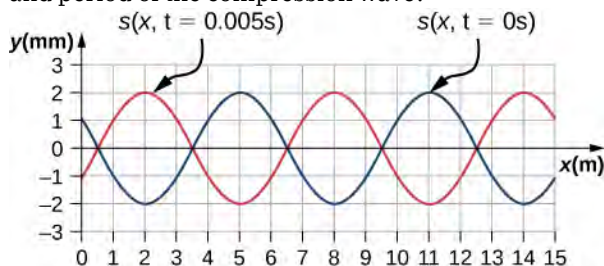
17.8 Shock Waves

26. What is the difference between a sonic boom and a shock wave?
27. Due to efficiency considerations related to its bow wake, the supersonic transport aircraft must maintain a cruising speed that is a constant ratio to the speed of sound (a constant Mach number). If the aircraft flies from warm air into colder air, should it increase or decrease its speed? Explain your answer.
28. When you hear a sonic boom, you often cannot see the plane that made it. Why is that?

35. A speaker is placed at the opening of a long horizontal tube. The speaker oscillates at a frequency f , creating a sound wave that moves down the tube. The wave moves through the tube at a speed of $v = 340.00 \text{ m/s}$. The sound wave is modeled with the wave function $s(x, t) = s_{\text{max}} \cos(kx - \omega t + \phi)$. At time $t = 0.00 \text{ s}$, an air molecule at $x = 3.5 \text{ m}$ is at the maximum displacement of 7.00 nm . At the same time, another molecule at $x = 3.7 \text{ m}$ has a displacement of 3.00 nm . What is the frequency at which the speaker is oscillating?
36. A 250-Hz tuning fork is struck and begins to vibrate. A sound-level meter is located 34.00 m away. It takes the sound $\Delta t = 0.10 \text{ s}$ to reach the meter. The maximum displacement of the tuning fork is 1.00 mm . Write a wave function for the sound.
37. A sound wave produced by an ultrasonic transducer, moving in air, is modeled with the wave equation $s(x, t) = 4.50 \text{ nm} \cos(9.15 \times 10^4 \text{ m}^{-1}x - 2\pi(5.00 \text{ MHz})t)$. The transducer is to be used in nondestructive testing to test for fractures in steel beams. The speed of sound in the steel

beam is $v = 5950$ m/s. Find the wave function for the sound wave in the steel beam.

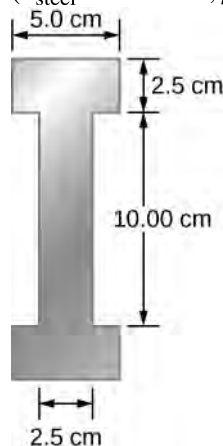
38. Porpoises emit sound waves that they use for navigation. If the wavelength of the sound wave emitted is 4.5 cm, and the speed of sound in the water is $v = 1530$ m/s, what is the period of the sound?
39. Bats use sound waves to catch insects. Bats can detect frequencies up to 100 kHz. If the sound waves travel through air at a speed of $v = 343$ m/s, what is the wavelength of the sound waves?
40. A bat sends of a sound wave 100 kHz and the sound waves travel through air at a speed of $v = 343$ m/s. (a) If the maximum pressure difference is 1.30 Pa, what is a wave function that would model the sound wave, assuming the wave is sinusoidal? (Assume the phase shift is zero.) (b) What are the period and wavelength of the sound wave?
41. Consider the graph shown below of a compression wave. Shown are snapshots of the wave function for $t = 0.000$ s (blue) and $t = 0.005$ s (orange). What are the wavelength, maximum displacement, velocity, and period of the compression wave?



42. Consider the graph in the preceding problem of a compression wave. Shown are snapshots of the wave function for $t = 0.000$ s (blue) and $t = 0.005$ s (orange). Given that the displacement of the molecule at time $t = 0.00$ s and position $x = 0.00$ m is $s(0.00 \text{ m}, 0.00 \text{ s}) = 1.08$ mm, derive a wave function to model the compression wave.
43. A guitar string oscillates at a frequency of 100 Hz and produces a sound wave. (a) What do you think the frequency of the sound wave is that the vibrating string produces? (b) If the speed of the sound wave is $v = 343$ m/s, what is the wavelength of the sound wave?

17.2 Speed of Sound

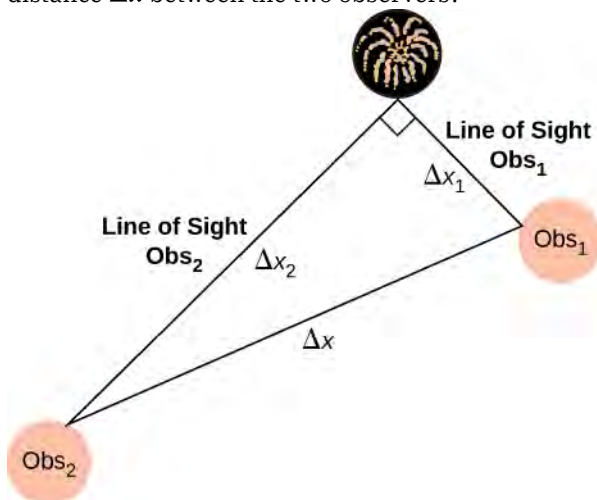
44. When poked by a spear, an operatic soprano lets out a 1200-Hz shriek. What is its wavelength if the speed of sound is 345 m/s?
45. What frequency sound has a 0.10-m wavelength when the speed of sound is 340 m/s?
46. Calculate the speed of sound on a day when a 1500-Hz frequency has a wavelength of 0.221 m.
47. (a) What is the speed of sound in a medium where a 100-kHz frequency produces a 5.96-cm wavelength? (b) Which substance in [Table 17.1](#) is this likely to be?
48. Show that the speed of sound in 20.0°C air is 343 m/s, as claimed in the text.
49. Air temperature in the Sahara Desert can reach 56.0°C (about 134°F). What is the speed of sound in air at that temperature?
50. Dolphins make sounds in air and water. What is the ratio of the wavelength of a sound in air to its wavelength in seawater? Assume air temperature is 20.0°C .
51. A sonar echo returns to a submarine 1.20 s after being emitted. What is the distance to the object creating the echo? (Assume that the submarine is in the ocean, not in fresh water.)
52. (a) If a submarine's sonar can measure echo times with a precision of 0.0100 s, what is the smallest difference in distances it can detect? (Assume that the submarine is in the ocean, not in fresh water.) (b) Discuss the limits this time resolution imposes on the ability of the sonar system to detect the size and shape of the object creating the echo.
53. Ultrasonic sound waves are often used in methods of nondestructive testing. For example, this method can be used to find structural faults in a steel I-beams used in building. Consider a 10.00 meter long, steel I-beam with a cross-section shown below. The weight of the I-beam is 3846.50 N. What would be the speed of sound through in the I-beam? ($Y_{\text{steel}} = 200$ GPa, $\beta_{\text{steel}} = 159$ GPa).



54. A physicist at a fireworks display times the lag

between seeing an explosion and hearing its sound, and finds it to be 0.400 s. (a) How far away is the explosion if air temperature is 24.0°C and if you neglect the time taken for light to reach the physicist? (b) Calculate the distance to the explosion taking the speed of light into account. Note that this distance is negligibly greater.

55. During a 4th of July celebration, an M80 firework explodes on the ground, producing a bright flash and a loud bang. The air temperature of the night air is $T_F = 95.00^{\circ}\text{F}$. Two observers see the flash and hear the bang. The first observer notes the time between the flash and the bang as 0.10 second. The second observer notes the difference as 0.15 seconds. The line of sight between the two observers meet at a right angle as shown below. What is the distance Δx between the two observers?



56. The density of a sample of water is $\rho = 998.00 \text{ kg/m}^3$ and the bulk modulus is $\beta = 2.15 \text{ GPa}$. What is the speed of sound through the sample?
57. Suppose a bat uses sound echoes to locate its insect prey, 3.00 m away. (See [Figure 17.6](#).) (a) Calculate the echo times for temperatures of 5.00°C and 35.0°C . (b) What percent uncertainty does this cause for the bat in locating the insect? (c) Discuss the significance of this uncertainty and whether it could cause difficulties for the bat. (In practice, the bat continues to use sound as it closes in, eliminating most of any difficulties imposed by this and other effects, such as motion of the prey.)

17.3 Sound Intensity

58. What is the intensity in watts per meter squared of a 85.0-dB sound?
59. The warning tag on a lawn mower states that it produces noise at a level of 91.0 dB. What is this in watts per meter squared?
60. A sound wave traveling in air has a pressure amplitude of 0.5 Pa. What is the intensity of the wave?
61. What intensity level does the sound in the preceding problem correspond to?
62. What sound intensity level in dB is produced by earphones that create an intensity of $4.00 \times 10^{-2} \text{ W/m}^2$?
63. What is the decibel level of a sound that is twice as intense as a 90.0-dB sound? (b) What is the decibel level of a sound that is one-fifth as intense as a 90.0-dB sound?
64. What is the intensity of a sound that has a level 7.00 dB lower than a $4.00 \times 10^{-9} \text{ W/m}^2$ sound? (b) What is the intensity of a sound that is 3.00 dB higher than a $4.00 \times 10^{-9} \text{ W/m}^2$ sound?
65. People with good hearing can perceive sounds as low as -8.00 dB at a frequency of 3000 Hz. What is the intensity of this sound in watts per meter squared?
66. If a large housefly 3.0 m away from you makes a noise of 40.0 dB, what is the noise level of 1000 flies at that distance, assuming interference has a negligible effect?
67. Ten cars in a circle at a boom box competition produce a 120-dB sound intensity level at the center of the circle. What is the average sound intensity level produced there by each stereo, assuming interference effects can be neglected?
68. The amplitude of a sound wave is measured in terms of its maximum gauge pressure. By what factor does the amplitude of a sound wave increase if the sound intensity level goes up by 40.0 dB?
69. If a sound intensity level of 0 dB at 1000 Hz corresponds to a maximum gauge pressure (sound amplitude) of 10^{-9} atm , what is the maximum gauge pressure in a 60-dB sound? What is the maximum gauge pressure in a 120-dB sound?
70. An 8-hour exposure to a sound intensity level of 90.0 dB may cause hearing damage. What energy in joules falls on a 0.800-cm-diameter eardrum so exposed?
71. Sound is more effectively transmitted into a stethoscope by direct contact rather than through the air, and it is further intensified by being concentrated on the smaller area of the eardrum. It is reasonable to assume that sound

is transmitted into a stethoscope 100 times as effectively compared with transmission through the air. What, then, is the gain in decibels produced by a stethoscope that has a sound gathering area of 15.0 cm^2 , and concentrates the sound onto two eardrums with a total area of 0.900 cm^2 with an efficiency of 40.0%?

72. Loudspeakers can produce intense sounds with surprisingly small energy input in spite of their low efficiencies. Calculate the power input needed to produce a 90.0-dB sound intensity level for a 12.0-cm-diameter speaker that has an efficiency of 1.00%. (This value is the sound intensity level right at the speaker.)
73. The factor of 10^{-12} in the range of intensities to which the ear can respond, from threshold to that causing damage after brief exposure, is truly remarkable. If you could measure distances over the same range with a single instrument and the smallest distance you could measure was 1 mm, what would the largest be?
74. What are the closest frequencies to 500 Hz that an average person can clearly distinguish as being different in frequency from 500 Hz? The sounds are not present simultaneously.
75. Can you tell that your roommate turned up the sound on the TV if its average sound intensity level goes from 70 to 73 dB?
76. If a woman needs an amplification of 5.0×10^5 times the threshold intensity to enable her to hear at all frequencies, what is her overall hearing loss in dB? Note that smaller amplification is appropriate for more intense sounds to avoid further damage to her hearing from levels above 90 dB.
77. A person has a hearing threshold 10 dB above normal at 100 Hz and 50 dB above normal at 4000 Hz. How much more intense must a 100-Hz tone be than a 4000-Hz tone if they are both barely audible to this person?

17.4 Normal Modes of a Standing Sound Wave

78. (a) What is the fundamental frequency of a 0.672-m-long tube, open at both ends, on a day when the speed of sound is 344 m/s? (b) What is the frequency of its second harmonic?
79. What is the length of a tube that has a fundamental frequency of 176 Hz and a first overtone of 352 Hz if the speed of sound is 343 m/s?
80. The ear canal resonates like a tube closed at one

end. (See [Figure 17_03_HumEar](#).) If ear canals range in length from 1.80 to 2.60 cm in an average population, what is the range of fundamental resonant frequencies? Take air temperature to be 37.0°C , which is the same as body temperature.

81. Calculate the first overtone in an ear canal, which resonates like a 2.40-cm-long tube closed at one end, by taking air temperature to be 37.0°C . Is the ear particularly sensitive to such a frequency? (The resonances of the ear canal are complicated by its nonuniform shape, which we shall ignore.)
82. A crude approximation of voice production is to consider the breathing passages and mouth to be a resonating tube closed at one end. (a) What is the fundamental frequency if the tube is 0.240 m long, by taking air temperature to be 37.0°C ? (b) What would this frequency become if the person replaced the air with helium? Assume the same temperature dependence for helium as for air.
83. A 4.0-m-long pipe, open at one end and closed at the other, is in a room where the temperature is $T = 22^\circ\text{C}$. A speaker capable of producing variable frequencies is placed at the open end and is used to cause the tube to resonate. (a) What is the wavelength and the frequency of the fundamental frequency? (b) What is the frequency and wavelength of the first overtone?
84. A 4.0-m-long pipe, open at both ends, is placed in a room where the temperature is $T = 25^\circ\text{C}$. A speaker capable of producing variable frequencies is placed at the open end and is used to cause the tube to resonate. (a) What are the wavelength and the frequency of the fundamental frequency? (b) What are the frequency and wavelength of the first overtone?
85. A nylon guitar string is fixed between two lab posts 2.00 m apart. The string has a linear mass density of $\mu = 7.20 \text{ g/m}$ and is placed under a tension of 160.00 N. The string is placed next to a tube, open at both ends, of length L . The string is plucked and resonates at the fundamental frequency while the tube resonates at the $n = 3$ mode. The speed of sound is 343 m/s. What is the length of the tube?
86. A 512-Hz tuning fork is struck and placed next to a tube with a movable piston, creating a tube with a variable length. The piston is slid down the pipe and resonance is reached when the piston is 115.50 cm from the open end. The next resonance is reached when the piston is

82.50 cm from the open end. (a) What is the speed of sound in the tube? (b) How far from the open end will the piston cause the next mode of resonance?

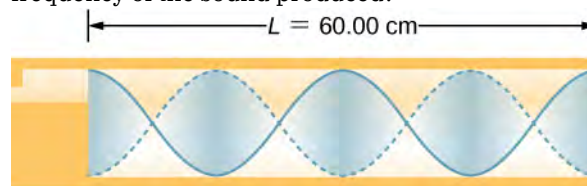
87. Students in a physics lab are asked to find the length of an air column in a tube closed at one end that has a fundamental frequency of 256 Hz. They hold the tube vertically and fill it with water to the top, then lower the water while a 256-Hz tuning fork is rung and listen for the first resonance. (a) What is the air temperature if the resonance occurs for a length of 0.336 m? (b) At what length will they observe the second resonance (first overtone)?

17.5 Sources of Musical Sound

88. If a wind instrument, such as a tuba, has a fundamental frequency of 32.0 Hz, what are its first three overtones? It is closed at one end. (The overtones of a real tuba are more complex than this example, because it is a tapered tube.)
89. What are the first three overtones of a bassoon that has a fundamental frequency of 90.0 Hz? It is open at both ends. (The overtones of a real bassoon are more complex than this example, because its double reed makes it act more like a tube closed at one end.)
90. How long must a flute be in order to have a fundamental frequency of 262 Hz (this frequency corresponds to middle C on the evenly tempered chromatic scale) on a day when air temperature is 20.0°C? It is open at both ends.
91. What length should an oboe have to produce a fundamental frequency of 110 Hz on a day when the speed of sound is 343 m/s? It is open at both ends.
92. (a) Find the length of an organ pipe closed at one end that produces a fundamental frequency of 256 Hz when air temperature is 18.0°C. (b) What is its fundamental frequency at 25.0°C?
93. An organ pipe ($L = 3.00$ m) is closed at both ends. Compute the wavelengths and frequencies of the first three modes of resonance. Assume the speed of sound is $v = 343.00$ m/s.
94. An organ pipe ($L = 3.00$ m) is closed at one end. Compute the wavelengths and frequencies of the first three modes of resonance. Assume the speed of sound is $v = 343.00$ m/s.
95. A sound wave of a frequency of 2.00 kHz is produced by a string oscillating in the $n = 6$ mode. The linear mass density of the string is

$\mu = 0.0065$ kg/m and the length of the string is 1.50 m. What is the tension in the string?

96. Consider the sound created by resonating the tube shown below. The air temperature is $T_C = 30.00^\circ\text{C}$. What are the wavelength, wave speed, and frequency of the sound produced?



97. A student holds an 80.00-cm lab pole one quarter of the length from the end of the pole. The lab pole is made of aluminum. The student strikes the lab pole with a hammer. The pole resonates at the lowest possible frequency. What is that frequency?
98. A string on the violin has a length of 24.00 cm and a mass of 0.860 g. The fundamental frequency of the string is 1.00 kHz. (a) What is the speed of the wave on the string? (b) What is the tension in the string?
99. By what fraction will the frequencies produced by a wind instrument change when air temperature goes from 10.0°C to 30.0°C? That is, find the ratio of the frequencies at those temperatures.

17.6 Beats

100. What beat frequencies are present: (a) If the musical notes A and C are played together (frequencies of 220 and 264 Hz)? (b) If D and F are played together (frequencies of 297 and 352 Hz)? (c) If all four are played together?
101. What beat frequencies result if a piano hammer hits three strings that emit frequencies of 127.8, 128.1, and 128.3 Hz?
102. A piano tuner hears a beat every 2.00 s when listening to a 264.0-Hz tuning fork and a single piano string. What are the two possible frequencies of the string?
103. Two identical strings, of identical lengths of 2.00 m and linear mass density of $\mu = 0.0065$ kg/m, are fixed on both ends. String A is under a tension of 120.00 N. String B is under a tension of 130.00 N. They are each plucked and produce sound at the $n = 10$ mode. What is the beat frequency?
104. A piano tuner uses a 512-Hz tuning fork to tune a piano. He strikes the fork and hits a key on the piano and hears a beat frequency of 5 Hz. He tightens the string of the piano, and

repeats the procedure. Once again he hears a beat frequency of 5 Hz. What happened?

105. A string with a linear mass density of $\mu = 0.0062 \text{ kg/m}$ is stretched between two posts 1.30 m apart. The tension in the string is 150.00 N. The string oscillates and produces a sound wave. A 1024-Hz tuning fork is struck and the beat frequency between the two sources is 52.83 Hz. What are the possible frequency and wavelength of the wave on the string?
106. A car has two horns, one emitting a frequency of 199 Hz and the other emitting a frequency of 203 Hz. What beat frequency do they produce?
107. The middle C hammer of a piano hits two strings, producing beats of 1.50 Hz. One of the strings is tuned to 260.00 Hz. What frequencies could the other string have?
108. Two tuning forks having frequencies of 460 and 464 Hz are struck simultaneously. What average frequency will you hear, and what will the beat frequency be?
109. Twin jet engines on an airplane are producing an average sound frequency of 4100 Hz with a beat frequency of 0.500 Hz. What are their individual frequencies?
110. Three adjacent keys on a piano (F, F-sharp, and G) are struck simultaneously, producing frequencies of 349, 370, and 392 Hz. What beat frequencies are produced by this discordant combination?

17.7 The Doppler Effect

111. (a) What frequency is received by a person watching an oncoming ambulance moving at 110 km/h and emitting a steady 800-Hz sound from its siren? The speed of sound on this day is 345 m/s. (b) What frequency does she receive after the ambulance has passed?
112. (a) At an air show a jet flies directly toward the stands at a speed of 1200 km/h, emitting a frequency of 3500 Hz, on a day when the speed of sound is 342 m/s. What frequency is received by the observers? (b) What frequency do they receive as the plane flies directly away from them?
113. What frequency is received by a mouse just before being dispatched by a hawk flying at it at 25.0 m/s and emitting a screech of frequency 3500 Hz? Take the speed of sound to be 331 m/s.
114. A spectator at a parade receives an 888-Hz tone from an oncoming trumpeter who is playing an 880-Hz note. At what speed is the musician approaching if the speed of sound is 338 m/s?
115. A commuter train blows its 200-Hz horn as it approaches a crossing. The speed of sound is 335 m/s. (a) An observer waiting at the crossing receives a frequency of 208 Hz. What is the speed of the train? (b) What frequency does the observer receive as the train moves away?
116. Can you perceive the shift in frequency produced when you pull a tuning fork toward you at 10.0 m/s on a day when the speed of sound is 344 m/s? To answer this question, calculate the factor by which the frequency shifts and see if it is greater than 0.300%.
117. Two eagles fly directly toward one another, the first at 15.0 m/s and the second at 20.0 m/s. Both screech, the first one emitting a frequency of 3200 Hz and the second one emitting a frequency of 3800 Hz. What frequencies do they receive if the speed of sound is 330 m/s?
118. Student A runs down the hallway of the school at a speed of $v_o = 5.00 \text{ m/s}$, carrying a ringing 1024.00-Hz tuning fork toward a concrete wall. The speed of sound is $v = 343.00 \text{ m/s}$. Student B stands at rest at the wall. (a) What is the frequency heard by student B? (b) What is the beat frequency heard by student A?
119. An ambulance with a siren ($f = 1.00 \text{ kHz}$) blaring is approaching an accident scene. The ambulance is moving at 70.00 mph. A nurse is approaching the scene from the opposite direction, running at $v_o = 7.00 \text{ m/s}$. What frequency does the nurse observe? Assume the speed of sound is $v = 343.00 \text{ m/s}$.
120. The frequency of the siren of an ambulance is 900 Hz and is approaching you. You are standing on a corner and observe a frequency of 960 Hz. What is the speed of the ambulance (in mph) if the speed of sound is $v = 340.00 \text{ m/s}$?
121. What is the minimum speed at which a source must travel toward you for you to be able to hear that its frequency is Doppler shifted? That is, what speed produces a shift of 0.300% on a day when the speed of sound is 331 m/s?

17.8 Shock Waves

122. An airplane is flying at Mach 1.50 at an altitude of 7500.00 meters, where the speed of sound is

$v = 343.00$ m/s. How far away from a stationary observer will the plane be when the observer hears the sonic boom?

- 123.** A jet flying at an altitude of 8.50 km has a speed of Mach 2.00, where the speed of sound is $v = 340.00$ m/s. How long after the jet is directly overhead, will a stationary observer hear a sonic boom?
- 124.** The shock wave off the front of a fighter jet has an angle of $\theta = 70.00^\circ$. The jet is flying at 1200 km/h. What is the speed of sound?
- 125.** A plane is flying at Mach 1.2, and an observer on the ground hears the sonic boom 15.00 seconds after the plane is directly overhead. What is the altitude of the plane? Assume the speed of sound is $v_w = 343.00$ m/s.
- 126.** A bullet is fired and moves at a speed of 1342 mph. Assume the speed of sound is $v = 340.00$ m/s. What is the angle of the shock

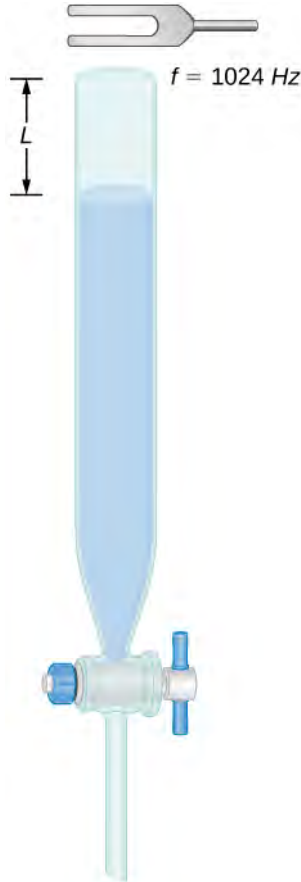
wave produced?

- 127.** A speaker is placed at the opening of a long horizontal tube. The speaker oscillates at a frequency of f , creating a sound wave that moves down the tube. The wave moves through the tube at a speed of $v = 340.00$ m/s. The sound wave is modeled with the wave function $s(x, t) = s_{\max} \cos(kx - \omega t + \phi)$. At time $t = 0.00$ s, an air molecule at $x = 2.3$ m is at the maximum displacement of 6.34 nm. At the same time, another molecule at $x = 2.7$ m has a displacement of 2.30 nm. What is the wave function of the sound wave, that is, find the wave number, angular frequency, and the initial phase shift?
- 128.** An airplane moves at Mach 1.2 and produces a shock wave. (a) What is the speed of the plane in meters per second? (b) What is the angle that the shock wave moves?

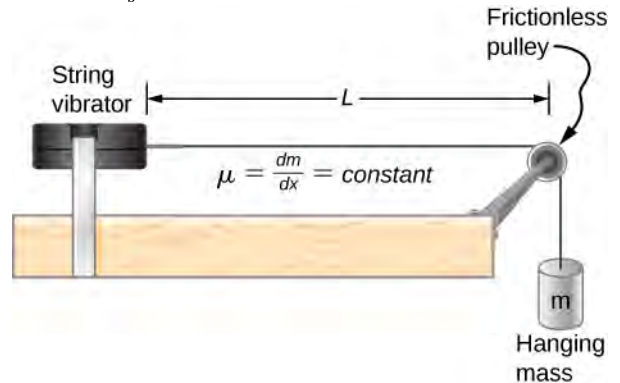
Additional Problems

- 129.** A 0.80-m-long tube is opened at both ends. The air temperature is 26°C . The air in the tube is oscillated using a speaker attached to a signal generator. What are the wavelengths and frequencies of first two modes of sound waves that resonate in the tube?

- 130.** A tube filled with water has a valve at the bottom to allow the water to flow out of the tube. As the water is emptied from the tube, the length L of the air column changes. A 1024-Hz tuning fork is placed at the opening of the tube. Water is removed from the tube until the $n = 5$ mode of a sound wave resonates. What is the length of the air column if the temperature of the air in the room is 18°C ?



- 131.** Consider the following figure. The length of the string between the string vibrator and the pulley is $L = 1.00$ m. The linear density of the string is $\mu = 0.006$ kg/m. The string vibrator can oscillate at any frequency. The hanging mass is 2.00 kg. (a) What are the wavelength and frequency of $n = 6$ mode? (b) The string oscillates the air around the string. What is the wavelength of the sound if the speed of the sound is $v_s = 343.00$ m/s?



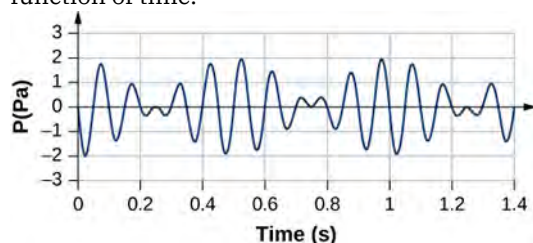
- 132.** Early Doppler shift experiments were conducted using a band playing music on a train. A trumpet player on a moving railroad flatcar plays a 320-Hz note. The sound waves heard by a stationary observer on a train platform hears a frequency of 350 Hz. What is the flatcar's speed in mph? The temperature of the air is $T_C = 22^\circ\text{C}$.
- 133.** Two cars move toward one another, both sounding their horns ($f_s = 800$ Hz). Car A is moving at 65 mph and Car B is at 75 mph. What is the beat frequency heard by each driver? The air temperature is $T_C = 22.00^\circ\text{C}$.
- 134.** Student A runs after Student B. Student A carries a tuning fork ringing at 1024 Hz, and student B carries a tuning fork ringing at 1000 Hz. Student A is running at a speed of $v_A = 5.00$ m/s and Student B is running at $v_B = 6.00$ m/s. What is the beat frequency heard by each student? The speed of sound is $v = 343.00$ m/s.
- 135.** Suppose that the sound level from a source is 75 dB and then drops to 52 dB, with a frequency of 600 Hz. Determine the (a) initial and (b) final sound intensities and the (c) initial and (d) final sound wave amplitudes. The air temperature is $T_C = 24.00^\circ\text{C}$ and the air density is $\rho = 1.184$ kg/m³.

- 136.** The Doppler shift for a Doppler radar is found by $f = f_R \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)$, where f_R is the frequency of the radar, f is the frequency observed by the radar, c is the speed of light, and v is the speed of the target. What is the beat frequency observed at the radar, assuming the speed of the target is much slower than the speed of light?

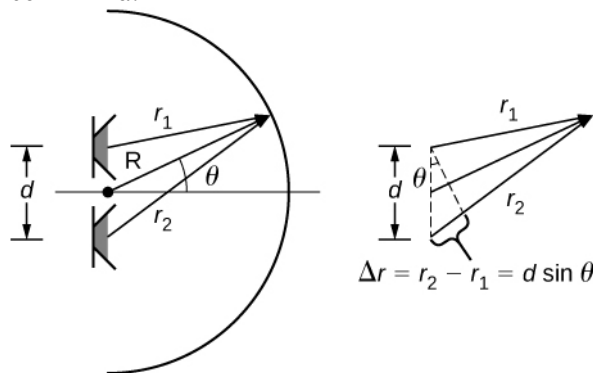
- 137.** A stationary observer hears a frequency of 1000.00 Hz as a source approaches and a frequency of 850.00 Hz as a source departs. The source moves at a constant velocity of 75 mph. What is the temperature of the air?
- 138.** A flute plays a note with a frequency of 600 Hz. The flute can be modeled as a pipe open at both ends, where the flute player changes the length with his finger positions. What is the length of the tube if this is the fundamental frequency?

Challenge Problems

- 139.** Two sound speakers are separated by a distance d , each sounding a frequency f . An observer stands at one speaker and walks in a straight line a distance x , perpendicular to the line between the two speakers, until he comes to the first maximum intensity of sound. The speed of sound is v . How far is he from the speaker?
- 140.** Consider the beats shown below. This is a graph of the gauge pressure versus time for the position $x = 0.00$ m. The wave moves with a speed of $v = 343.00$ m/s. (a) How many beats are there per second? (b) How many times does the wave oscillate per second? (c) Write a wave function for the gauge pressure as a function of time.



- 141.** Two speakers producing the same frequency of sound are a distance of d apart. Consider an arc along a circle of radius R , centered at the midpoint of the speakers, as shown below. (a) At what angles will there be maxima? (b) At what angle will there be minima?



- 142.** A string has a length of 1.5 m, a linear mass density $\mu = 0.008$ kg/m, and a tension of 120 N. If the air temperature is $T = 22^\circ\text{C}$, what should the length of a pipe open at both ends for it to have the same frequency for the $n = 3$ mode?
- 143.** A string ($\mu = 0.006 \frac{\text{kg}}{\text{m}}$, $L = 1.50$ m) is fixed at both ends and is under a tension of 155 N. It oscillates in the $n = 10$ mode and produces sound. A tuning fork is ringing nearby, producing a beat frequency of 23.76 Hz. (a) What is the frequency of the sound from the string? (b) What is the frequency of the tuning fork if the tuning fork frequency is lower? (c) What should be the tension of the string for the beat frequency to be zero?
- 144.** A string has a linear mass density μ , a length L , and a tension of F_T , and oscillates in a mode n at a frequency f . Find the ratio of $\frac{\Delta f}{f}$ for a small change in tension.
- 145.** A string has a linear mass density $\mu = 0.007$ kg/m, a length $L = 0.70$ m, a tension of $F_T = 110$ N, and oscillates in a mode $n = 3$. (a) What is the frequency of the oscillations? (b) Use the result in the preceding problem to find the change in the frequency when the tension is increased by 1.00%.
- 146.** A speaker powered by a signal generator is used to study resonance in a tube. The signal generator can be adjusted from a frequency of 1000 Hz to 1800 Hz. First, a 0.75-m-long tube, open at both ends, is studied. The temperature in the room is $T_F = 85.00^\circ\text{F}$. (a) Which normal modes of the pipe can be studied? What are the frequencies and wavelengths? Next a cap is placed on one end of the 0.75-meter-long pipe. (b) Which normal modes of the pipe can be studied? What are the frequencies and wavelengths?

- 147.** A string on the violin has a length of 23.00 cm and a mass of 0.900 grams. The tension in the string is 850.00 N. The temperature in the room is $T_C = 24.00^\circ\text{C}$. The string is plucked and oscillates in the $n = 9$ mode. (a) What is the speed of the wave on the string? (b) What is the wavelength of the sounding wave produced? (c) What is the frequency of the oscillating string? (d) What is the frequency of the sound produced? (e) What is the wavelength of the sound produced?

Flow is not laminar.

b. Hose: $v = 1.96 \frac{\text{m}}{\text{s}}$

$N_R = 35,100 > 2000 \Rightarrow$

Flow is not laminar.

103. $3.16 \times 10^{-4} \text{ m}^3/\text{s}$

Additional Problems

105. 30.6 m

107. a. $p_{120} = 1.60 \times 10^4 \text{ N/m}^2$;

$p_{80} = 1.07 \times 10^4 \text{ N/m}^2$

b. Since an infant is only approximately 20 inches tall, while an adult is approximately 70 inches tall, the blood pressure for an infant would be expected to be smaller than that of an adult. The blood only feels a pressure of 20 inches rather than 70 inches, so the pressure should be smaller.

109. a. 41.4 g; b. 41.4 cm^3 ; c. 1.09 g/cm^3 . This is clearly not the density of the bone everywhere. The air pockets will have a density of approximately $1.29 \times 10^{-3} \text{ g/cm}^3$, while the bone will be substantially denser.

111. 12.3 N

113. a. $3.02 \times 10^{-2} \text{ cm/s}$. (This small speed allows time for diffusion of materials to and from the blood.) b. 2.37×10^{10} capillaries. (This large number is an overestimate, but it is still reasonable.)

115. a. $2.76 \times 10^5 \text{ N/m}^2$; b. $P_2 = 2.81 \times 10^5 \text{ N/m}^2$

117. $8.7 \times 10^{-2} \text{ mm}^3/\text{s}$

119. a. 1.52; b. Turbulence would decrease the flow rate of the blood, which would require an even larger increase in the pressure difference, leading to higher blood pressure.

Challenge Problems

121. $p = 0.99 \times 10^5 \text{ Pa}$

123. 800 kg/m^3

125. 11.2 m/s

127. a. 71.8 m/s; b. 257 m/s

129. a. $150 \text{ cm}^3/\text{s}$; b. $33.3 \text{ cm}^3/\text{s}$; c. $25.0 \text{ cm}^3/\text{s}$; d. $0.0100 \text{ cm}^3/\text{s}$; e. $0.0300 \text{ cm}^3/\text{s}$

131. a. $1.20 \times 10^5 \text{ N/m}^2$; b. The flow rate in the main increases by 90%. c. There are approximately 38 more users in the afternoon.

Chapter 15

Check Your Understanding

15.1 The ruler is a stiffer system, which carries greater force for the same amount of displacement. The ruler snaps your hand with greater force, which hurts more.

15.2 You could increase the mass of the object that is oscillating. Other options would be to reduce the amplitude, or use a less stiff spring.

15.3 A ketchup bottle sits on a lazy Susan in the center of the dinner table. You set it rotating in uniform circular motion. A set of lights shine on the bottle, producing a shadow on the wall.

15.4 The movement of the pendulums will not differ at all because the mass of the bob has no effect on the motion of a simple pendulum. The pendulums are only affected by the period (which is related to the pendulum's length) and by the acceleration due to gravity.

15.5 Friction often comes into play whenever an object is moving. Friction causes damping in a harmonic oscillator.

15.6 The performer must be singing a note that corresponds to the natural frequency of the glass. As the sound wave is directed at the glass, the glass responds by resonating at the same frequency as the sound

wave. With enough energy introduced into the system, the glass begins to vibrate and eventually shatters.

Conceptual Questions

1. The restoring force must be proportional to the displacement and act opposite to the direction of motion with no drag forces or friction. The frequency of oscillation does not depend on the amplitude.
3. Examples: Mass attached to a spring on a frictionless table, a mass hanging from a string, a simple pendulum with a small amplitude of motion. All of these examples have frequencies of oscillation that are independent of amplitude.
5. Since the frequency is proportional to the square root of the force constant and inversely proportional to the square root of the mass, it is likely that the truck is heavily loaded, since the force constant would be the same whether the truck is empty or heavily loaded.
7. In a car, elastic potential energy is stored when the shock is extended or compressed. In some running shoes elastic potential energy is stored in the compression of the material of the soles of the running shoes. In pole vaulting, elastic potential energy is stored in the bending of the pole.
9. The overall system is stable. There may be times when the stability is interrupted by a storm, but the driving force provided by the sun bring the atmosphere back into a stable pattern.
11. The maximum speed is equal to $v_{\max} = A\omega$ and the angular frequency is independent of the amplitude, so the amplitude would be affected. The radius of the circle represents the amplitude of the circle, so make the amplitude larger.
13. The period of the pendulum is $T = 2\pi\sqrt{L/g}$. In summer, the length increases, and the period increases. If the period should be one second, but period is longer than one second in the summer, it will oscillate fewer than 60 times a minute and clock will run slow. In the winter it will run fast.
15. A car shock absorber.
17. The second law of thermodynamics states that perpetual motion machines are impossible. Eventually the ordered motion of the system decreases and returns to equilibrium.
19. All harmonic motion is damped harmonic motion, but the damping may be negligible. This is due to friction and drag forces. It is easy to come up with five examples of damped motion: (1) A mass oscillating on a hanging on a spring (it eventually comes to rest). (2) Shock absorbers in a car (thankfully they also come to rest). (3) A pendulum is a grandfather clock (weights are added to add energy to the oscillations). (4) A child on a swing (eventually comes to rest unless energy is added by pushing the child). (5) A marble rolling in a bowl (eventually comes to rest). As for the undamped motion, even a mass on a spring in a vacuum will eventually come to rest due to internal forces in the spring. Damping may be negligible, but cannot be eliminated.

Problems

21. Proof
23. 0.400 s/beat
25. 12,500 Hz
27. a. 340 km/hr; b. 11.3×10^3 rev/min
29. $f = \frac{1}{3}f_0$
31. 0.009 kg; 2%
33. a. 1.57×10^5 N/m; b. 77 kg, yes, he is eligible to play
35. a. 6.53×10^3 N/m; b. yes, when the man is at his lowest point in his hopping the spring will be compressed the most
37. a. 1.99 Hz; b. 44.3 cm; c. 65.0 cm
39. a. 0.335 m/s; b. 5.61×10^{-4} J
41. a. $x(t) = 2 \cos(0.52s^{-1}t)$; b. $v(t) = (-1.05 \text{ m/s})\sin(0.52s^{-1}t)$
43. 24.8 cm
45. 4.01 s
47. 1.58 s

49. 9.82002 m/s^2
 51. 6%
 53. 141 J
 55. a. $4.90 \times 10^{-3} \text{ m}$; b. $1.15 \times 10^{-2} \text{ m}$

Additional Problems

57. 94.7 kg
 59. a. 314 N/m; b. 1.00 s; c. 1.25 m/s
 61. ratio of 2.45
 63. The length must increase by 0.0116%.
 65. $\theta = (0.31 \text{ rad})\sin(3.13 \text{ s}^{-1}t)$
 67. a. 0.99 s; b. 0.11 m

Challenge Problems

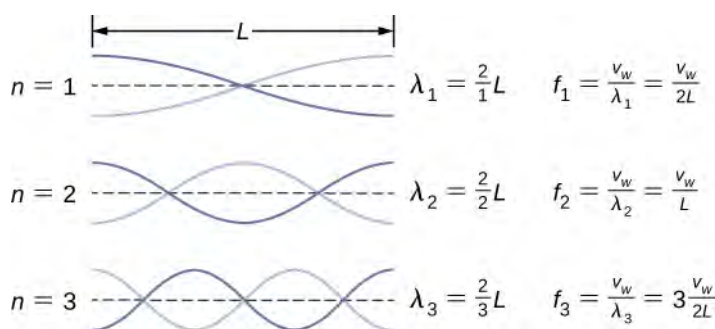
69. a. $3.95 \times 10^6 \text{ N/m}$; b. $7.90 \times 10^6 \text{ J}$
 71. $F \approx -\text{constant } r'$
 73. a. 7.54 cm; b. $3.25 \times 10^4 \text{ N/m}$

Chapter 16

Check Your Understanding

- 16.1** The wavelength of the waves depends on the frequency and the velocity of the wave. The frequency of the sound wave is equal to the frequency of the wave on the string. The wavelengths of the sound waves and the waves on the string are equal only if the velocities of the waves are the same, which is not always the case. If the speed of the sound wave is different from the speed of the wave on the string, the wavelengths are different. This velocity of sound waves will be discussed in [Sound](#).
- 16.2** In a transverse wave, the wave may move at a constant propagation velocity through the medium, but the medium oscillates perpendicular to the motion of the wave. If the wave moves in the positive x -direction, the medium oscillates up and down in the y -direction. The velocity of the medium is therefore not constant, but the medium's velocity and acceleration are similar to that of the simple harmonic motion of a mass on a spring.
- 16.3** Yes, a cosine function is equal to a sine function with a phase shift, and either function can be used in a wave function. Which function is more convenient to use depends on the initial conditions. In [Figure 16.11](#), the wave has an initial height of $y(0.00, 0.00) = 0$ and then the wave height increases to the maximum height at the crest. If the initial height at the initial time was equal to the amplitude of the wave $y(0.00, 0.00) = +A$, then it might be more convenient to model the wave with a cosine function.
- 16.4** This wave, with amplitude $A = 0.5 \text{ m}$, wavelength $\lambda = 10.00 \text{ m}$, period $T = 0.50 \text{ s}$, is a solution to the wave equation with a wave velocity $v = 20.00 \text{ m/s}$.
- 16.5** Since the speed of a wave on a taut string is proportional to the square root of the tension divided by the linear density, the wave speed would increase by $\sqrt{2}$.
- 16.6** At first glance, the time-averaged power of a sinusoidal wave on a string may look proportional to the linear density of the string because $P = \frac{1}{2}\mu A^2 \omega^2 v$; however, the speed of the wave depends on the linear density. Replacing the wave speed with $\sqrt{\frac{F_T}{\mu}}$ shows that the power is proportional to the square root of tension and proportional to the square root of the linear mass density:

$$P = \frac{1}{2}\mu A^2 \omega^2 v = \frac{1}{2}\mu A^2 \omega^2 \sqrt{\frac{F_T}{\mu}} = \frac{1}{2}A^2 \omega^2 \sqrt{\mu F_T}.$$
- 16.7** Yes, the equations would work equally well for symmetric boundary conditions of a medium free to oscillate on each end where there was an antinode on each end. The normal modes of the first three modes are shown below. The dotted line shows the equilibrium position of the medium.

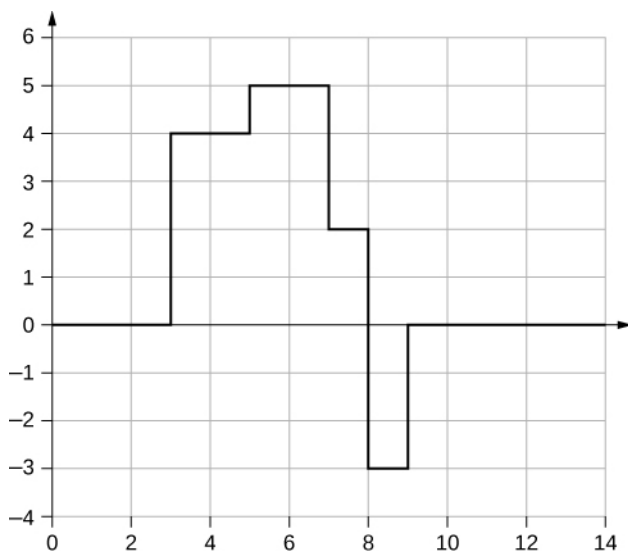


Note that the first mode is two quarters, or one half, of a wavelength. The second mode is one quarter of a wavelength, followed by one half of a wavelength, followed by one quarter of a wavelength, or one full wavelength. The third mode is one and a half wavelengths. These are the same result as the string with a node on each end. The equations for symmetrical boundary conditions work equally well for fixed boundary conditions and free boundary conditions. These results will be revisited in the next chapter when discussing sound wave in an open tube.

Conceptual Questions

1. A wave on a guitar string is an example of a transverse wave. The disturbance of the string moves perpendicular to the propagation of the wave. The sound produced by the string is a longitudinal wave where the disturbance of the air moves parallel to the propagation of the wave.
3. Propagation speed is the speed of the wave propagating through the medium. If the wave speed is constant, the speed can be found by $v = \frac{\lambda}{T} = \lambda f$. The frequency is the number of wave that pass a point per unit time. The wavelength is directly proportional to the wave speed and inversely proportional to the frequency.
5. No, the distance you move your hand up and down will determine the amplitude of the wave. The wavelength will depend on the frequency you move your hand up and down, and the speed of the wave through the spring.
7. Light from the Sun and stars reach Earth through empty space where there is no medium present.
9. The wavelength is equal to the velocity of the wave times the frequency and the wave number is equal to $k = \frac{2\pi}{\lambda}$, so yes, the wave number will depend on the frequency and also depend on the velocity of the wave propagating through the spring.
11. The medium moves in simple harmonic motion as the wave propagates through the medium, continuously changing speed, therefore it accelerates. The acceleration of the medium is due to the restoring force of the medium, which acts in the opposite direction of the displacement.
13. The wave speed is proportional to the square root of the tension, so the speed is doubled.
15. Since the speed of a wave on a string is inversely proportional to the square root of the linear mass density, the speed would be higher in the low linear mass density of the string.
17. The tension in the wire is due to the weight of the electrical power cable.
19. The time averaged power is $P = \frac{E\lambda}{T} = \frac{1}{2}\mu A^2 \omega^2 \frac{\lambda}{T} = \frac{1}{2}\mu A^2 \omega^2 v$. If the frequency or amplitude is halved, the power decreases by a factor of 4.
21. As a portion on the string moves vertically, it exerts a force on the neighboring portion of the string, doing work on the portion and transferring the energy.
23. The intensity of a spherical wave is $I = \frac{P}{4\pi r^2}$, if no energy is dissipated the intensity will decrease by a factor of nine at three meters.
25. At the interface, the incident pulse produces a reflected pulse and a transmitted pulse. The reflected pulse would be out of phase with respect to the incident pulse, and would move at the same propagation speed as the incident pulse, but would move in the opposite direction. The transmitted pulse would travel in the same direction as the incident pulse, but at half the speed. The transmitted pulse would be in phase with the incident pulse. Both the reflected pulse and the transmitted pulse would have amplitudes less than the amplitude of the incident pulse.

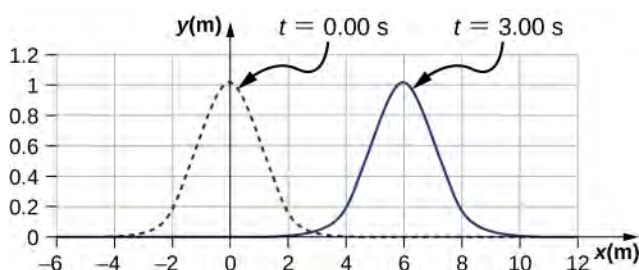
27.



29. It may be as easy as changing the length and/or the density a small amount so that the parts do not resonate at the frequency of the motor.
31. Energy is supplied to the glass by the work done by the force of your finger on the glass. When supplied at the right frequency, standing waves form. The glass resonates and the vibrations produce sound.
33. For the equation $y(x, t) = 4.00 \text{ cm} \sin(3 \text{ m}^{-1}x) \cos(4 \text{ s}^{-1}t)$, there is a node because when $x = 0.00 \text{ m}$, $\sin(3 \text{ m}^{-1}(0.00 \text{ m})) = 0.00$, so $y(0.00 \text{ m}, t) = 0.00 \text{ m}$ for all time. For the equation $y(x, t) = 4.00 \text{ cm} \sin(3 \text{ m}^{-1}x + \frac{\pi}{2}) \cos(4 \text{ s}^{-1}t)$, there is an antinode because when $x = 0.00 \text{ m}$, $\sin(3 \text{ m}^{-1}(0.00 \text{ m}) + \frac{\pi}{2}) = +1.00$, so $y(0.00 \text{ m}, t)$ oscillates between $+A$ and $-A$ as the cosine term oscillates between $+1$ and -1 .

Problems

35. $2d = vt \Rightarrow d = 11.25 \text{ m}$
 $v = f\lambda$, so that $f = 0.125 \text{ Hz}$, so that
37. $N = 7.50$ times
39. $v = f\lambda \Rightarrow \lambda = 0.400 \text{ m}$
41. $v = f\lambda \Rightarrow f = 2.50 \times 10^9 \text{ Hz}$
43. a. The P-waves outrun the S-waves by a speed of $v = 3.20 \text{ km/s}$; therefore, $\Delta d = 0.320 \text{ km}$. b. Since the uncertainty in the distance is less than a kilometer, our answer to part (a) does not seem to limit the detection of nuclear bomb detonations. However, if the velocities are uncertain, then the uncertainty in the distance would increase and could then make it difficult to identify the source of the seismic waves.
 $v = 1900 \text{ m/s}$
45. $\Delta t = 1.05 \mu\text{s}$
47. $y(x, t) = -0.037 \text{ cm}$
- 49.



The pulse will move $\Delta x = 6.00 \text{ m}$.

51. a. $A = 0.25 \text{ m}$; b. $k = 0.30 \text{ m}^{-1}$; c. $\omega = 0.90 \text{ s}^{-1}$; d. $v = 3.0 \text{ m/s}$; e. $\phi = \pi/3 \text{ rad}$; f. $\lambda = 20.93 \text{ m}$; g.

$$T = 6.98 \text{ s}$$

53. $A = 0.30 \text{ m}$, $\lambda = 4.50 \text{ m}$, $v = 18.00 \text{ m/s}$, $f = 4.00 \text{ Hz}$, $T = 0.25 \text{ s}$

55. $y(x, t) = 0.23 \text{ m} \sin(3.49 \text{ m}^{-1}x - 0.63 \text{ s}^{-1}t)$

57. They have the same angular frequency, frequency, and period. They are traveling in opposite directions and $y_2(x, t)$ has twice the wavelength as $y_1(x, t)$ and is moving at half the wave speed.

59. Each particle of the medium moves a distance of $4A$ each period. The period can be found by dividing the velocity by the wavelength: $t = 10.42 \text{ s}$

61. a. $\mu = 0.040 \text{ kg/m}$; b. $v = 15.75 \text{ m/s}$

63. $v = 180 \text{ m/s}$

65. $v = 547.723 \text{ m/s}$, $\Delta t = 5.48 \text{ ms}$

67. 0.707

69. $v_1 t + v_2 t = 2.00 \text{ m}$, $t = 1.69 \text{ ms}$

71. $v = 288.68 \text{ m/s}$, $\lambda = 0.73 \text{ m}$

73. a. $A = 0.0125 \text{ cm}$; b. $F_T = 0.96 \text{ N}$

75. $v = 74.54 \text{ m/s}$, $P_\lambda = 91.85 \text{ W}$

77. a. $I = 20.0 \text{ W/m}^2$; b. $I = \frac{P}{A}$, $A = 10.0 \text{ m}^2$

$$A = 4\pi r^2, r = 0.892 \text{ m}$$

79. $I = 650 \text{ W/m}^2$

81. $P \propto E \propto I \propto X^2 \Rightarrow \frac{P_2}{P_1} = \left(\frac{X_2}{X_1}\right)^2$

$$P_2 = 2.50 \text{ kW}$$

83. $I \propto X^2 \Rightarrow \frac{I_1}{I_2} = \left(\frac{X_1}{X_2}\right)^2 \Rightarrow$

$$I_2 = 3.38 \times 10^{-5} \text{ W/m}^2$$

85. $f = 100.00 \text{ Hz}$, $A = 1.10 \text{ cm}$

87. a. $I_2 = 0.063 I_1$; b. $I_1 4\pi r_1^2 = I_2 4\pi r_2^2$

$$r_2 = 3.16 \text{ m}$$

89. $2\pi r_1 A_1^2 = 2\pi r_2 A_2^2$, $A_1 = \left(\frac{r_2}{r_1}\right)^{1/2} A_2 = 0.17 \text{ m}$

91. $y(x, t) = 0.63 \text{ m}$

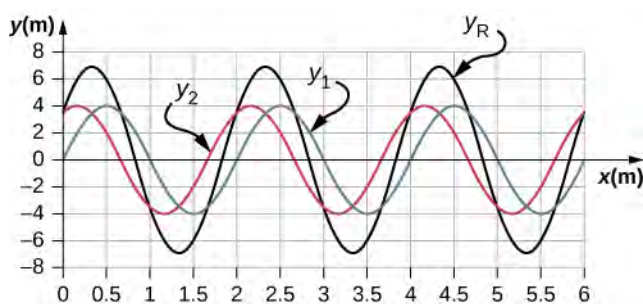
93. $A_R = 2A \cos\left(\frac{\phi}{2}\right)$, $\phi = 1.17 \text{ rad}$

95. $y_R = 1.90 \text{ cm}$

$$\omega = 6.28 \text{ s}^{-1}, k = 3.00 \text{ m}^{-1}, \phi = \frac{\pi}{8} \text{ rad},$$

97. $A_R = 2A \cos\left(\frac{\phi}{2}\right)$, $A = 0.37 \text{ m}$

99. a.



;

b. $\lambda = 2.0$ m, $A = 4$ m; c. $\lambda_R = 2.0$ m, $A_R = 6.93$ m

101. $y_R(x, t) = 2A \cos\left(\frac{\phi}{2}\right) \cos\left(kx - \omega t + \frac{\phi}{2}\right)$; The result is not surprising because $\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$.

103. $\lambda_n = \frac{2.00}{n} L$, $f_n = \frac{v}{\lambda_n}$
 $\lambda_1 = 4.00$ m, $f_1 = 12.5$ Hz
 $\lambda_2 = 2.00$ m, $f_2 = 25.00$ Hz
 $\lambda_3 = 1.33$ m, $f_3 = 37.59$ Hz

105. $v = 158.11$ m/s, $\lambda = 4.44$ m, $f = 35.61$ Hz
 $\lambda_s = 9.63$ m

107. $y(x, t) = [0.60 \text{ cm} \sin(3 \text{ m}^{-1}x)] \cos(4 \text{ s}^{-1}t)$

109. $\lambda_{100} = 0.06$ m

$v = 56.8$ m/s, $f_n = nf_1$, $n = 1, 2, 3, 4, 5, \dots$

$f_{100} = 947$ Hz

111. $T = 2\Delta t$, $v = \frac{\lambda}{T}$, $\lambda = 2.12$ m

113. $\lambda_1 = 6.00$ m, $\lambda_2 = 3.00$ m, $\lambda_3 = 2.00$ m, $\lambda_4 = 1.50$ m

$v = 258.20$ m/s $= \lambda f$

$f_1 = 43.03$ Hz, $f_2 = 86.07$ Hz, $f_3 = 129.10$ Hz, $f_4 = 172.13$ Hz

115. $v = 134.16$ m/s, $\lambda = 1.4$ m, $f = 95.83$ Hz, $T = 0.0104$ s

Additional Problems

117. $\lambda = 0.10$ m

119. a. $f = 4.74 \times 10^{14}$ Hz; b. $\lambda = 422$ nm

121. $\lambda = 16.00$ m, $f = 0.10$ Hz, $T = 10.00$ s, $v = 1.6$ m/s

123. $\lambda = (v_b + v)t_b$, $v = 3.75$ m/s, $\lambda = 3.00$ m

$$\frac{\partial^2 (y_1 + y_2)}{\partial t^2} = -A\omega^2 \sin(kx - \omega t) - A\omega^2 \sin(kx - \omega t + \phi)$$

$$\frac{\partial^2 (y_1 + y_2)}{\partial x^2} = -Ak^2 \sin(kx - \omega t) - Ak^2 \sin(kx - \omega t + \phi)$$

125.

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$-A\omega^2 \sin(kx - \omega t) - A\omega^2 \sin(kx - \omega t + \phi) = \left(\frac{1}{v^2}\right) (-Ak^2 \sin(kx - \omega t) - Ak^2 \sin(kx - \omega t + \phi))$$

$$v = \frac{\omega}{k}$$

127. $y(x, t) = 0.40 \text{ m} \sin(0.015 \text{ m}^{-1}x + 1.5 \text{ s}^{-1}t)$

129. $v = 223.61 \text{ m/s}$, $k = 1.57 \text{ m}^{-1}$, $\omega = 142.43 \text{ s}^{-1}$

131.

$$P = \frac{1}{2} A^2 (2\pi f)^2 \sqrt{\mu F_T}$$

$$\mu = 2.00 \times 10^{-4} \text{ kg/m}$$

133. $P = \frac{1}{2} \mu A^2 \omega^2 \frac{\lambda}{T}$, $\mu = 0.0018 \text{ kg/m}$

135. a. $A_R = 2A \cos\left(\frac{\phi}{2}\right)$, $\cos\left(\frac{\phi}{2}\right) = 1$, $\phi = 0, 2\pi, 4\pi, \dots$; b.

$$A_R = 2A \cos\left(\frac{\phi}{2}\right)$$
, $\cos\left(\frac{\phi}{2}\right) = 0$, $\phi = 0, \pi, 3\pi, 5\pi, \dots$

137. $y_R(x, t) = 0.6 \text{ m} \sin(4 \text{ m}^{-1}x) \cos(3 \text{ s}^{-1}t)$

139. a. (1) $F_T - 20.00 \text{ kg} (9.80 \text{ m/s}^2) \cos 45^\circ = 0$; b. $F_T = 138.57 \text{ N}$
 (2) $m (9.80 \text{ m/s}^2) - F_T = 0$ $v = 74.45 \text{ m/s}$

$$m = 14.14 \text{ kg}$$

141. $F_T = 2 \text{ N}$, $v = 6.73 \text{ m/s}$

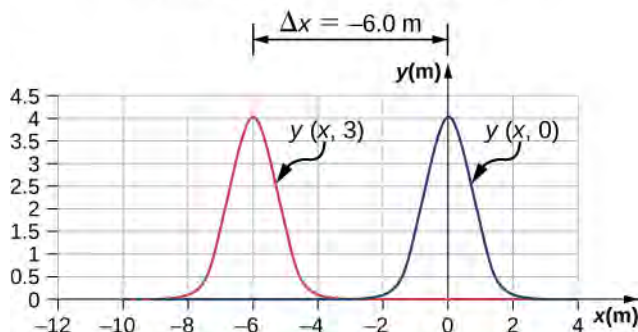
143. a. $f_n = \frac{nv}{2L}$, $v = \frac{2Lf_{n+1}}{n+1}$, $\frac{n+1}{n} = \frac{2Lf_{n+1}}{2Lf_n}$, $1 + \frac{1}{n} = 1.2$, $n = 5$; b. $F_T = 245.76 \text{ N}$

$$\lambda_n = \frac{2}{n} L$$
, $\lambda_5 = 1.6 \text{ m}$, $\lambda_6 = 1.33 \text{ m}$

Challenge Problems

145. a. Moves in the negative x direction at a propagation speed of $v = 2.00 \text{ m/s}$. b. $\Delta x = -6.00 \text{ m}$; c.

Wave Function vs. Time



$$\sin(kx - \omega t) = \sin\left(kx + \frac{\phi}{2}\right) \cos\left(\omega t + \frac{\phi}{2}\right) - \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$$

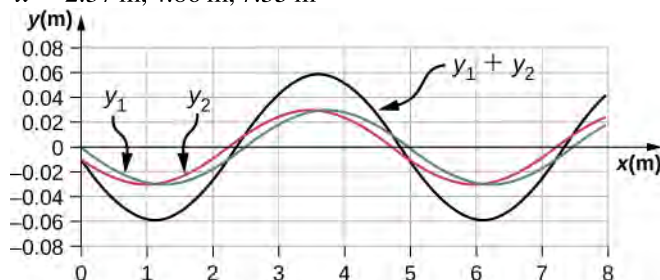
$$147. \sin(kx - \omega t + \phi) = \sin\left(kx + \frac{\phi}{2}\right) \cos\left(\omega t + \frac{\phi}{2}\right) + \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$$

$$\sin(kx - \omega t) + \sin(kx + \omega t + \phi) = 2 \sin\left(kx + \frac{\phi}{2}\right) \cos\left(\omega t + \frac{\phi}{2}\right)$$

$$y_R = 2 A \sin\left(kx + \frac{\phi}{2}\right) \cos\left(\omega t + \frac{\phi}{2}\right)$$

$$149. \sin\left(kx + \frac{\phi}{2}\right) = 0, kx + \frac{\phi}{2} = 0, \pi, 2\pi, 1.26 \text{ m}^{-1} x + \frac{\pi}{20} = \pi, 2\pi, 3\pi;$$

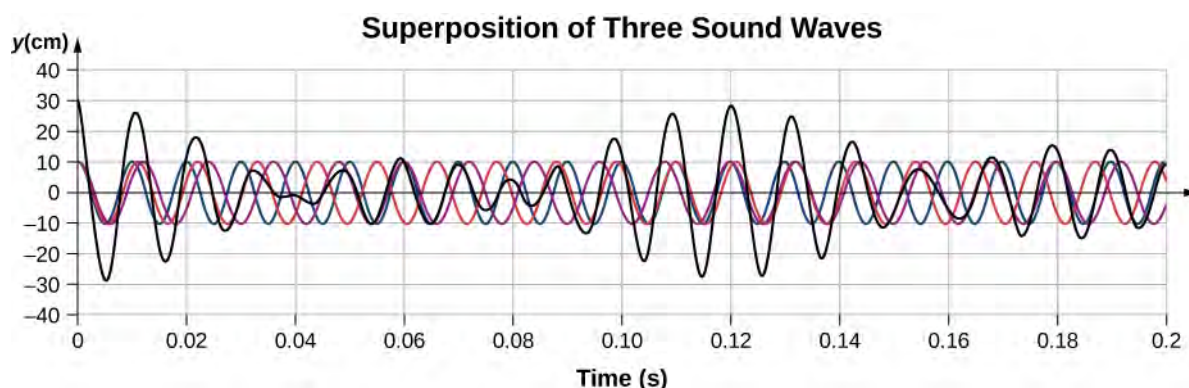
$$x = 2.37 \text{ m}, 4.86 \text{ m}, 7.35 \text{ m}$$



Chapter 17

Check Your Understanding

- 17.1** Sound and light both travel at definite speeds, and the speed of sound is slower than the speed of light. The first shell is probably very close by, so the speed difference is not noticeable. The second shell is farther away, so the light arrives at your eyes noticeably sooner than the sound wave arrives at your ears.
- 17.2** 10 dB: rustle of leaves; 50 dB: average office; 100 dB: noisy factory
- 17.3** Amplitude is directly proportional to the experience of loudness. As amplitude increases, loudness increases.
- 17.4** In the example, the two speakers were producing sound at a single frequency. Music has various frequencies and wavelengths.
- 17.5** Regular headphones only block sound waves with a physical barrier. Noise-canceling headphones use destructive interference to reduce the loudness of outside sounds.
- 17.6** When the tube resonates at its natural frequency, the wave's node is located at the closed end of the tube, and the antinode is located at the open end. The length of the tube is equal to one-fourth of the wavelength of this wave. Thus, if we know the wavelength of the wave, we can determine the length of the tube.
- 17.7** Compare their sizes. High-pitch instruments are generally smaller than low-pitch instruments because they generate a smaller wavelength.
- 17.8** An easy way to understand this event is to use a graph, as shown below. It appears that beats are produced, but with a more complex pattern of interference.



- 17.9** If I am driving and I hear Doppler shift in an ambulance siren, I would be able to tell when it was getting closer and also if it has passed by. This would help me to know whether I needed to pull over and let the ambulance through.

Conceptual Questions

1. Sound is a disturbance of matter (a pressure wave) that is transmitted from its source outward. Hearing is the human perception of sound.
3. Consider a sound wave moving through air. The pressure of the air is the equilibrium condition, it is the change in pressure that produces the sound wave.
5. The frequency does not change as the sound wave moves from one medium to another. Since the speed changes and the frequency does not, the wavelength must change. This is similar to the driving force of a harmonic oscillator or a wave on the string.
7. The transducer sends out a sound wave, which reflects off the object in question and measures the time it takes for the sound wave to return. Since the speed of sound is constant, the distance to the object can found by multiplying the velocity of sound by half the time interval measured.
9. The ear plugs reduce the intensity of the sound both in water and on land, but Navy researchers have found that sound under water is heard through vibrations mastoid, which is the bone behind the ear.
11. The fundamental wavelength of a tube open at each end is $2L$, where the wavelength of a tube open at one end and closed at one end is $4L$. The tube open at one end has the lower fundamental frequency, assuming the speed of sound is the same in both tubes.
13. The wavelength in each is twice the length of the tube. The frequency depends on the wavelength and the speed of the sound waves. The frequency in room *B* is higher because the speed of sound is higher where the temperature is higher.
15. When resonating at the fundamental frequency, the wavelength for pipe *C* is $4L$, and for pipes *A* and *B* is $2L$. The frequency is equal to $f = v/\lambda$. Pipe *C* has the lowest frequency and pipes *A* and *B* have equal frequencies, higher than the one in pipe *C*.
17. Since the boundary conditions are both symmetric, the frequencies are $f_n = \frac{nv}{2L}$. Since the speed is the same in each, the frequencies are the same. If the wave speed were doubled in the string, the frequencies in the string would be twice the frequencies in the tube.
19. The frequency of the unknown fork is 255 Hz. No, if only the 250 Hz fork is used, listening to the beat frequency could only limit the possible frequencies to 245 Hz or 255 Hz.
21. The beat frequency is 0.7 Hz.
23. Observer 1 will observe the highest frequency. Observer 2 will observe the lowest frequency. Observer 3 will hear a higher frequency than the source frequency, but lower than the frequency observed by observer 1, as the source approaches and a lower frequency than the source frequency, but higher than the frequency observed by observer 1, as the source moves away from observer 3.
25. Doppler radar can not only detect the distance to a storm, but also the speed and direction at which the storm is traveling.
27. The speed of sound decreases as the temperature decreases. The Mach number is equal to $M = \frac{v_s}{v}$, so the plane should slow down.

Problems

29. $s_{\max} = 4.00 \text{ nm}$, $\lambda = 1.72 \text{ m}$, $f = 200 \text{ Hz}$, $v = 343.17 \text{ m/s}$

31. a. $\lambda = 68.60 \text{ } \mu\text{m}$; b. $\lambda = 360.00 \text{ } \mu\text{m}$

33. a. $k = 183.09 \text{ m}^{-1}$;

b. $\Delta P = -1.11 \text{ Pa}$

35. $s_1 = 7.00 \text{ nm}$, $s_2 = 3.00 \text{ nm}$, $kx_1 + \phi = 0 \text{ rad}$

$kx_2 + \phi = 1.128 \text{ rad}$

$k(x_2 - x_1) = 1.128 \text{ rad}$, $k = 5.64 \text{ m}^{-1}$

$\lambda = 1.11 \text{ m}$, $f = 306.31 \text{ Hz}$

37. $k = 5.28 \times 10^3 \text{ m}$

$s(x, t) = 4.50 \text{ nm} \cos(5.28 \times 10^3 \text{ m}^{-1}x - 2\pi(5.00 \text{ MHz})t)$

39. $\lambda = 3.43 \text{ mm}$

$\lambda = 6.00 \text{ m}$

41. $s_{\max} = 2.00 \text{ mm}$

$v = 600 \text{ m/s}$

$T = 0.01 \text{ s}$

43. (a) $f = 100 \text{ Hz}$, (b) $\lambda = 3.43 \text{ m}$

45. $f = 3400 \text{ Hz}$

47. a. $v = 5.96 \times 10^3 \text{ m/s}$; b. steel (from value in [Table 17.1](#))

49. $v = 363 \frac{\text{m}}{\text{s}}$

51. $\Delta x = 924 \text{ m}$

53. $V = 0.05 \text{ m}^3$

$m = 392.5 \text{ kg}$

$\rho = 7850 \text{ kg/m}^3$

$v = 5047.54 \text{ m/s}$

55. $T_C = 35^\circ\text{C}$, $v = 351.58 \text{ m/s}$

$\Delta x_1 = 35.16 \text{ m}$, $\Delta x_2 = 52.74 \text{ m}$

$\Delta x = 63.39 \text{ m}$

57. a. $t_{5.00^\circ\text{C}} = 0.0180 \text{ s}$, $t_{35.0^\circ\text{C}} = 0.0171 \text{ s}$; b. % uncertainty = 5.00%; c. This uncertainty could definitely cause difficulties for the bat, if it didn't continue to use sound as it closed in on its prey. A 5% uncertainty could be the difference between catching the prey around the neck or around the chest, which means that it could miss grabbing its prey.

59. $1.26 \times 10^{-3} \text{ W/m}^2$

61. 85 dB

63. a. 93 dB; b. 83 dB
65. $1.58 \times 10^{-13} \text{ W/m}^2$
67. A decrease of a factor of 10 in intensity corresponds to a reduction of 10 dB in sound level:
 $120 \text{ dB} - 10 \text{ dB} = 110 \text{ dB}$.
69. We know that 60 dB corresponds to a factor of 10^6 increase in intensity. Therefore,
 $I \propto X^2 \Rightarrow \frac{I_2}{I_1} = \left(\frac{X_2}{X_1}\right)^2$, so that $X_2 = 10^{-6} \text{ atm}$.
 120 dB corresponds to a factor of 10^{12} increase $\Rightarrow 10^{-9} \text{ atm}(10^{12})^{1/2} = 10^{-3} \text{ atm}$.
71. 28.2 dB
73. $1 \times 10^6 \text{ km}$
75. $73 \text{ dB} - 70 \text{ dB} = 3 \text{ dB}$; Such a change in sound level is easily noticed.
77. 2.5; The 100-Hz tone must be 2.5 times more intense than the 4000-Hz sound to be audible by this person.
79. 0.974 m
81. 11.0 kHz; The ear is not particularly sensitive to this frequency, so we don't hear overtones due to the ear canal.
83. a. $v = 344.08 \text{ m/s}$, $\lambda_1 = 16.00 \text{ m}$, $f_1 = 21.51 \text{ Hz}$;
 b. $\lambda_3 = 5.33 \text{ m}$, $f_3 = 64.56 \text{ Hz}$
85.
 $v_{\text{string}} = 149.07 \text{ m/s}$, $\lambda_3 = 1.33 \text{ m}$, $f_3 = 112.08 \text{ Hz}$
 $\lambda_1 = \frac{v}{f_1}$, $L = 1.53 \text{ m}$
87. a. 22.0°C ; b. 1.01 m
89. first overtone = 180 Hz;
 second overtone = 270 Hz;
 third overtone = 360 Hz
91. 1.56 m
93. The pipe has symmetrical boundary conditions;
- $\lambda_n = \frac{2}{n}L$, $f_n = \frac{nv}{2L}$, $n = 1, 2, 3$
 $\lambda_1 = 6.00 \text{ m}$, $\lambda_2 = 3.00 \text{ m}$, $\lambda_3 = 2.00 \text{ m}$
 $f_1 = 57.17 \text{ Hz}$, $f_2 = 114.33 \text{ Hz}$, $f_3 = 171.50 \text{ Hz}$
95. $\lambda_6 = 0.5 \text{ m}$
 $v = 1000 \text{ m/s}$
 $F_T = 6500 \text{ N}$
97. $f = 6.40 \text{ kHz}$
99. 1.03 or 3%
101. $f_B = |f_1 - f_2|$
 $|128.3 \text{ Hz} - 128.1 \text{ Hz}| = 0.2 \text{ Hz}$;
 $|128.3 \text{ Hz} - 127.8 \text{ Hz}| = 0.5 \text{ Hz}$;
 $|128.1 \text{ Hz} - 127.8 \text{ Hz}| = 0.3 \text{ Hz}$

103. $v_A = 135.87 \text{ m/s}$, $v_B = 141.42 \text{ m/s}$,
 $\lambda_A = \lambda_B = 0.40 \text{ m}$
 $\Delta f = 15.00 \text{ Hz}$

105. $v = 155.54 \text{ m/s}$,
 $f_{\text{string}} = 971.17 \text{ Hz}$, $n = 16.23$
 $f_{\text{string}} = 1076.83 \text{ Hz}$, $n = 18.00$
 The frequency is 1076.83 Hz and the wavelength is 0.14 m.

107. $f_2 = f_1 \pm f_B = 260.00 \text{ Hz} \pm 1.50 \text{ Hz}$,
 so that $f_2 = 261.50 \text{ Hz}$ or $f_2 = 258.50 \text{ Hz}$

109. $f_{\text{ace}} = \frac{f_1 + f_2}{2}$; $f_B = f_1 - f_2$ (assume $f_1 > f_2$)
 $f_{\text{ace}} = \frac{(f_B + f_2) + f_2}{2} \Rightarrow$
 $f_2 = 4099.750 \text{ Hz}$
 $f_1 = 4100.250 \text{ Hz}$

111. a. 878 Hz; b. 735 Hz

113. $3.79 \times 10^3 \text{ Hz}$

115. a. 12.9 m/s; b. 193 Hz

117. The first eagle hears $4.23 \times 10^3 \text{ Hz}$. The second eagle hears $3.56 \times 10^3 \text{ Hz}$.

119. $v_s = 31.29 \text{ m/s}$
 $f_o = 1.12 \text{ kHz}$

121. An audible shift occurs when $\frac{f_{\text{obs}}}{f_s} \geq 1.003$;
 $f_{\text{obs}} = f_s \frac{v}{v - v_s} \Rightarrow \frac{f_{\text{obs}}}{f_s} = \frac{v}{v - v_s} \Rightarrow$
 $v_s = 0.990 \text{ m/s}$

123. $\theta = 30.02^\circ$
 $v_s = 680.00 \text{ m/s}$
 $\tan \theta = \frac{y}{v_s t}$, $t = 21.65 \text{ s}$

125. $\sin \theta = \frac{1}{M}$, $\theta = 56.47^\circ$
 $y = 9.31 \text{ km}$

$$s_1 = 6.34 \text{ nm}$$

$$s_2 = 2.30 \text{ nm}$$

$$kx_1 + \phi = 0 \text{ rad}$$

$$kx_2 + \phi = 1.20 \text{ rad}$$

$$127. \quad k(x_2 - x_1) = 1.20 \text{ rad}$$

$$k = 3.00 \text{ m}^{-1}$$

$$\omega = 1019.62 \text{ s}^{-1}$$

$$s_1 = s_{\max} \cos(kx_1 - \phi)$$

$$\phi = 5.66 \text{ rad}$$

$$s(x, t) = 6.30 \text{ nm} \cos(3.00 \text{ m}^{-1}x - 1019.62 \text{ s}^{-1}t + 5.66)$$

Additional Problems

$$129. \quad v_s = 346.40 \text{ m/s};$$

$$\lambda_n = \frac{2}{n}L \quad f_n = \frac{v_s}{\lambda_n}$$

$$\lambda_1 = 1.60 \text{ m} \quad f_1 = 216.50 \text{ Hz}$$

$$\lambda_2 = 0.80 \text{ m} \quad f_1 = 433.00 \text{ Hz}$$

$$\lambda_6 = 0.40 \text{ m}$$

$$131. \quad \text{a.} \quad ; \text{ b. } \lambda_s = 2.40 \text{ m}$$

$$v = 57.15 \frac{\text{m}}{\text{s}}$$

$$f_6 = 142.89 \text{ Hz}$$

$$v = 344.08 \frac{\text{m}}{\text{s}}$$

$$133. \quad v_A = 29.05 \frac{\text{m}}{\text{s}}, \quad v_B = 33.52 \text{ m/s}$$

$$f_A = 961.18 \text{ Hz},$$

$$f_B = 958.89 \text{ Hz}$$

$$f_{A,\text{beat}} = 161.18 \text{ Hz}, \quad f_{B,\text{beat}} = 158.89 \text{ Hz}$$

$$135. \quad v = 345.24 \frac{\text{m}}{\text{s}}; \text{ a. } I = 31.62 \frac{\mu\text{W}}{\text{m}^2}; \text{ b. } I = 0.16 \frac{\mu\text{W}}{\text{m}^2}; \text{ c. } s_{\max} = 104.39 \mu\text{m}; \text{ d. } s_{\max} = 7.43 \mu\text{m}$$

$$137. \quad \frac{f_A}{f_D} = \frac{v+v_s}{v-v_s}, \quad (v-v_s) \frac{f_A}{f_D} = v+v_s, \quad v = 347.39 \frac{\text{m}}{\text{s}}$$

$$T_C = 27.70^\circ$$

Challenge Problems

$$\sqrt{x^2 + d^2} - x = \lambda, \quad x^2 + d^2 = (\lambda + x)^2$$

139. $x^2 + d^2 = \lambda^2 + 2x\lambda + x^2, \quad d^2 = \lambda^2 + 2x\lambda$

$$x = \frac{d^2 - \left(\frac{v}{f}\right)^2}{2\frac{v}{f}}$$

141. a. For maxima $\Delta r = d \sin \theta$

$$d \sin \theta = n\lambda \quad n = 0, \pm 1, \pm 2, \dots, \quad \theta = \sin^{-1} \left(n \frac{\lambda}{d} \right) \quad n = 0, \pm 1, \pm 2, \dots$$

b. For minima, $\Delta r = d \sin \theta$

$$d \sin \theta = \left(n + \frac{1}{2} \right) \lambda \quad n = 0, \pm 1, \pm 2, \dots$$

$$\theta = \sin^{-1} \left(\left(n + \frac{1}{2} \right) \frac{\lambda}{d} \right) \quad n = 0, \pm 1, \pm 2, \dots$$

143. a. $v_{\text{string}} = 160.73 \frac{\text{m}}{\text{s}}, \quad f_{\text{string}} = 535.77 \text{ Hz};$ b. $f_{\text{fork}} = 512 \text{ Hz};$ c. $f_{\text{fork}} = \frac{n\sqrt{\frac{F_T}{\mu}}}{2L}, \quad F_T = 141.56 \text{ N}$

145. a. $f = 268.62 \text{ Hz};$ b. $\Delta f \approx \frac{1}{2} \frac{\Delta F_T}{F_T} f = 1.34 \text{ Hz}$

147. a. $v = 466.07 \frac{\text{m}}{\text{s}};$ b. $\lambda_9 = 51.11 \text{ mm};$ c. $f_9 = 9.12 \text{ kHz};$

d. $f_{\text{sound}} = 9.12 \text{ kHz};$ e. $\lambda_{\text{air}} = 37.86 \text{ mm}$

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CHAPTER 1

Temperature and Heat



Figure 1.1 These snowshoers on Mount Hood in Oregon are enjoying the heat flow and light caused by high temperature. All three mechanisms of heat transfer are relevant to this picture. The heat flowing out of the fire also turns the solid snow to liquid water and vapor. (credit: modification of work by “Mt. Hood Territory”/Flickr)

Chapter Outline

[1.1 Temperature and Thermal Equilibrium](#)

[1.2 Thermometers and Temperature Scales](#)

[1.3 Thermal Expansion](#)

[1.4 Heat Transfer, Specific Heat, and Calorimetry](#)

[1.5 Phase Changes](#)

[1.6 Mechanisms of Heat Transfer](#)

INTRODUCTION Heat and temperature are important concepts for each of us, every day. How we dress in the morning depends on whether the day is hot or cold, and most of what we do requires energy that ultimately comes from the Sun. The study of heat and temperature is part of an area of physics known as thermodynamics. The laws of thermodynamics govern the flow of energy throughout the universe. They are studied in all areas of science and engineering, from chemistry to biology to environmental science.

In this chapter, we explore heat and temperature. It is not always easy to distinguish these terms. Heat is the flow of energy from one object to another. This flow of energy is caused by a difference in temperature. The

✓ CHECK YOUR UNDERSTANDING 1.5

Why does snow often remain even when daytime temperatures are higher than the freezing temperature?

1.6 Mechanisms of Heat Transfer

Learning Objectives

By the end of this section, you will be able to:

- Explain some phenomena that involve conductive, convective, and radiative heat transfer
- Solve problems on the relationships between heat transfer, time, and rate of heat transfer
- Solve problems using the formulas for conduction and radiation

Just as interesting as the effects of heat transfer on a system are the methods by which it occurs. Whenever there is a temperature difference, heat transfer occurs. It may occur rapidly, as through a cooking pan, or slowly, as through the walls of a picnic ice chest. So many processes involve heat transfer that it is hard to imagine a situation where no heat transfer occurs. Yet every heat transfer takes place by only three methods:

1. **Conduction** is heat transfer through stationary matter by physical contact. (The matter is stationary on a macroscopic scale—we know that thermal motion of the atoms and molecules occurs at any temperature above absolute zero.) Heat transferred from the burner of a stove through the bottom of a pan to food in the pan is transferred by conduction.
2. **Convection** is the heat transfer by the macroscopic movement of a fluid. This type of transfer takes place in a forced-air furnace and in weather systems, for example.
3. Heat transfer by **radiation** occurs when microwaves, infrared radiation, visible light, or another form of electromagnetic radiation is emitted or absorbed. An obvious example is the warming of Earth by the Sun. A less obvious example is thermal radiation from the human body.

In the illustration at the beginning of this chapter, the fire warms the snowshoers' faces largely by radiation. Convection carries some heat to them, but most of the air flow from the fire is upward (creating the familiar shape of flames), carrying heat to the food being cooked and into the sky. The snowshoers wear clothes designed with low conductivity to prevent heat flow out of their bodies.

In this section, we examine these methods in some detail. Each method has unique and interesting characteristics, but all three have two things in common: They transfer heat solely because of a temperature difference, and the greater the temperature difference, the faster the heat transfer ([Figure 1.19](#)).

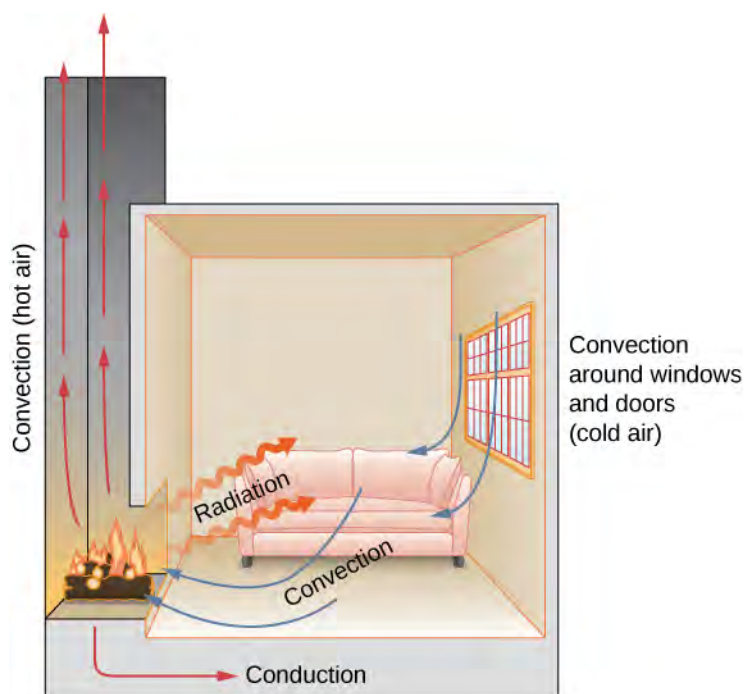


Figure 1.19 In a fireplace, heat transfer occurs by all three methods: conduction, convection, and radiation. Radiation is responsible for most of the heat transferred into the room. Heat transfer also occurs through conduction into the room, but much slower. Heat transfer by convection also occurs through cold air entering the room around windows and hot air leaving the room by rising up the chimney.

✓ CHECK YOUR UNDERSTANDING 1.6

Name an example from daily life (different from the text) for each mechanism of heat transfer.

Conduction

As you walk barefoot across the living room carpet in a cold house and then step onto the kitchen tile floor, your feet feel colder on the tile. This result is intriguing, since the carpet and tile floor are both at the same temperature. The different sensation is explained by the different rates of heat transfer: The heat loss is faster for skin in contact with the tiles than with the carpet, so the sensation of cold is more intense.

Some materials conduct thermal energy faster than others. [Figure 1.20](#) shows a material that conducts heat slowly—it is a good thermal insulator, or poor heat conductor—used to reduce heat flow into and out of a house.



Figure 1.20 Insulation is used to limit the conduction of heat from the inside to the outside (in winter) and from the outside to the inside (in summer). (credit: Giles Douglas)

A molecular picture of heat conduction will help justify the equation that describes it. [Figure 1.21](#) shows molecules in two bodies at different temperatures, T_h and T_c , for “hot” and “cold.” The average kinetic energy of a molecule in the hot body is higher than in the colder body. If two molecules collide, energy transfers from the high-energy to the low-energy molecule. In a metal, the picture would also include free valence electrons colliding with each other and with atoms, likewise transferring energy. The cumulative effect of all collisions is a net flux of heat from the hotter body to the colder body. Thus, the rate of heat transfer increases with increasing temperature difference $\Delta T = T_h - T_c$. If the temperatures are the same, the net heat transfer rate is zero. Because the number of collisions increases with increasing area, heat conduction is proportional to the cross-sectional area—a second factor in the equation.

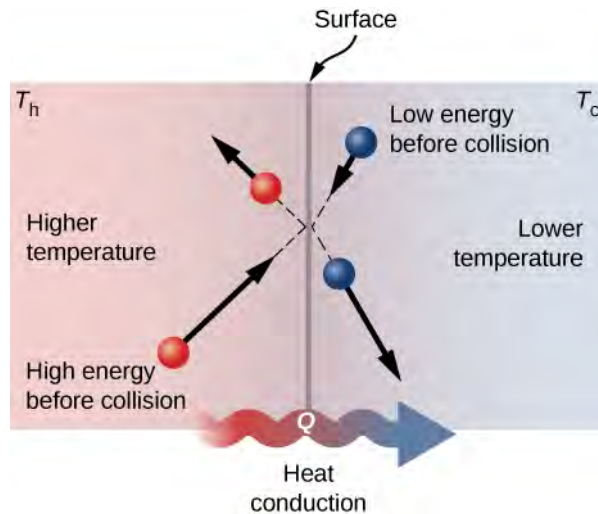


Figure 1.21 Molecules in two bodies at different temperatures have different average kinetic energies. Collisions occurring at the contact surface tend to transfer energy from high-temperature regions to low-temperature regions. In this illustration, a molecule in the lower-temperature region (right side) has low energy before collision, but its energy increases after colliding with a high-energy molecule at the contact surface. In contrast, a molecule in the higher-temperature region (left side) has high energy before collision, but its energy

decreases after colliding with a low-energy molecule at the contact surface.

A third quantity that affects the conduction rate is the thickness of the material through which heat transfers. [Figure 1.22](#) shows a slab of material with a higher temperature on the left than on the right. Heat transfers from the left to the right by a series of molecular collisions. The greater the distance between hot and cold, the more time the material takes to transfer the same amount of heat.

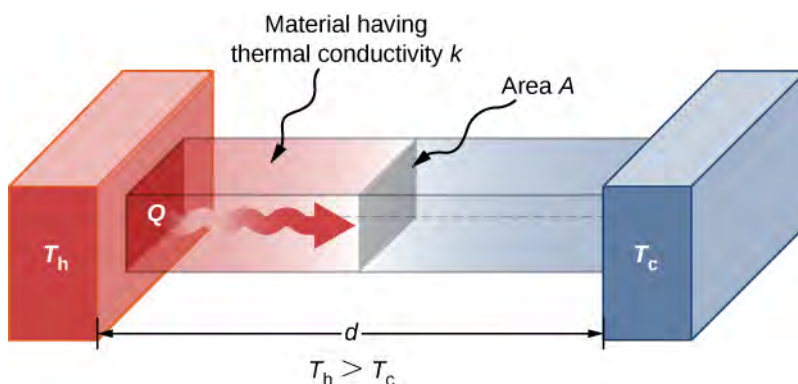


Figure 1.22 Heat conduction occurs through any material, represented here by a rectangular bar, whether window glass or walrus blubber.

All four of these quantities appear in a simple equation deduced from and confirmed by experiments. The **rate of conductive heat transfer** through a slab of material, such as the one in [Figure 1.22](#), is given by

$$P = \frac{dQ}{dt} = \frac{kA(T_h - T_c)}{d} \quad 1.9$$

where P is the power or rate of heat transfer in watts or in kilocalories per second, A and d are its surface area and thickness, as shown in [Figure 1.22](#), $T_h - T_c$ is the temperature difference across the slab, and k is the **thermal conductivity** of the material. [Table 1.5](#) gives representative values of thermal conductivity.

More generally, we can write

$$P = -kA \frac{dT}{dx},$$

where x is the coordinate in the direction of heat flow. Since in [Figure 1.22](#), the power and area are constant, dT/dx is constant, and the temperature decreases linearly from T_h to T_c .

Substance	Thermal Conductivity k (W/m · °C)
Diamond	2000
Silver	420
Copper	390
Gold	318
Aluminum	220
Steel iron	80
Steel (stainless)	14

Substance	Thermal Conductivity k (W/m · °C)
Ice	2.2
Glass (average)	0.84
Concrete brick	0.84
Water	0.6
Fatty tissue (without blood)	0.2
Asbestos	0.16
Plasterboard	0.16
Wood	0.08–0.16
Snow (dry)	0.10
Cork	0.042
Glass wool	0.042
Wool	0.04
Down feathers	0.025
Air	0.023
Polystyrene foam	0.010

Table 1.5 Thermal Conductivities of Common Substances Values are given for temperatures near 0 °C.



EXAMPLE 1.10

Calculating Heat Transfer through Conduction

A polystyrene foam icebox has a total area of 0.950 m^2 and walls with an average thickness of 2.50 cm. The box contains ice, water, and canned beverages at 0 °C. The inside of the box is kept cold by melting ice. How much ice melts in one day if the icebox is kept in the trunk of a car at 35.0 °C?

Strategy

This question involves both heat for a phase change (melting of ice) and the transfer of heat by conduction. To find the amount of ice melted, we must find the net heat transferred. This value can be obtained by calculating the rate of heat transfer by conduction and multiplying by time.

Solution

First we identify the knowns.

$k = 0.010 \text{ W/m} \cdot ^\circ\text{C}$ for polystyrene foam; $A = 0.950 \text{ m}^2$; $d = 2.50 \text{ cm} = 0.0250 \text{ m}$; $T_c = 0 ^\circ\text{C}$; $T_h = 35.0 ^\circ\text{C}$;
 $t = 1 \text{ day} = 24 \text{ hours} = 86,400 \text{ s}$.

Then we identify the unknowns. We need to solve for the mass of the ice, m . We also need to solve for the net heat transferred to melt the ice, Q . The rate of heat transfer by conduction is given by

$$P = \frac{dQ}{dt} = \frac{kA(T_h - T_c)}{d}.$$

The heat used to melt the ice is $Q = mL_f$. We insert the known values:

$$P = \frac{(0.010 \text{ W/m} \cdot ^\circ\text{C})(0.950 \text{ m}^2)(35.0^\circ\text{C} - 0^\circ\text{C})}{0.0250 \text{ m}} = 13.3 \text{ W}.$$

Multiplying the rate of heat transfer by the time we obtain

$$Q = Pt = (13.3 \text{ W})(86.400 \text{ s}) = 1.15 \times 10^6 \text{ J}.$$

We set this equal to the heat transferred to melt the ice, $Q = mL_f$, and solve for the mass m :

$$m = \frac{Q}{L_f} = \frac{1.15 \times 10^6 \text{ J}}{334 \times 10^3 \text{ J/kg}} = 3.44 \text{ kg}.$$

Significance

The result of 3.44 kg, or about 7.6 lb, seems about right, based on experience. You might expect to use about a 4 kg (7–10 lb) bag of ice per day. A little extra ice is required if you add any warm food or beverages.

[Table 1.5](#) shows that polystyrene foam is a very poor conductor and thus a good insulator. Other good insulators include fiberglass, wool, and goosedown feathers. Like polystyrene foam, these all contain many small pockets of air, taking advantage of air's poor thermal conductivity.

In developing insulation, the smaller the conductivity k and the larger the thickness d , the better. Thus, the ratio d/k , called the *R factor*, is large for a good insulator. The rate of conductive heat transfer is inversely proportional to R . R factors are most commonly quoted for household insulation, refrigerators, and the like. Unfortunately, in the United States, R is still in non-metric units of $\text{ft}^2 \cdot ^\circ\text{F} \cdot \text{h/Btu}$, although the unit usually goes unstated [1 British thermal unit (Btu) is the amount of energy needed to change the temperature of 1.0 lb of water by 1.0°F , which is 1055.1 J]. A couple of representative values are an R factor of 11 for 3.5-inch-thick fiberglass batts (pieces) of insulation and an R factor of 19 for 6.5-inch-thick fiberglass batts ([Figure 1.23](#)). In the US, walls are usually insulated with 3.5-inch batts, whereas ceilings are usually insulated with 6.5-inch batts. In cold climates, thicker batts may be used.



Figure 1.23 The fiberglass batt is used for insulation of walls and ceilings to prevent heat transfer between the inside of the building and the outside environment. (credit: Tracey Nicholls)

Note that in [Table 1.5](#), most of the best thermal conductors—silver, copper, gold, and aluminum—are also the best electrical conductors, because they contain many free electrons that can transport thermal energy. (Diamond, an electrical insulator, conducts heat by atomic vibrations.) Cooking utensils are typically made from good conductors, but the handles of those used on the stove are made from good insulators (bad conductors).



EXAMPLE 1.11

Two Conductors End to End

A steel rod and an aluminum rod, each of diameter 1.00 cm and length 25.0 cm, are welded end to end. One end of the steel rod is placed in a large tank of boiling water at 100 °C, while the far end of the aluminum rod is placed in a large tank of water at 20 °C. The rods are insulated so that no heat escapes from their surfaces. What is the temperature at the joint, and what is the rate of heat conduction through this composite rod?

Strategy

The heat that enters the steel rod from the boiling water has no place to go but through the steel rod, then through the aluminum rod, to the cold water. Therefore, we can equate the rate of conduction through the steel to the rate of conduction through the aluminum.

We repeat the calculation with a second method, in which we use the thermal resistance R of the rod, since it simply adds when two rods are joined end to end. (We will use a similar method in the chapter on direct-current circuits.)

Solution

1. Identify the knowns and convert them to SI units.

The length of each rod is $L_{Al} = L_{steel} = 0.25$ m, the cross-sectional area of each rod is

$A_{Al} = A_{steel} = 7.85 \times 10^{-5}$ m², the thermal conductivity of aluminum is $k_{Al} = 220$ W/m · °C, the thermal conductivity of steel is $k_{steel} = 80$ W/m · °C, the temperature at the hot end is $T = 100$ °C, and the temperature at the cold end is $T = 20$ °C.

2. Calculate the heat-conduction rate through the steel rod and the heat-conduction rate through the aluminum rod in terms of the unknown temperature T at the joint:

$$\begin{aligned}
 P_{\text{steel}} &= \frac{k_{\text{steel}} A_{\text{steel}} \Delta T_{\text{steel}}}{L_{\text{steel}}} \\
 &= \frac{(80 \text{ W/m}\cdot\text{C}) (7.85 \times 10^{-5} \text{ m}^2) (100^\circ\text{C} - T)}{0.25 \text{ m}} \\
 &= (0.0251 \text{ W/}^\circ\text{C}) (100^\circ\text{C} - T); \\
 P_{\text{Al}} &= \frac{k_{\text{Al}} A_{\text{Al}} \Delta T_{\text{Al}}}{L_{\text{Al}}} \\
 &= \frac{(220 \text{ W/m}\cdot\text{C}) (7.85 \times 10^{-5} \text{ m}^2) (T - 20^\circ\text{C})}{0.25 \text{ m}} \\
 &= (0.0691 \text{ W/}^\circ\text{C}) (T - 20^\circ\text{C}).
 \end{aligned}$$

3. Set the two rates equal and solve for the unknown temperature:

$$\begin{aligned}
 (0.0691 \text{ W/}^\circ\text{C}) (T - 20^\circ\text{C}) &= (0.0251 \text{ W/}^\circ\text{C}) (100^\circ\text{C} - T) \\
 T &= 41.3^\circ\text{C}.
 \end{aligned}$$

4. Calculate either rate:

$$P_{\text{steel}} = (0.0251 \text{ W/}^\circ\text{C}) (100^\circ\text{C} - 41.3^\circ\text{C}) = 1.47 \text{ W}.$$

5. If desired, check your answer by calculating the other rate.

Solution

- Recall that $R = L/k$. Now $P = A\Delta T/R$, or $\Delta T = PR/A$.
- We know that $\Delta T_{\text{steel}} + \Delta T_{\text{Al}} = 100^\circ\text{C} - 20^\circ\text{C} = 80^\circ\text{C}$. We also know that $P_{\text{steel}} = P_{\text{Al}}$, and we denote that rate of heat flow by P . Combine the equations:

$$\frac{PR_{\text{steel}}}{A} + \frac{PR_{\text{Al}}}{A} = 80^\circ\text{C}.$$

Thus, we can simply add R factors. Now, $P = \frac{80^\circ\text{C}}{A(R_{\text{steel}} + R_{\text{Al}})}$.

3. Find the R_s from the known quantities:

$$R_{\text{steel}} = 3.13 \times 10^{-3} \text{ m}^2 \cdot ^\circ\text{C/W}$$

and

$$R_{\text{Al}} = 1.14 \times 10^{-3} \text{ m}^2 \cdot ^\circ\text{C/W}.$$

- Substitute these values in to find $P = 1.47 \text{ W}$ as before.
- Determine ΔT for the aluminum rod (or for the steel rod) and use it to find T at the joint.

$$\Delta T_{\text{Al}} = \frac{PR_{\text{Al}}}{A} = \frac{(1.47 \text{ W}) (1.14 \times 10^{-3} \text{ m}^2 \cdot ^\circ\text{C/W})}{7.85 \times 10^{-5} \text{ m}^2} = 21.3^\circ\text{C},$$

so $T = 20^\circ\text{C} + 21.3^\circ\text{C} = 41.3^\circ\text{C}$, as in Solution 1.

6. If desired, check by determining ΔT for the other rod.

Significance

In practice, adding R values is common, as in calculating the R value of an insulated wall. In the analogous situation in electronics, the resistance corresponds to AR in this problem and is additive even when the areas are unequal, as is common in electronics. Our equation for heat conduction can be used only when the areas are equal; otherwise, we would have a problem in three-dimensional heat flow, which is beyond our scope.

CHECK YOUR UNDERSTANDING 1.7

How does the rate of heat transfer by conduction change when all spatial dimensions are doubled?

Conduction is caused by the random motion of atoms and molecules. As such, it is an ineffective mechanism for heat transport over macroscopic distances and short times. For example, the temperature on Earth would be unbearably cold during the night and extremely hot during the day if heat transport in the atmosphere were only through conduction. Also, car engines would overheat unless there was a more efficient way to remove excess heat from the pistons. The next module discusses the important heat-transfer mechanism in such situations.

Convection

In convection, thermal energy is carried by the large-scale flow of matter. It can be divided into two types. In *forced convection*, the flow is driven by fans, pumps, and the like. A simple example is a fan that blows air past you in hot surroundings and cools you by replacing the air heated by your body with cooler air. A more complicated example is the cooling system of a typical car, in which a pump moves coolant through the radiator and engine to cool the engine and a fan blows air to cool the radiator.

In *free or natural convection*, the flow is driven by buoyant forces: hot fluid rises and cold fluid sinks because density decreases as temperature increases. The house in [Figure 1.24](#) is kept warm by natural convection, as is the pot of water on the stove in [Figure 1.25](#). Ocean currents and large-scale atmospheric circulation, which result from the buoyancy of warm air and water, transfer hot air from the tropics toward the poles and cold air from the poles toward the tropics. (Earth's rotation interacts with those flows, causing the observed eastward flow of air in the temperate zones.)

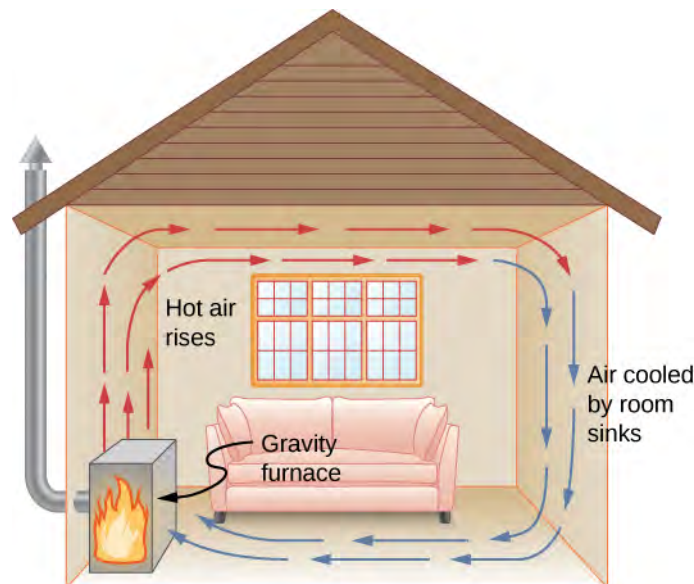


Figure 1.24 Air heated by a so-called gravity furnace expands and rises, forming a convective loop that transfers energy to other parts of the room. As the air is cooled at the ceiling and outside walls, it contracts, eventually becoming denser than room air and sinking to the floor. A properly designed heating system using natural convection, like this one, can heat a home quite efficiently.

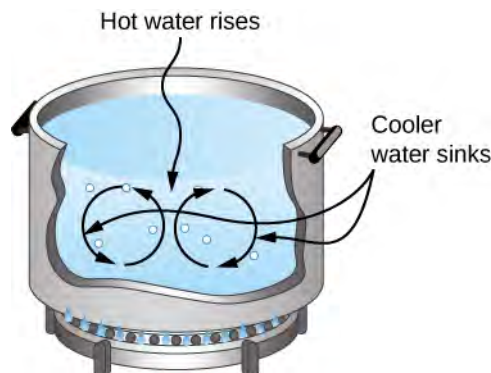


Figure 1.25 Natural convection plays an important role in heat transfer inside this pot of water. Once conducted to the inside, heat

transfer to other parts of the pot is mostly by convection. The hotter water expands, decreases in density, and rises to transfer heat to other regions of the water, while colder water sinks to the bottom. This process keeps repeating.

INTERACTIVE

Natural convection like that of [Figure 1.24](#) and [Figure 1.25](#), but acting on rock in Earth's mantle, drives [plate tectonics](#) (<https://openstax.org/l/21platetecton>) that are the motions that have shaped Earth's surface.

Convection is usually more complicated than conduction. Beyond noting that the convection rate is often approximately proportional to the temperature difference, we will not do any quantitative work comparable to the formula for conduction. However, we can describe convection qualitatively and relate convection rates to heat and time. Air is a poor conductor, so convection dominates heat transfer by air. Therefore, the amount of available space for airflow determines whether air transfers heat rapidly or slowly. There is little heat transfer in a space filled with air with a small amount of other material that prevents flow. The space between the inside and outside walls of a typical American house, for example, is about 9 cm (3.5 in.)—large enough for convection to work effectively. The addition of wall insulation prevents airflow, so heat loss (or gain) is decreased. On the other hand, the gap between the two panes of a double-paned window is about 1 cm, which largely prevents convection and takes advantage of air's low conductivity to reduce heat loss. Fur, cloth, and fiberglass also take advantage of the low conductivity of air by trapping it in spaces too small to support convection ([Figure 1.26](#)).

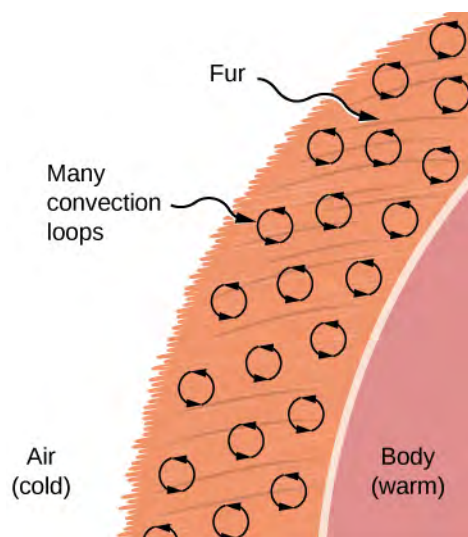


Figure 1.26 Fur is filled with air, breaking it up into many small pockets. Convection is very slow here, because the loops are so small. The low conductivity of air makes fur a very good lightweight insulator.

Some interesting phenomena happen when convection is accompanied by a phase change. The combination allows us to cool off by sweating even if the temperature of the surrounding air exceeds body temperature. Heat from the skin is required for sweat to evaporate from the skin, but without air flow, the air becomes saturated and evaporation stops. Air flow caused by convection replaces the saturated air by dry air and evaporation continues.



EXAMPLE 1.12

Calculating the Flow of Mass during Convection

The average person produces heat at the rate of about 120 W when at rest. At what rate must water evaporate from the body to get rid of all this energy? (For simplicity, we assume this evaporation occurs when a person is sitting in the shade and surrounding temperatures are the same as skin temperature, eliminating heat transfer by other methods.)

Strategy

Energy is needed for this phase change ($Q = mL_v$). Thus, the energy loss per unit time is

$$\frac{Q}{t} = \frac{mL_v}{t} = 120 \text{ W} = 120 \text{ J/s}.$$

We divide both sides of the equation by L_v to find that the mass evaporated per unit time is

$$\frac{m}{t} = \frac{120 \text{ J/s}}{L_v}.$$

Solution

Insert the value of the latent heat from [Table 1.4](#), $L_v = 2430 \text{ kJ/kg} = 2430 \text{ J/g}$. This yields

$$\frac{m}{t} = \frac{120 \text{ J/s}}{2430 \text{ J/g}} = 0.0494 \text{ g/s} = 2.96 \text{ g/min}.$$

Significance

Evaporating about 3 g/min seems reasonable. This would be about 180 g (about 7 oz.) per hour. If the air is very dry, the sweat may evaporate without even being noticed. A significant amount of evaporation also takes place in the lungs and breathing passages.

Another important example of the combination of phase change and convection occurs when water evaporates from the oceans. Heat is removed from the ocean when water evaporates. If the water vapor condenses in liquid droplets as clouds form, possibly far from the ocean, heat is released in the atmosphere. Thus, there is an overall transfer of heat from the ocean to the atmosphere. This process is the driving power behind thunderheads, those great cumulus clouds that rise as much as 20.0 km into the stratosphere ([Figure 1.27](#)). Water vapor carried in by convection condenses, releasing tremendous amounts of energy. This energy causes the air to expand and rise to colder altitudes. More condensation occurs in these regions, which in turn drives the cloud even higher. This mechanism is an example of positive feedback, since the process reinforces and accelerates itself. It sometimes produces violent storms, with lightning and hail. The same mechanism drives hurricanes.

INTERACTIVE

This [time-lapse video \(https://openstax.org/l/21convthuncurr\)](https://openstax.org/l/21convthuncurr) shows convection currents in a thunderstorm, including “rolling” motion similar to that of boiling water.



Figure 1.27 Cumulus clouds are caused by water vapor that rises because of convection. The rise of clouds is driven by a positive feedback mechanism. (credit: “Amada44”/Wikimedia Commons)

CHECK YOUR UNDERSTANDING 1.8

Explain why using a fan in the summer feels refreshing.

Radiation

You can feel the heat transfer from the Sun. The space between Earth and the Sun is largely empty, so the Sun warms us without any possibility of heat transfer by convection or conduction. Similarly, you can sometimes tell that the oven is hot without touching its door or looking inside—it may just warm you as you walk by. In these examples, heat is transferred by radiation ([Figure 1.28](#)). That is, the hot body emits electromagnetic waves that are absorbed by the skin. No medium is required for electromagnetic waves to propagate. Different names are used for electromagnetic waves of different wavelengths: radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and gamma rays.



Figure 1.28 Most of the heat transfer from this fire to the observers occurs through infrared radiation. The visible light, although dramatic, transfers relatively little thermal energy. Convection transfers energy away from the observers as hot air rises, while conduction is negligibly slow here. Skin is very sensitive to infrared radiation, so you can sense the presence of a fire without looking at it directly. (credit: Daniel O'Neil)

The energy of electromagnetic radiation varies over a wide range, depending on the wavelength: A shorter wavelength (or higher frequency) corresponds to a higher energy. Because more heat is radiated at higher temperatures, higher temperatures produce more intensity at every wavelength but especially at shorter wavelengths. In visible light, wavelength determines color—red has the longest wavelength and violet the shortest—so a temperature change is accompanied by a color change. For example, an electric heating element on a stove glows from red to orange, while the higher-temperature steel in a blast furnace glows from yellow to white. Infrared radiation is the predominant form radiated by objects cooler than the electric element and the steel. The radiated energy as a function of wavelength depends on its intensity, which is represented in [Figure 1.29](#) by the height of the distribution. ([Electromagnetic Waves](#) explains more about the electromagnetic spectrum, and [Photons and Matter Waves](#) discusses why the decrease in wavelength corresponds to an increase in energy.)

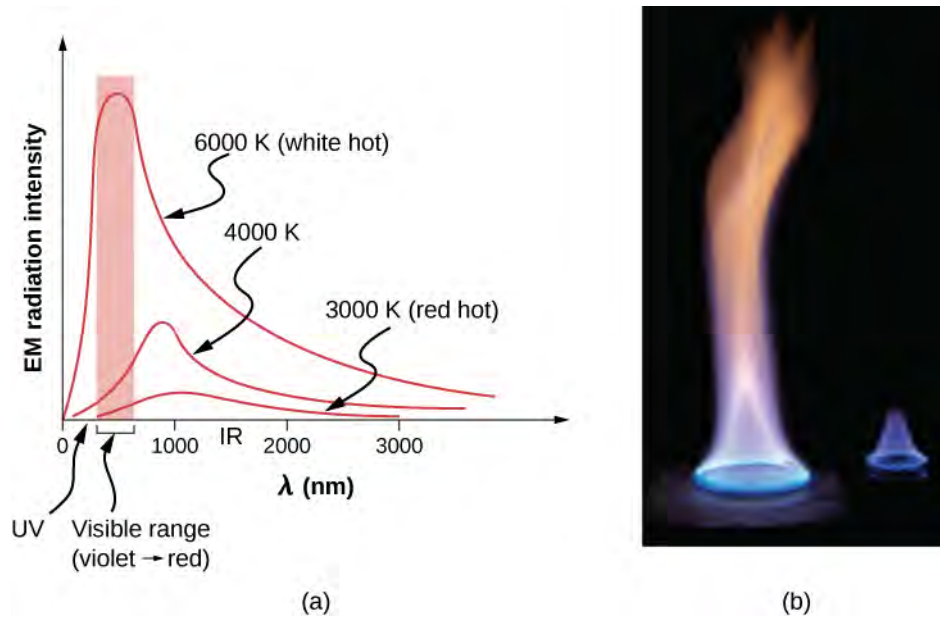


Figure 1.29 (a) A graph of the spectrum of electromagnetic waves emitted from an ideal radiator at three different temperatures. The intensity or rate of radiation emission increases dramatically with temperature, and the spectrum shifts down in wavelength toward the visible and ultraviolet parts of the spectrum. The shaded portion denotes the visible part of the spectrum. It is apparent that the shift toward the ultraviolet with temperature makes the visible appearance shift from red to white to blue as temperature increases. (b) Note the variations in color corresponding to variations in flame temperature.

The rate of heat transfer by radiation also depends on the object's color. Black is the most effective, and white is the least effective. On a clear summer day, black asphalt in a parking lot is hotter than adjacent gray sidewalk, because black absorbs better than gray (Figure 1.30). The reverse is also true—black radiates better than gray. Thus, on a clear summer night, the asphalt is colder than the gray sidewalk, because black radiates the energy more rapidly than gray. A perfectly black object would be an *ideal radiator* and an *ideal absorber*, as it would capture all the radiation that falls on it. In contrast, a perfectly white object or a perfect mirror would reflect all radiation, and a perfectly transparent object would transmit it all (Figure 1.31). Such objects would not emit any radiation. Mathematically, the color is represented by the **emissivity** e . A “blackbody” radiator would have an $e = 1$, whereas a perfect reflector or transmitter would have $e = 0$. For real examples, tungsten light bulb filaments have an e of about 0.5, and carbon black (a material used in printer toner) has an emissivity of about 0.95.

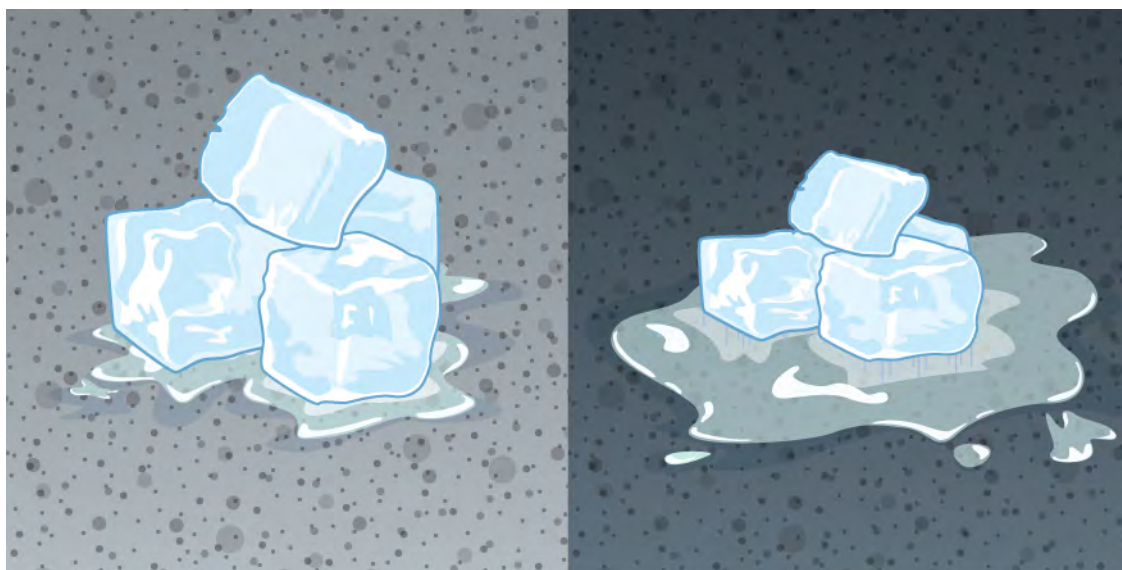


Figure 1.30 The darker pavement is hotter than the lighter pavement (much more of the ice on the right has melted), although both have been in the sunlight for the same time. The thermal conductivities of the pavements are the same.

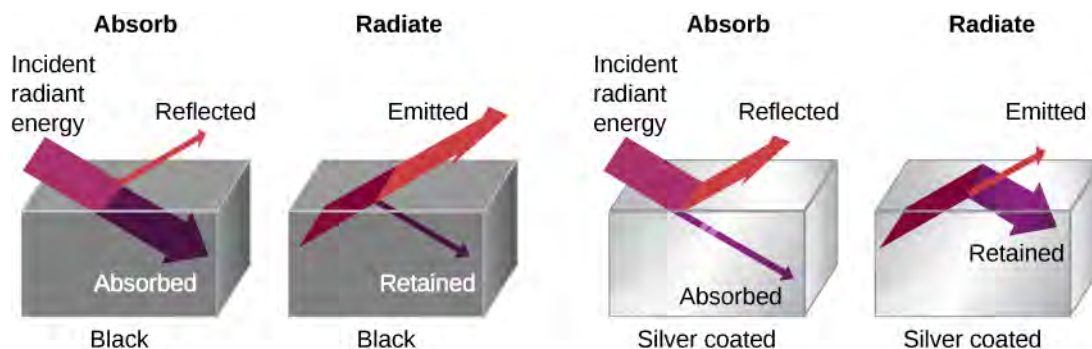


Figure 1.31 A black object is a good absorber and a good radiator, whereas a white, clear, or silver object is a poor absorber and a poor radiator.

To see that, consider a silver object and a black object that can exchange heat by radiation and are in thermal equilibrium. We know from experience that they will stay in equilibrium (the result of a principle that will be discussed at length in [Second Law of Thermodynamics](#)). For the black object's temperature to stay constant, it must emit as much radiation as it absorbs, so it must be as good at radiating as absorbing. Similar considerations show that the silver object must radiate as little as it absorbs. Thus, one property, emissivity, controls both radiation and absorption.

Finally, the radiated heat is proportional to the object's surface area, since every part of the surface radiates. If you knock apart the coals of a fire, the radiation increases noticeably due to an increase in radiating surface area.

The rate of heat transfer by emitted radiation is described by the **Stefan-Boltzmann law of radiation**:

$$P = \sigma A e T^4,$$

where $\sigma = 5.67 \times 10^{-8} \text{ J/s} \cdot \text{m}^2 \cdot \text{K}^4$ is the Stefan-Boltzmann constant, a combination of fundamental constants of nature; A is the surface area of the object; and T is its temperature in kelvins.

The proportionality to the *fourth power* of the absolute temperature is a remarkably strong temperature dependence. It allows the detection of even small temperature variations. Images called *thermographs* can be used medically to detect regions of abnormally high temperature in the body, perhaps indicative of disease. Similar techniques can be used to detect heat leaks in homes ([Figure 1.32](#)), optimize performance of blast furnaces, improve comfort levels in work environments, and even remotely map Earth's temperature profile.



Figure 1.32 A thermograph of part of a building shows temperature variations, indicating where heat transfer to the outside is most severe. Windows are a major region of heat transfer to the outside of homes. (credit: US Army)

The Stefan-Boltzmann equation needs only slight refinement to deal with a simple case of an object's absorption of radiation from its surroundings. Assuming that an object with a temperature T_1 is surrounded by an environment with uniform temperature T_2 , the **net rate of heat transfer by radiation** is

$$P_{\text{net}} = \sigma e A (T_2^4 - T_1^4), \quad 1.10$$

where e is the emissivity of the object alone. In other words, it does not matter whether the surroundings are white, gray, or black: The balance of radiation into and out of the object depends on how well it emits and absorbs radiation. When $T_2 > T_1$, the quantity P_{net} is positive, that is, the net heat transfer is from hot to cold.

Before doing an example, we have a complication to discuss: different emissivities at different wavelengths. If the fraction of incident radiation an object reflects is the same at all visible wavelengths, the object is gray; if the fraction depends on the wavelength, the object has some other color. For instance, a red or reddish object reflects red light more strongly than other visible wavelengths. Because it absorbs less red, it radiates less red when hot. Differential reflection and absorption of wavelengths outside the visible range have no effect on what we see, but they may have physically important effects. Skin is a very good absorber and emitter of infrared radiation, having an emissivity of 0.97 in the infrared spectrum. Thus, in spite of the obvious variations in skin color, we are all nearly black in the infrared. This high infrared emissivity is why we can so easily feel radiation on our skin. It is also the basis for the effectiveness of night-vision scopes used by law enforcement and the military to detect human beings.



EXAMPLE 1.13

Calculating the Net Heat Transfer of a Person

What is the rate of heat transfer by radiation of an unclothed person standing in a dark room whose ambient temperature is 22.0°C ? The person has a normal skin temperature of 33.0°C and a surface area of 1.50 m^2 . The emissivity of skin is 0.97 in the infrared, the part of the spectrum where the radiation takes place.

Strategy

We can solve this by using the equation for the rate of radiative heat transfer.

Solution

Insert the temperature values $T_2 = 295\text{ K}$ and $T_1 = 306\text{ K}$, so that

$$\begin{aligned}
 \frac{Q}{t} &= \sigma e A (T_2^4 - T_1^4) \\
 &= (5.67 \times 10^{-8} \text{ J/s} \cdot \text{m}^2 \cdot \text{K}^4) (0.97) (1.50 \text{ m}^2) [(295 \text{ K})^4 - (306 \text{ K})^4] \\
 &= -99 \text{ J/s} = -99 \text{ W}.
 \end{aligned}$$

Significance

This value is a significant rate of heat transfer to the environment (note the minus sign), considering that a person at rest may produce energy at the rate of 125 W and that conduction and convection are also transferring energy to the environment. Indeed, we would probably expect this person to feel cold. Clothing significantly reduces heat transfer to the environment by all mechanisms, because clothing slows down both conduction and convection, and has a lower emissivity (especially if it is light-colored) than skin.

The average temperature of Earth is the subject of much current discussion. Earth is in radiative contact with both the Sun and dark space, so we cannot use the equation for an environment at a uniform temperature. Earth receives almost all its energy from radiation of the Sun and reflects some of it back into outer space. Conversely, dark space is very cold, about 3 K, so that Earth radiates energy into the dark sky. The rate of heat transfer from soil and grasses can be so rapid that frost may occur on clear summer evenings, even in warm latitudes.

The average temperature of Earth is determined by its energy balance. To a first approximation, it is the temperature at which Earth radiates heat to space as fast as it receives energy from the Sun.

An important parameter in calculating the temperature of Earth is its emissivity (e). On average, it is about 0.65, but calculation of this value is complicated by the great day-to-day variation in the highly reflective cloud coverage. Because clouds have lower emissivity than either oceans or land masses, they emit some of the radiation back to the surface, greatly reducing heat transfer into dark space, just as they greatly reduce heat transfer into the atmosphere during the day. There is negative feedback (in which a change produces an effect that opposes that change) between clouds and heat transfer; higher temperatures evaporate more water to form more clouds, which reflect more radiation back into space, reducing the temperature.

The often-mentioned **greenhouse effect** is directly related to the variation of Earth's emissivity with wavelength ([Figure 1.33](#)). The greenhouse effect is a natural phenomenon responsible for providing temperatures suitable for life on Earth and for making Venus unsuitable for human life. Most of the infrared radiation emitted from Earth is absorbed by carbon dioxide (CO_2) and water (H_2O) in the atmosphere and then re-radiated into outer space or back to Earth. Re-radiation back to Earth maintains its surface temperature about 40 °C higher than it would be if there were no atmosphere. (The glass walls and roof of a greenhouse increase the temperature inside by blocking convective heat losses, not radiative losses.)

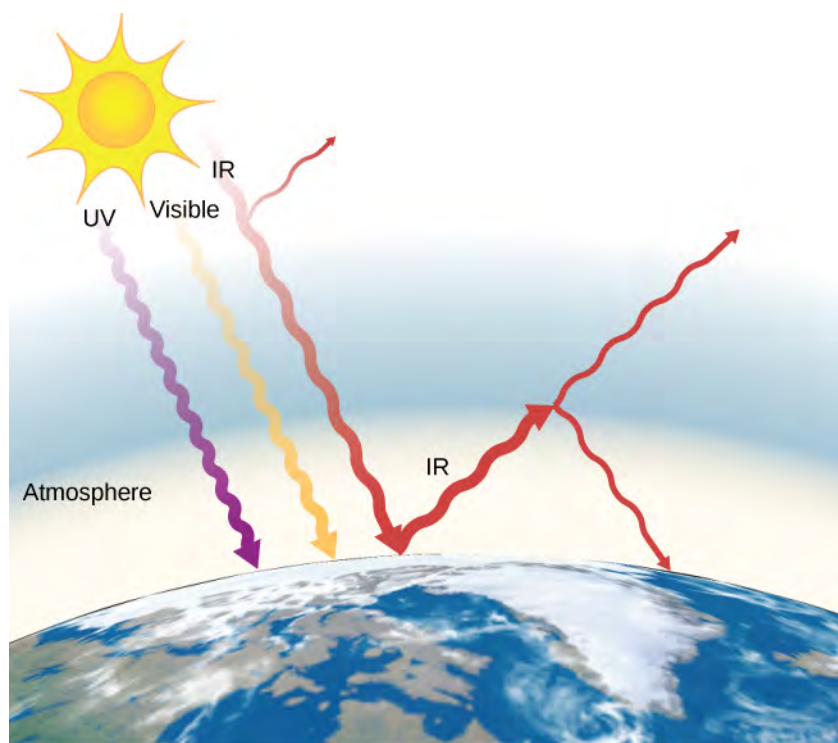


Figure 1.33 The greenhouse effect is the name given to the increase of Earth's temperature due to absorption of radiation in the atmosphere. The atmosphere is transparent to incoming visible radiation and most of the Sun's infrared. The Earth absorbs that energy and re-emits it. Since Earth's temperature is much lower than the Sun's, it re-emits the energy at much longer wavelengths, in the infrared. The atmosphere absorbs much of that infrared radiation and radiates about half of the energy back down, keeping Earth warmer than it would otherwise be. The amount of trapping depends on concentrations of trace gases such as carbon dioxide, and an increase in the concentration of these gases increases Earth's surface temperature.

The greenhouse effect is central to the discussion of global warming due to emission of carbon dioxide and methane (and other greenhouse gases) into Earth's atmosphere from industry, transportation, and farming. Changes in global climate could lead to more intense storms, precipitation changes (affecting agriculture), reduction in rain forest biodiversity, and rising sea levels.

INTERACTIVE

You can explore [a simulation of the greenhouse effect \(https://openstax.org/l/21simgreeneff\)](https://openstax.org/l/21simgreeneff) that takes the point of view that the atmosphere scatters (redirects) infrared radiation rather than absorbing it and reradiating it. You may want to run the simulation first with no greenhouse gases in the atmosphere and then look at how adding greenhouse gases affects the infrared radiation from the Earth and the Earth's temperature.



PROBLEM-SOLVING STRATEGY

Effects of Heat Transfer

1. Examine the situation to determine what type of heat transfer is involved.
2. Identify the type(s) of heat transfer—conduction, convection, or radiation.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
4. Make a list of what is given or what can be inferred from the problem as stated (identify the knowns).
5. Solve the appropriate equation for the quantity to be determined (the unknown).
6. For conduction, use the equation $P = \frac{kA\Delta T}{d}$. Table 1.5 lists thermal conductivities. For convection, determine the amount of matter moved and the equation $Q = mc\Delta T$, along with $Q = mL_f$ or $Q = mL_v$ if

a substance changes phase. For radiation, the equation $P_{\text{net}} = \sigma e A (T_2^4 - T_1^4)$ gives the net heat transfer rate.

7. Substitute the knowns along with their units into the appropriate equation and obtain numerical solutions complete with units.
 8. Check the answer to see if it is reasonable. Does it make sense?
-

CHECK YOUR UNDERSTANDING 1.9

How much greater is the rate of heat radiation when a body is at the temperature 40°C than when it is at the temperature 20°C ?

CHAPTER REVIEW

Key Terms

absolute temperature scale scale, such as Kelvin, with a zero point that is absolute zero

absolute zero temperature at which the average kinetic energy of molecules is zero

calorie (cal) energy needed to change the temperature of 1.00 g of water by 1.00 °C

calorimeter container that prevents heat transfer in or out

calorimetry study of heat transfer inside a container impervious to heat

Celsius scale temperature scale in which the freezing point of water is 0 °C and the boiling point of water is 100 °C

coefficient of linear expansion (α) material property that gives the change in length, per unit length, per 1-°C change in temperature; a constant used in the calculation of linear expansion; the coefficient of linear expansion depends to some degree on the temperature of the material

coefficient of volume expansion (β) similar to α but gives the change in volume, per unit volume, per 1-°C change in temperature

conduction heat transfer through stationary matter by physical contact

convection heat transfer by the macroscopic movement of fluid

critical point for a given substance, the combination of temperature and pressure above which the liquid and gas phases are indistinguishable

critical pressure pressure at the critical point

critical temperature temperature at the critical point

degree Celsius (°C) unit on the Celsius temperature scale

degree Fahrenheit (°F) unit on the Fahrenheit temperature scale

emissivity measure of how well an object radiates

Fahrenheit scale temperature scale in which the freezing point of water is 32 °F and the boiling point of water is 212 °F

greenhouse effect warming of the earth that is due to gases such as carbon dioxide and methane that absorb infrared radiation from Earth's surface and reradiate it in all directions, thus sending some of it back toward Earth

heat energy transferred solely due to a temperature difference

heat of fusion energy per unit mass required to

change a substance from the solid phase to the liquid phase, or released when the substance changes from liquid to solid

heat of sublimation energy per unit mass required to change a substance from the solid phase to the vapor phase

heat of vaporization energy per unit mass required to change a substance from the liquid phase to the vapor phase

heat transfer movement of energy from one place or material to another as a result of a difference in temperature

Kelvin scale (K) temperature scale in which 0 K is the lowest possible temperature, representing absolute zero

kilocalorie (kcal) energy needed to change the temperature of 1.00 kg of water between 14.5 °C and 15.5 °C

latent heat coefficient general term for the heats of fusion, vaporization, and sublimation

mechanical equivalent of heat work needed to produce the same effects as heat transfer

net rate of heat transfer by radiation

$$P_{\text{net}} = \sigma e A (T_2^4 - T_1^4)$$

phase diagram graph of pressure vs. temperature of a particular substance, showing at which pressures and temperatures the phases of the substance occur

radiation energy transferred by electromagnetic waves directly as a result of a temperature difference

rate of conductive heat transfer rate of heat transfer from one material to another

specific heat amount of heat necessary to change the temperature of 1.00 kg of a substance by 1.00 °C; also called "specific heat capacity"

Stefan-Boltzmann law of radiation $P = \sigma A e T^4$, where $\sigma = 5.67 \times 10^{-8} \text{ J/s} \cdot \text{m}^2 \cdot \text{K}^4$ is the Stefan-Boltzmann constant, A is the surface area of the object, T is the absolute temperature, and e is the emissivity

sublimation phase change from solid to gas

temperature quantity measured by a thermometer, which reflects the mechanical energy of molecules in a system

thermal conductivity property of a material describing its ability to conduct heat

thermal equilibrium condition in which heat no longer flows between two objects that are in contact; the two objects have the same

temperature

thermal expansion change in size or volume of an object with change in temperature

thermal stress stress caused by thermal expansion or contraction

triple point pressure and temperature at which a substance exists in equilibrium as a solid, liquid, and gas

vapor gas at a temperature below the boiling

temperature

vapor pressure pressure at which a gas coexists with its solid or liquid phase

zeroth law of thermodynamics law that states that if two objects are in thermal equilibrium, and a third object is in thermal equilibrium with one of those objects, it is also in thermal equilibrium with the other object

Key Equations

Linear thermal expansion

$$\Delta L = \alpha L \Delta T$$

Thermal expansion in two dimensions

$$\Delta A = 2\alpha A \Delta T$$

Thermal expansion in three dimensions

$$\Delta V = \beta V \Delta T$$

Heat transfer

$$Q = mc\Delta T$$

Transfer of heat in a calorimeter

$$Q_{\text{cold}} + Q_{\text{hot}} = 0$$

Heat due to phase change (melting and freezing)

$$Q = mL_f$$

Heat due to phase change (evaporation and condensation)

$$Q = mL_v$$

Rate of conductive heat transfer

$$P = \frac{kA(T_h - T_c)}{d}$$

Net rate of heat transfer by radiation

$$P_{\text{net}} = \sigma eA(T_2^4 - T_1^4)$$

Summary

1.1 Temperature and Thermal Equilibrium

- Temperature is operationally defined as the quantity measured by a thermometer. It is proportional to the average kinetic energy of atoms and molecules in a system.
- Thermal equilibrium occurs when two bodies are in contact with each other and can freely exchange energy. Systems are in thermal equilibrium when they have the same temperature.
- The zeroth law of thermodynamics states that when two systems, *A* and *B*, are in thermal equilibrium with each other, and *B* is in thermal equilibrium with a third system *C*, then *A* is also in thermal equilibrium with *C*.

1.2 Thermometers and Temperature Scales

- Three types of thermometers are alcohol, liquid crystal, and infrared radiation (pyrometer).

- The three main temperature scales are Celsius, Fahrenheit, and Kelvin. Temperatures can be converted from one scale to another using temperature conversion equations.
- The three phases of water (ice, liquid water, and water vapor) can coexist at a single pressure and temperature known as the triple point.

1.3 Thermal Expansion

- Thermal expansion is the increase of the size (length, area, or volume) of a body due to a change in temperature, usually a rise. Thermal contraction is the decrease in size due to a change in temperature, usually a fall in temperature.
- Thermal stress is created when thermal expansion or contraction is constrained.

1.4 Heat Transfer, Specific Heat, and Calorimetry

- Heat and work are the two distinct methods of energy transfer.
- Heat transfer to an object when its temperature changes is often approximated well by $Q = mc\Delta T$, where m is the object's mass and c is the specific heat of the substance.

1.5 Phase Changes

- Most substances have three distinct phases (under ordinary conditions on Earth), and they depend on temperature and pressure.
- Two phases coexist (i.e., they are in thermal equilibrium) at a set of pressures and temperatures.
- Phase changes occur at fixed temperatures for a given substance at a given pressure, and these temperatures are called boiling, freezing (or melting), and sublimation points.

1.6 Mechanisms of Heat Transfer

- Heat is transferred by three different methods: conduction, convection, and radiation.
- Heat conduction is the transfer of heat between two objects in direct contact with each other.
- The rate of heat transfer P (energy per unit time) is proportional to the temperature difference $T_h - T_c$ and the contact area A and inversely

proportional to the distance d between the objects.

- Convection is heat transfer by the macroscopic movement of mass. Convection can be natural or forced, and generally transfers thermal energy faster than conduction. Convection that occurs along with a phase change can transfer energy from cold regions to warm ones.
- Radiation is heat transfer through the emission or absorption of electromagnetic waves.
- The rate of radiative heat transfer is proportional to the emissivity e . For a perfect blackbody, $e = 1$, whereas a perfectly white, clear, or reflective body has $e = 0$, with real objects having values of e between 1 and 0.
- The rate of heat transfer depends on the surface area and the fourth power of the absolute temperature:

$$P = \sigma e A T^4,$$

where $\sigma = 5.67 \times 10^{-8} \text{ J/s} \cdot \text{m}^2 \cdot \text{K}^4$ is the Stefan-Boltzmann constant and e is the emissivity of the body. The net rate of heat transfer from an object by radiation is

$$\frac{Q_{\text{net}}}{t} = \sigma e A (T_2^4 - T_1^4),$$

where T_1 is the temperature of the object surrounded by an environment with uniform temperature T_2 and e is the emissivity of the object.

Conceptual Questions

1.1 Temperature and Thermal Equilibrium

- What does it mean to say that two systems are in thermal equilibrium?
- Give an example in which A has some kind of non-thermal equilibrium relationship with B , and B has the same relationship with C , but A does not have that relationship with C .

1.2 Thermometers and Temperature Scales

- If a thermometer is allowed to come to equilibrium with the air, and a glass of water is not in equilibrium with the air, what will happen to the thermometer reading when it is placed in the water?
- Give an example of a physical property that varies with temperature and describe how it is used to measure temperature.

1.3 Thermal Expansion

- Pouring cold water into hot glass or ceramic cookware can easily break it. What causes the breaking? Explain why Pyrex®, a glass with a small coefficient of linear expansion, is less susceptible.
- One method of getting a tight fit, say of a metal peg in a hole in a metal block, is to manufacture the peg slightly larger than the hole. The peg is then inserted when at a different temperature than the block. Should the block be hotter or colder than the peg during insertion? Explain your answer.
- Does it really help to run hot water over a tight metal lid on a glass jar before trying to open it? Explain your answer.
- When a cold alcohol thermometer is placed in a hot liquid, the column of alcohol goes *down* slightly before going up. Explain why.
- Calculate the length of a 1-meter rod of a material

with thermal expansion coefficient α when the temperature is raised from 300 K to 600 K.

Taking your answer as the new initial length, find the length after the rod is cooled back down to 300 K. Is your answer 1 meter? Should it be? How can you account for the result you got?

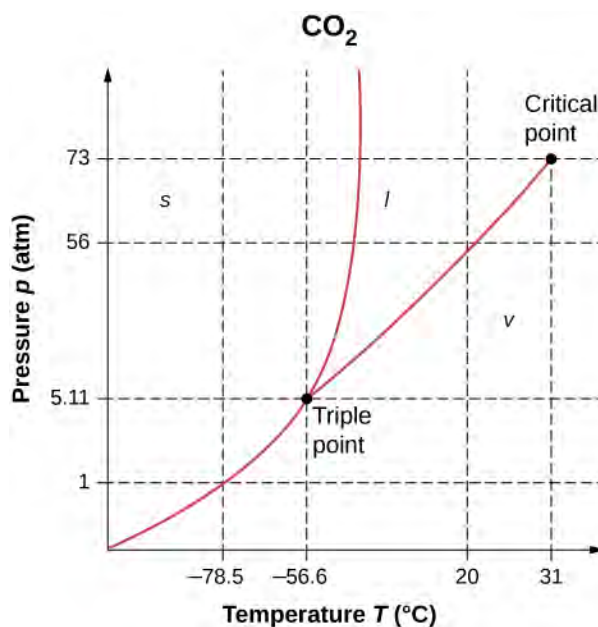
10. Noting the large stresses that can be caused by thermal expansion, an amateur weapon inventor decides to use it to make a new kind of gun. He plans to jam a bullet against an aluminum rod inside a closed invar tube. When he heats the tube, the rod will expand more than the tube and a very strong force will build up. Then, by a method yet to be determined, he will open the tube in a split second and let the force of the rod launch the bullet at very high speed. What is he overlooking?

1.4 Heat Transfer, Specific Heat, and Calorimetry

11. How is heat transfer related to temperature?
12. Describe a situation in which heat transfer occurs.
13. When heat transfers into a system, is the energy stored as heat? Explain briefly.
14. The brakes in a car increase in temperature by ΔT when bringing the car to rest from a speed v . How much greater would ΔT be if the car initially had twice the speed? You may assume the car stops fast enough that no heat transfers out of the brakes.

1.5 Phase Changes

15. A pressure cooker contains water and steam in equilibrium at a pressure greater than atmospheric pressure. How does this greater pressure increase cooking speed?
16. As shown below, which is the phase diagram for carbon dioxide, what is the vapor pressure of solid carbon dioxide (dry ice) at -78.5°C ? (Note that the axes in the figure are nonlinear and the graph is not to scale.)



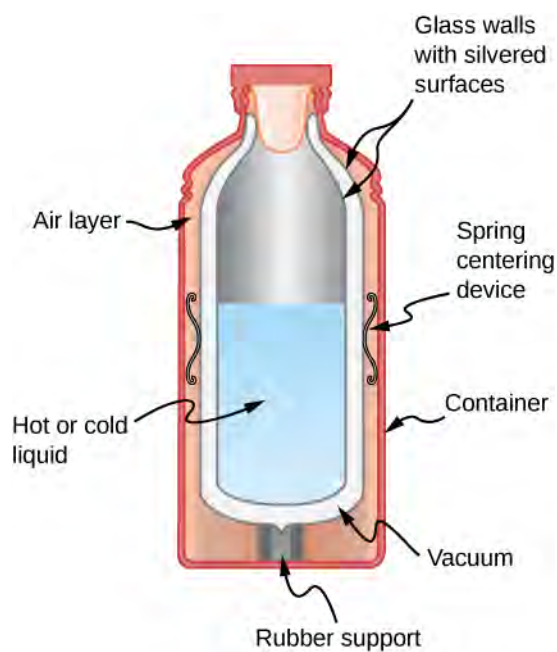
17. Can carbon dioxide be liquefied at room temperature (20°C)? If so, how? If not, why not? (See the phase diagram in the preceding problem.)
18. What is the distinction between gas and vapor?
19. Heat transfer can cause temperature and phase changes. What else can cause these changes?
20. How does the latent heat of fusion of water help slow the decrease of air temperatures, perhaps preventing temperatures from falling significantly below 0°C , in the vicinity of large bodies of water?
21. What is the temperature of ice right after it is formed by freezing water?
22. If you place 0°C ice into 0°C water in an insulated container, what will the net result be? Will there be less ice and more liquid water, or more ice and less liquid water, or will the amounts stay the same?
23. What effect does condensation on a glass of ice water have on the rate at which the ice melts? Will the condensation speed up the melting process or slow it down?
24. In Miami, Florida, which has a very humid climate and numerous bodies of water nearby, it is unusual for temperatures to rise above about 38°C (100°F). In the desert climate of Phoenix, Arizona, however, temperatures rise above that almost every day in July and August. Explain how the evaporation of water helps limit high temperatures in humid climates.
25. In winter, it is often warmer in San Francisco than in Sacramento, 150 km inland. In summer, it is nearly always hotter in Sacramento. Explain how the bodies of water surrounding San

Francisco moderate its extreme temperatures.

26. Freeze-dried foods have been dehydrated in a vacuum. During the process, the food freezes and must be heated to facilitate dehydration. Explain both how the vacuum speeds up dehydration and why the food freezes as a result.
27. In a physics classroom demonstration, an instructor inflates a balloon by mouth and then cools it in liquid nitrogen. When cold, the shrunken balloon has a small amount of light blue liquid in it, as well as some snow-like crystals. As it warms up, the liquid boils, and part of the crystals sublime, with some crystals lingering for a while and then producing a liquid. Identify the blue liquid and the two solids in the cold balloon. Justify your identifications using data from [Table 1.4](#).

1.6 Mechanisms of Heat Transfer

28. What are the main methods of heat transfer from the hot core of Earth to its surface? From Earth's surface to outer space?
29. When our bodies get too warm, they respond by sweating and increasing blood circulation to the surface to transfer thermal energy away from the core. What effect will those processes have on a person in a 40.0°C hot tub?
30. Shown below is a cut-away drawing of a thermos bottle (also known as a Dewar flask), which is a device designed specifically to slow down all forms of heat transfer. Explain the functions of the various parts, such as the vacuum, the silvering of the walls, the thin-walled long glass neck, the rubber support, the air layer, and the stopper.



31. Some electric stoves have a flat ceramic surface with heating elements hidden beneath. A pot placed over a heating element will be heated, while the surface only a few centimeters away is safe to touch. Why is ceramic, with a conductivity less than that of a metal but greater than that of a good insulator, an ideal choice for the stove top?
32. Loose-fitting white clothing covering most of the body, shown below, is ideal for desert dwellers, both in the hot Sun and during cold evenings. Explain how such clothing is advantageous during both day and night.



33. One way to make a fireplace more energy-efficient is to have room air circulate around the outside of the fire box and back into the room. Detail the methods of heat transfer involved.
34. On cold, clear nights horses will sleep under the cover of large trees. How does this help them keep warm?
35. When watching a circus during the day in a large, dark-colored tent, you sense significant heat transfer from the tent. Explain why this occurs.
36. Satellites designed to observe the radiation from cold (3 K) dark space have sensors that are shaded from the Sun, Earth, and the Moon and are cooled to very low temperatures. Why must the sensors be at low temperature?
37. Why are thermometers that are used in weather stations shielded from the sunshine? What does a thermometer measure if it is shielded from the sunshine? What does it measure if it is not?
38. Putting a lid on a boiling pot greatly reduces the heat transfer necessary to keep it boiling. Explain why.
39. Your house will be empty for a while in cold weather, and you want to save energy and money. Should you turn the thermostat down to the lowest level that will protect the house from damage such as freezing pipes, or leave it at the normal temperature? (If you don't like coming back to a cold house, imagine that a timer controls the heating system so the house will be warm when you get back.) Explain your answer.
40. You pour coffee into an unlidged cup, intending to drink it 5 minutes later. You can add cream when you pour the cup or right before you drink it. (The cream is at the same temperature either way. Assume that the cream and coffee come into thermal equilibrium with each other very quickly.) Which way will give you hotter coffee? What feature of this question is different from the previous one?
41. Broiling is a method of cooking by radiation, which produces somewhat different results from cooking by conduction or convection. A gas flame or electric heating element produces a very high temperature close to the food and *above* it. Why is radiation the dominant heat-transfer method in this situation?
42. On a cold winter morning, why does the metal of a bike feel colder than the wood of a porch?

Problems

1.2 Thermometers and Temperature Scales

43. While traveling outside the United States, you feel sick. A companion gets you a thermometer, which says your temperature is 39. What scale is that on? What is your Fahrenheit temperature? Should you seek medical help?
44. What are the following temperatures on the Kelvin scale?
 - (a) 68.0 °F, an indoor temperature sometimes recommended for energy conservation in winter
 - (b) 134 °F, one of the highest atmospheric temperatures ever recorded on Earth (Death Valley, California, 1913)
 - (c) 9890 °F, the temperature of the surface of the Sun
45. (a) Suppose a cold front blows into your locale and drops the temperature by 40.0 Fahrenheit degrees. How many degrees Celsius does the temperature decrease when it decreases by 40.0 °F? (b) Show that any change in temperature in Fahrenheit degrees is nine-fifths the change in Celsius degrees
46. An Associated Press article on climate change said, "Some of the ice shelf's disappearance was probably during times when the planet was 36 degrees Fahrenheit (2 degrees Celsius) to 37 degrees Fahrenheit (3 degrees Celsius) warmer than it is today." What mistake did the reporter make?
47. (a) At what temperature do the Fahrenheit and Celsius scales have the same numerical value? (b) At what temperature do the Fahrenheit and Kelvin scales have the same numerical value?
48. A person taking a reading of the temperature in a freezer in Celsius makes two mistakes: first omitting the negative sign and then thinking the temperature is Fahrenheit. That is, the person reads $-x$ °C as x °F. Oddly enough, the result is the correct Fahrenheit temperature. What is the original Celsius reading? Round your answer to three significant figures.

1.3 Thermal Expansion

49. The height of the Washington Monument is measured to be 170.00 m on a day when the temperature is 35.0 °C. What will its height be on a day when the temperature falls to -10.0 °C? Although the monument is made of limestone, assume that its coefficient of thermal

- expansion is the same as that of marble. Give your answer to five significant figures.
50. How much taller does the Eiffel Tower become at the end of a day when the temperature has increased by 15°C ? Its original height is 321 m and you can assume it is made of steel.
 51. What is the change in length of a 3.00-cm-long column of mercury if its temperature changes from 37.0°C to 40.0°C , assuming the mercury is constrained to a cylinder but unconstrained in length? Your answer will show why thermometers contain bulbs at the bottom instead of simple columns of liquid.
 52. How large an expansion gap should be left between steel railroad rails if they may reach a maximum temperature 35.0°C greater than when they were laid? Their original length is 10.0 m.
 53. You are looking to buy a small piece of land in Hong Kong. The price is “only” \$60,000 per square meter. The land title says the dimensions are $20\text{ m} \times 30\text{ m}$. By how much would the total price change if you measured the parcel with a steel tape measure on a day when the temperature was 20°C above the temperature that the tape measure was designed for? The dimensions of the land do not change.
 54. Global warming will produce rising sea levels partly due to melting ice caps and partly due to the expansion of water as average ocean temperatures rise. To get some idea of the size of this effect, calculate the change in length of a column of water 1.00 km high for a temperature increase of 1.00°C . Assume the column is not free to expand sideways. As a model of the ocean, that is a reasonable approximation, as only parts of the ocean very close to the surface can expand sideways onto land, and only to a limited degree. As another approximation, neglect the fact that ocean warming is not uniform with depth.
 55. (a) Suppose a meter stick made of steel and one made of aluminum are the same length at 0°C . What is their difference in length at 22.0°C ? (b) Repeat the calculation for two 30.0-m-long surveyor’s tapes.
 56. (a) If a 500-mL glass beaker is filled to the brim with ethyl alcohol at a temperature of 5.00°C , how much will overflow when the alcohol’s temperature reaches the room temperature of 22.0°C ? (b) How much less water would overflow under the same conditions?
 57. Most cars have a coolant reservoir to catch radiator fluid that may overflow when the engine is hot. A radiator is made of copper and is filled to its 16.0-L capacity when at 10.0°C . What volume of radiator fluid will overflow when the radiator and fluid reach a temperature of 95.0°C , given that the fluid’s volume coefficient of expansion is $\beta = 400 \times 10^{-6}/^{\circ}\text{C}$? (Your answer will be a conservative estimate, as most car radiators have operating temperatures greater than 95.0°C).
 58. A physicist makes a cup of instant coffee and notices that, as the coffee cools, its level drops 3.00 mm in the glass cup. Show that this decrease cannot be due to thermal contraction by calculating the decrease in level if the 350 cm^3 of coffee is in a 7.00-cm-diameter cup and decreases in temperature from 95.0°C to 45.0°C . (Most of the drop in level is actually due to escaping bubbles of air.)
 59. The density of water at 0°C is very nearly 1000 kg/m^3 (it is actually 999.84 kg/m^3), whereas the density of ice at 0°C is 917 kg/m^3 . Calculate the pressure necessary to keep ice from expanding when it freezes, neglecting the effect such a large pressure would have on the freezing temperature. (This problem gives you only an indication of how large the forces associated with freezing water might be.)
 60. Show that $\beta = 3\alpha$, by calculating the infinitesimal change in volume dV of a cube with sides of length L when the temperature changes by dT .

1.4 Heat Transfer, Specific Heat, and Calorimetry

61. On a hot day, the temperature of an 80,000-L swimming pool increases by 1.50°C . What is the net heat transfer during this heating? Ignore any complications, such as loss of water by evaporation.
62. To sterilize a 50.0-g glass baby bottle, we must raise its temperature from 22.0°C to 95.0°C . How much heat transfer is required?
63. The same heat transfer into identical masses of different substances produces different temperature changes. Calculate the final temperature when 1.00 kcal of heat transfers into 1.00 kg of the following, originally at 20.0°C : (a) water; (b) concrete; (c) steel; and (d) mercury.
64. Rubbing your hands together warms them by

converting work into thermal energy. If a woman rubs her hands back and forth for a total of 20 rubs, at a distance of 7.50 cm per rub, and with an average frictional force of 40.0 N, what is the temperature increase? The mass of tissues warmed is only 0.100 kg, mostly in the palms and fingers.

65. A 0.250-kg block of a pure material is heated from 20.0 °C to 65.0 °C by the addition of 4.35 kJ of energy. Calculate its specific heat and identify the substance of which it is most likely composed.
66. Suppose identical amounts of heat transfer into different masses of copper and water, causing identical changes in temperature. What is the ratio of the mass of copper to water?
67. (a) The number of kilocalories in food is determined by calorimetry techniques in which the food is burned and the amount of heat transfer is measured. How many kilocalories per gram are there in a 5.00-g peanut if the energy from burning it is transferred to 0.500 kg of water held in a 0.100-kg aluminum cup, causing a 54.9-°C temperature increase? Assume the process takes place in an ideal calorimeter, in other words a perfectly insulated container. (b) Compare your answer to the following labeling information found on a package of dry roasted peanuts: a serving of 33 g contains 200 calories. Comment on whether the values are consistent.
68. Following vigorous exercise, the body temperature of an 80.0 kg person is 40.0 °C. At what rate in watts must the person transfer thermal energy to reduce the body temperature to 37.0 °C in 30.0 min, assuming the body continues to produce energy at the rate of 150 W? (1 watt = 1 joule/second or 1 W = 1 J/s)
69. In a study of healthy young men¹, doing 20 push-ups in 1 minute burned an amount of energy per kg that for a 70.0-kg man corresponds to 8.06 calories (kcal). How much would a 70.0-kg man's temperature rise if he did not lose any heat during that time?
70. A 1.28-kg sample of water at 10.0 °C is in a calorimeter. You drop a piece of steel with a mass of 0.385 kg at 215 °C into it. After the sizzling subsides, what is the final equilibrium temperature? (Make the reasonable

assumptions that any steam produced condenses into liquid water during the process of equilibration and that the evaporation and condensation don't affect the outcome, as we'll see in the next section.)

71. Repeat the preceding problem, assuming the water is in a glass beaker with a mass of 0.200 kg, which in turn is in a calorimeter. The beaker is initially at the same temperature as the water. Before doing the problem, should the answer be higher or lower than the preceding answer? Comparing the mass and specific heat of the beaker to those of the water, do you think the beaker will make much difference?

1.5 Phase Changes

72. How much heat transfer (in kilocalories) is required to thaw a 0.450-kg package of frozen vegetables originally at 0 °C if their heat of fusion is the same as that of water?
73. A bag containing 0 °C ice is much more effective in absorbing energy than one containing the same amount of 0 °C water. (a) How much heat transfer is necessary to raise the temperature of 0.800 kg of water from 0 °C to 30.0 °C? (b) How much heat transfer is required to first melt 0.800 kg of 0 °C ice and then raise its temperature? (c) Explain how your answer supports the contention that the ice is more effective.
74. (a) How much heat transfer is required to raise the temperature of a 0.750-kg aluminum pot containing 2.50 kg of water from 30.0 °C to the boiling point and then boil away 0.750 kg of water? (b) How long does this take if the rate of heat transfer is 500 W?
75. Condensation on a glass of ice water causes the ice to melt faster than it would otherwise. If 8.00 g of vapor condense on a glass containing both water and 200 g of ice, how many grams of the ice will melt as a result? Assume no other heat transfer occurs. Use L_v for water at 37 °C as a better approximation than L_v for water at 100 °C.)
76. On a trip, you notice that a 3.50-kg bag of ice lasts an average of one day in your cooler. What is the average power in watts entering the ice if it starts at 0 °C and completely melts to 0 °C water in exactly one day?
77. On a certain dry sunny day, a swimming pool's

1 JW Vezina, "An examination of the differences between two methods of estimating energy expenditure in resistance training activities," *Journal of Strength and Conditioning Research*, April 28, 2014, <http://www.ncbi.nlm.nih.gov/pubmed/24402448>

- temperature would rise by 1.50°C if not for evaporation. What fraction of the water must evaporate to carry away precisely enough energy to keep the temperature constant?
78. (a) How much heat transfer is necessary to raise the temperature of a 0.200-kg piece of ice from -20.0°C to 130.0°C , including the energy needed for phase changes? (b) How much time is required for each stage, assuming a constant 20.0 kJ/s rate of heat transfer? (c) Make a graph of temperature versus time for this process.
 79. In 1986, an enormous iceberg broke away from the Ross Ice Shelf in Antarctica. It was an approximately rectangular prism 160 km long, 40.0 km wide, and 250 m thick. (a) What is the mass of this iceberg, given that the density of ice is 917 kg/m^3 ? (b) How much heat transfer (in joules) is needed to melt it? (c) How many years would it take sunlight alone to melt ice this thick, if the ice absorbs an average of 100 W/m^2 , 12.00 h per day?
 80. How many grams of coffee must evaporate from 350 g of coffee in a 100-g glass cup to cool the coffee and the cup from 95.0°C to 45.0°C ? Assume the coffee has the same thermal properties as water and that the average heat of vaporization is 2340 kJ/kg (560 kcal/g). Neglect heat losses through processes other than evaporation, as well as the change in mass of the coffee as it cools. Do the latter two assumptions cause your answer to be higher or lower than the true answer?
 81. (a) It is difficult to extinguish a fire on a crude oil tanker, because each liter of crude oil releases $2.80 \times 10^7\text{ J}$ of energy when burned. To illustrate this difficulty, calculate the number of liters of water that must be expended to absorb the energy released by burning 1.00 L of crude oil, if the water's temperature rises from 20.0°C to 100°C , it boils, and the resulting steam's temperature rises to 300°C at constant pressure. (b) Discuss additional complications caused by the fact that crude oil is less dense than water.
 82. The energy released from condensation in thunderstorms can be very large. Calculate the energy released into the atmosphere for a small storm of radius 1 km, assuming that 1.0 cm of rain is precipitated uniformly over this area.
 83. To help prevent frost damage, 4.00 kg of water at 0°C is sprayed onto a fruit tree. (a) How much heat transfer occurs as the water freezes? (b) How much would the temperature of the 200-kg tree decrease if this amount of heat transferred from the tree? Take the specific heat to be $3.35\text{ kJ/kg} \cdot ^{\circ}\text{C}$, and assume that no phase change occurs in the tree.
 84. A 0.250-kg aluminum bowl holding 0.800 kg of soup at 25.0°C is placed in a freezer. What is the final temperature if 388 kJ of energy is transferred from the bowl and soup, assuming the soup's thermal properties are the same as that of water?
 85. A 0.0500-kg ice cube at -30.0°C is placed in 0.400 kg of 35.0°C water in a very well-insulated container. What is the final temperature?
 86. If you pour 0.0100 kg of 20.0°C water onto a 1.20-kg block of ice (which is initially at -15.0°C), what is the final temperature? You may assume that the water cools so rapidly that effects of the surroundings are negligible.
 87. Indigenous people sometimes cook in watertight baskets by placing hot rocks into water to bring it to a boil. What mass of 500°C granite must be placed in 4.00 kg of 15.0°C water to bring its temperature to 100°C , if 0.0250 kg of water escapes as vapor from the initial sizzle? You may neglect the effects of the surroundings.
 88. What would the final temperature of the pan and water be in [Example 1.7](#) if 0.260 kg of water were placed in the pan and 0.0100 kg of the water evaporated immediately, leaving the remainder to come to a common temperature with the pan?

1.6 Mechanisms of Heat Transfer

89. (a) Calculate the rate of heat conduction through house walls that are 13.0 cm thick and have an average thermal conductivity twice that of glass wool. Assume there are no windows or doors. The walls' surface area is 120 m^2 and their inside surface is at 18.0°C , while their outside surface is at 5.00°C . (b) How many 1-kW room heaters would be needed to balance the heat transfer due to conduction?
90. The rate of heat conduction out of a window on a winter day is rapid enough to chill the air next to it. To see just how rapidly the windows transfer heat by conduction, calculate the rate of conduction in watts through a 3.00-m^2 window that is 0.634 cm thick ($1/4\text{ in.}$) if the temperatures of the inner and outer surfaces are 5.00°C and -10.0°C , respectively. (This rapid rate will not be maintained—the inner

surface will cool, even to the point of frost formation.)

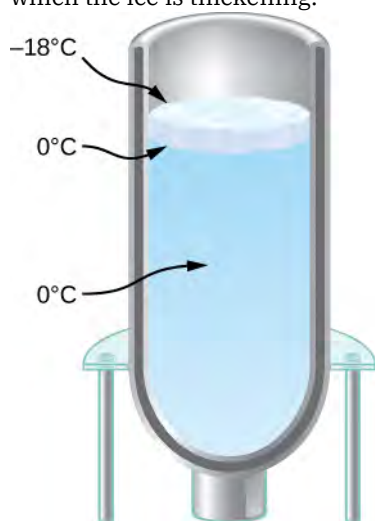
91. Calculate the rate of heat conduction out of the human body, assuming that the core internal temperature is 37.0°C , the skin temperature is 34.0°C , the thickness of the fatty tissues between the core and the skin averages 1.00 cm , and the surface area is 1.40 m^2 .
92. Suppose you stand with one foot on ceramic flooring and one foot on a wool carpet, making contact over an area of 80.0 cm^2 with each foot. Both the ceramic and the carpet are 2.00 cm thick and are 10.0°C on their bottom sides. At what rate must heat transfer occur from each foot to keep the top of the ceramic and carpet at 33.0°C ?
93. A man consumes 3000 kcal of food in one day, converting most of it to thermal energy to maintain body temperature. If he loses half this energy by evaporating water (through breathing and sweating), how many kilograms of water evaporate?
94. A firewalker runs across a bed of hot coals without sustaining burns. Calculate the heat transferred by conduction into the sole of one foot of a firewalker given that the bottom of the foot is a 3.00-mm -thick callus with a conductivity at the low end of the range for wood and its density is 300 kg/m^3 . The area of contact is 25.0 cm^2 , the temperature of the coals is 700°C , and the time in contact is 1.00 s . Ignore the evaporative cooling of sweat.
95. (a) What is the rate of heat conduction through the 3.00-cm -thick fur of a large animal having a 1.40-m^2 surface area? Assume that the animal's skin temperature is 32.0°C , that the air temperature is -5.00°C , and that fur has the same thermal conductivity as air. (b) What food intake will the animal need in one day to replace this heat transfer?
96. A walrus transfers energy by conduction through its blubber at the rate of 150 W when immersed in -1.00°C water. The walrus's internal core temperature is 37.0°C , and it has a surface area of 2.00 m^2 . What is the average thickness of its blubber, which has the conductivity of fatty tissues without blood?
97. Compare the rate of heat conduction through a 13.0-cm -thick wall that has an area of 10.0 m^2 and a thermal conductivity twice that of glass wool with the rate of heat conduction through a 0.750-cm -thick window that has an area of 2.00 m^2 , assuming the same temperature difference across each.
98. Suppose a person is covered head to foot by wool clothing with average thickness of 2.00 cm and is transferring energy by conduction through the clothing at the rate of 50.0 W . What is the temperature difference across the clothing, given the surface area is 1.40 m^2 ?
99. Some stove tops are smooth ceramic for easy cleaning. If the ceramic is 0.600 cm thick and heat conduction occurs through the same area and at the same rate as computed in [Example 1.11](#), what is the temperature difference across it? Ceramic has the same thermal conductivity as glass and brick.
100. One easy way to reduce heating (and cooling) costs is to add extra insulation in the attic of a house. Suppose a single-story cubical house already had 15 cm of fiberglass insulation in the attic and in all the exterior surfaces. If you added an extra 8.0 cm of fiberglass to the attic, by what percentage would the heating cost of the house drop? Take the house to have dimensions 10 m by 15 m by 3.0 m . Ignore air infiltration and heat loss through windows and doors, and assume that the interior is uniformly at one temperature and the exterior is uniformly at another.
101. Many decisions are made on the basis of the payback period: the time it will take through savings to equal the capital cost of an investment. Acceptable payback times depend upon the business or philosophy one has. (For some industries, a payback period is as small as 2 years.) Suppose you wish to install the extra insulation in the preceding problem. If energy cost $\$1.00$ per million joules and the insulation was $\$4.00$ per square meter, then calculate the simple payback time. Take the average ΔT for the 120-day heating season to be 15.0°C .

Additional Problems

- 102.** In 1701, the Danish astronomer Ole Rømer proposed a temperature scale with two fixed points, freezing water at 7.5 degrees, and boiling water at 60.0 degrees. What is the boiling point of oxygen, 90.2 K, on the Rømer scale?
- 103.** What is the percent error of thinking the melting point of tungsten is 3695 °C instead of the correct value of 3695 K?
- 104.** An engineer wants to design a structure in which the difference in length between a steel beam and an aluminum beam remains at 0.500 m regardless of temperature, for ordinary temperatures. What must the lengths of the beams be?
- 105.** How much stress is created in a steel beam if its temperature changes from -15 °C to 40 °C but it cannot expand? For steel, the Young's modulus $Y = 210 \times 10^9\text{ N/m}^2$ from [Stress, Strain, and Elastic Modulus](#). (Ignore the change in area resulting from the expansion.)
- 106.** A brass rod ($Y = 90 \times 10^9\text{ N/m}^2$), with a diameter of 0.800 cm and a length of 1.20 m when the temperature is 25 °C , is fixed at both ends. At what temperature is the force in it at 36,000 N?
- 107.** A mercury thermometer still in use for meteorology has a bulb with a volume of 0.780 cm^3 and a tube for the mercury to expand into of inside diameter 0.130 mm. (a) Neglecting the thermal expansion of the glass, what is the spacing between marks 1 °C apart? (b) If the thermometer is made of ordinary glass (not a good idea), what is the spacing?
- 108.** Even when shut down after a period of normal use, a large commercial nuclear reactor transfers thermal energy at the rate of 150 MW by the radioactive decay of fission products. This heat transfer causes a rapid increase in temperature if the cooling system fails ($1\text{ watt} = 1\text{ joule/second}$ or $1\text{ W} = 1\text{ J/s}$ and $1\text{ MW} = 1\text{ megawatt}$). (a) Calculate the rate of temperature increase in degrees Celsius per second (°C/s) if the mass of the reactor core is $1.60 \times 10^5\text{ kg}$ and it has an average specific heat of $0.3349\text{ kJ/kg} \cdot \text{°C}$. (b) How long would it take to obtain a temperature increase of 2000 °C , which could cause some metals holding the radioactive materials to melt? (The initial rate of temperature increase would be greater than that calculated here because the heat transfer is concentrated in a smaller mass. Later, however, the temperature increase would slow down because the 500,000-kg steel containment vessel would also begin to heat up.)
- 109.** You leave a pastry in the refrigerator on a plate and ask your roommate to take it out before you get home so you can eat it at room temperature, the way you like it. Instead, your roommate plays video games for hours. When you return, you notice that the pastry is still cold, but the game console has become hot. Annoyed, and knowing that the pastry will not be good if it is microwaved, you warm up the pastry by unplugging the console and putting it in a clean trash bag (which acts as a perfect calorimeter) with the pastry on the plate. After a while, you find that the equilibrium temperature is a nice, warm 38.3 °C . You know that the game console has a mass of 2.1 kg. Approximate it as having a uniform initial temperature of 45 °C . The pastry has a mass of 0.16 kg and a specific heat of $3.0\text{ kJ/(kg} \cdot \text{°C)}$, and is at a uniform initial temperature of 4.0 °C . The plate is at the same temperature and has a mass of 0.24 kg and a specific heat of $0.90\text{ kJ/(kg} \cdot \text{°C)}$. What is the specific heat of the console?
- 110.** Two solid spheres, A and B, made of the same material, are at temperatures of 0 °C and 100 °C , respectively. The spheres are placed in thermal contact in an ideal calorimeter, and they reach an equilibrium temperature of 20 °C . Which is the bigger sphere? What is the ratio of their diameters?

- 111.** In some countries, liquid nitrogen is used on dairy trucks instead of mechanical refrigerators. A 3.00-hour delivery trip requires 200 L of liquid nitrogen, which has a density of 808 kg/m^3 . (a) Calculate the heat transfer necessary to evaporate this amount of liquid nitrogen and raise its temperature to 3.00°C . (Use c_p and assume it is constant over the temperature range.) This value is the amount of cooling the liquid nitrogen supplies. (b) What is this heat transfer rate in kilowatt-hours? (c) Compare the amount of cooling obtained from melting an identical mass of 0°C ice with that from evaporating the liquid nitrogen.
- 112.** Some gun fanciers make their own bullets, which involves melting lead and casting it into lead slugs. How much heat transfer is needed to raise the temperature and melt 0.500 kg of lead, starting from 25.0°C ?
- 113.** A 0.800-kg iron cylinder at a temperature of $1.00 \times 10^3^\circ\text{C}$ is dropped into an insulated chest of 1.00 kg of ice at its melting point. What is the final temperature, and how much ice has melted?
- 114.** Repeat the preceding problem with 2.00 kg of ice instead of 1.00 kg .
- 115.** Repeat the preceding problem with 0.500 kg of ice, assuming that the ice is initially in a copper container of mass 1.50 kg in equilibrium with the ice.
- 116.** A 30.0-g ice cube at its melting point is dropped into an aluminum calorimeter of mass 100.0 g in equilibrium at 24.0°C with 300.0 g of an unknown liquid. The final temperature is 4.0°C . What is the heat capacity of the liquid?
- 117.** (a) Calculate the rate of heat conduction through a double-paned window that has a 1.50-m^2 area and is made of two panes of 0.800-cm -thick glass separated by a 1.00-cm air gap. The inside surface temperature is 15.0°C , while that on the outside is -10.0°C . (*Hint:* There are identical temperature drops across the two glass panes. First find these and then the temperature drop across the air gap. This problem ignores the increased heat transfer in the air gap due to convection.) (b) Calculate the rate of heat conduction through a 1.60-cm -thick window of the same area and with the same temperatures. Compare your answer with that for part (a).
- 118.** (a) An exterior wall of a house is 3 m tall and 10 m wide. It consists of a layer of drywall with an R factor of 0.56 , a layer 3.5 inches thick filled with fiberglass batts, and a layer of insulated siding with an R factor of 2.6 . The wall is built so well that there are no leaks of air through it. When the inside of the wall is at 22°C and the outside is at -2°C , what is the rate of heat flow through the wall? (b) More realistically, the 3.5 -inch space also contains 2-by-4 studs—wooden boards 1.5 inches by 3.5 inches oriented so that 3.5 -inch dimension extends from the drywall to the siding. They are “on 16-inch centers,” that is, the centers of the studs are 16 inches apart. What is the heat current in this situation? Don’t worry about one stud more or less.
- 119.** For the human body, what is the rate of heat transfer by conduction through the body’s tissue with the following conditions: the tissue thickness is 3.00 cm , the difference in temperature is 2.00°C , and the skin area is 1.50 m^2 . How does this compare with the average heat transfer rate to the body resulting from an energy intake of about 2400 kcal per day? (No exercise is included.)

- 120.** You have a Dewar flask (a laboratory vacuum flask) that has an open top and straight sides, as shown below. You fill it with water and put it into the freezer. It is effectively a perfect insulator, blocking all heat transfer, except on the top. After a time, ice forms on the surface of the water. The liquid water and the bottom surface of the ice, in contact with the liquid water, are at 0°C . The top surface of the ice is at the same temperature as the air in the freezer, -18°C . Set the rate of heat flow through the ice equal to the rate of loss of heat of fusion as the water freezes. When the ice layer is 0.700 cm thick, find the rate in m/s at which the ice is thickening.



- 121.** An infrared heater for a sauna has a surface area of 0.050 m^2 and an emissivity of 0.84 . What temperature must it run at if the required power is 360 W ? Neglect the temperature of the environment.
- 122.** (a) Determine the power of radiation from the Sun by noting that the intensity of the radiation at the distance of Earth is 1370 W/m^2 . *Hint:* That intensity will be found everywhere on a spherical surface with radius equal to that of Earth's orbit. (b) Assuming that the Sun's temperature is 5780 K and that its emissivity is 1 , find its radius.

Challenge Problems

- 123.** A pendulum is made of a rod of length L and negligible mass, but capable of thermal expansion, and a weight of negligible size. (a) Show that when the temperature increases by dT , the period of the pendulum increases by a fraction $\alpha dT/2$. (b) A clock controlled by a brass pendulum keeps time correctly at 10°C . If the room temperature is 30°C , does the clock run faster or slower? What is its error in seconds per day?
- 124.** At temperatures of a few hundred kelvins the specific heat capacity of copper approximately follows the empirical formula $c = \alpha + \beta T + \delta T^{-2}$, where $\alpha = 349\text{ J/kg} \cdot \text{K}$, $\beta = 0.107\text{ J/kg} \cdot \text{K}^2$, and $\delta = 4.58 \times 10^5\text{ J} \cdot \text{kg} \cdot \text{K}$. How much heat is needed to raise the temperature of a 2.00-kg piece of copper from 20°C to 250°C ?
- 125.** In a calorimeter of negligible heat capacity, 200 g of steam at 150°C and 100 g of ice at -40°C are mixed. The pressure is maintained at 1 atm . What is the final temperature, and how much steam, ice, and water are present?

- 126.** An astronaut performing an extra-vehicular activity (space walk) shaded from the Sun is wearing a spacesuit that can be approximated as perfectly white ($e = 0$) except for a $5\text{ cm} \times 8\text{ cm}$ patch in the form of the astronaut's national flag. The patch has emissivity 0.300. The spacesuit under the patch is 0.500 cm thick, with a thermal conductivity $k = 0.0600\text{ W/m}^\circ\text{C}$, and its inner surface is at a temperature of 20.0°C . What is the temperature of the patch, and what is the rate of heat loss through it? Assume the patch is so thin that its outer surface is at the same temperature as the outer surface of the spacesuit under it. Also assume the temperature of outer space is 0 K. You will get an equation that is very hard to solve in closed form, so you can solve it numerically with a graphing calculator, with software, or even by trial and error with a calculator.
- 127.** Find the growth of an ice layer as a function of time in a Dewar flask as seen in [Exercise 1.120](#). Call the thickness of the ice layer L . (a) Derive an equation for dL/dt in terms of L , the temperature T above the ice, and the properties of ice (which you can leave in symbolic form instead of substituting the numbers). (b) Solve this differential equation assuming that at $t = 0$, you have $L = 0$. If you have studied differential equations, you will know a technique for solving equations of this type: manipulate the equation to get dL/dt multiplied by a (very simple) function of L on one side, and integrate both sides with respect to time. Alternatively, you may be able to use your knowledge of the derivatives of various functions to guess the solution, which has a simple dependence on t . (c) Will the water eventually freeze to the bottom of the flask?
- 128.** As the very first rudiment of climatology, estimate the temperature of Earth. Assume it is a perfect sphere and its temperature is uniform. Ignore the greenhouse effect. Thermal radiation from the Sun has an intensity (the "solar constant" S) of about 1370 W/m^2 at the radius of Earth's orbit. (a) Assuming the Sun's rays are parallel, what area must S be multiplied by to get the total radiation intercepted by Earth? It will be easiest to answer in terms of Earth's radius, R . (b) Assume that Earth reflects about 30% of the solar energy it intercepts. In other words, Earth has an albedo with a value of $A = 0.3$. In terms of S , A , and R , what is the rate at which Earth absorbs energy from the Sun? (c) Find the temperature at which Earth radiates energy at the same rate. Assume that at the infrared wavelengths where it radiates, the emissivity e is 1. Does your result show that the greenhouse effect is important? (d) How does your answer depend on the the area of Earth?

- 129.** Let's stop ignoring the greenhouse effect and incorporate it into the previous problem in a very rough way. Assume the atmosphere is a single layer, a spherical shell around Earth, with an emissivity $e = 0.77$ (chosen simply to give the right answer) at infrared wavelengths emitted by Earth and by the atmosphere. However, the atmosphere is transparent to the Sun's radiation (that is, assume the radiation is at visible wavelengths with no infrared), so the Sun's radiation reaches the surface. The greenhouse effect comes from the difference between the atmosphere's transmission of visible light and its rather strong absorption of infrared. Note that the atmosphere's radius is not significantly different from Earth's, but since the atmosphere is a layer above Earth, it emits radiation both upward and downward, so it has twice Earth's area. There are three radiative energy transfers in this problem: solar radiation absorbed by Earth's surface; infrared radiation from the surface, which is absorbed by the atmosphere according to its emissivity; and infrared radiation from the atmosphere, half of which is absorbed by Earth and half of which goes out into space. Apply the method of the previous problem to get an equation for Earth's surface and one for the atmosphere, and solve them for the two unknown temperatures, surface and atmosphere.
- In terms of Earth's radius, the constant σ , and the unknown temperature T_s of the surface, what is the power of the infrared radiation from the surface?
 - What is the power of Earth's radiation absorbed by the atmosphere?
 - In terms of the unknown temperature T_e of the atmosphere, what is the power radiated from the atmosphere?
 - Write an equation that says the power of the radiation the atmosphere absorbs from Earth equals the power of the radiation it emits.
 - Half of the power radiated by the atmosphere hits Earth. Write an equation that says that the power Earth absorbs from the atmosphere and the Sun equals the power that it emits.
 - Solve your two equations for the unknown temperature of Earth.
- For steps that make this model less crude, see for example the [lectures](#)

(<https://openstax.org/l/21paulgormlec>) by Paul O'Gorman.

CHAPTER 16

Electromagnetic Waves



Figure 16.1 The pressure from sunlight predicted by Maxwell's equations helped produce the tail of Comet McNaught. (credit: modification of work by Sebastian Deiries—ESO)

Chapter Outline

[16.1 Maxwell's Equations and Electromagnetic Waves](#)

[16.2 Plane Electromagnetic Waves](#)

[16.3 Energy Carried by Electromagnetic Waves](#)

[16.4 Momentum and Radiation Pressure](#)

[16.5 The Electromagnetic Spectrum](#)

INTRODUCTION Our view of objects in the sky at night, the warm radiance of sunshine, the sting of sunburn, our cell phone conversations, and the X-rays revealing a broken bone—all are brought to us by electromagnetic waves. It would be hard to overstate the practical importance of electromagnetic waves, through their role in vision, through countless technological applications, and through their ability to transport the energy from the Sun through space to sustain life and almost all of its activities on Earth.

Theory predicted the general phenomenon of electromagnetic waves before anyone realized that light is a form of an electromagnetic wave. In the mid-nineteenth century, James Clerk Maxwell formulated a single theory combining all the electric and magnetic effects known at that time. Maxwell's equations, summarizing this theory, predicted the existence of electromagnetic waves that travel at the speed of light. His theory also predicted how these waves behave, and how they carry both energy and momentum. The tails of comets, such

as Comet McNaught in [Figure 16.1](#), provide a spectacular example. Energy carried by light from the Sun warms the comet to release dust and gas. The momentum carried by the light exerts a weak force that shapes the dust into a tail of the kind seen here. The flux of particles emitted by the Sun, called the solar wind, typically produces an additional, second tail, as described in detail in this chapter.

In this chapter, we explain Maxwell's theory and show how it leads to his prediction of electromagnetic waves. We use his theory to examine what electromagnetic waves are, how they are produced, and how they transport energy and momentum. We conclude by summarizing some of the many practical applications of electromagnetic waves.

16.1 Maxwell's Equations and Electromagnetic Waves

Learning Objectives

By the end of this section, you will be able to:

- Explain Maxwell's correction of Ampère's law by including the displacement current
- State and apply Maxwell's equations in integral form
- Describe how the symmetry between changing electric and changing magnetic fields explains Maxwell's prediction of electromagnetic waves
- Describe how Hertz confirmed Maxwell's prediction of electromagnetic waves

James Clerk Maxwell (1831–1879) was one of the major contributors to physics in the nineteenth century ([Figure 16.2](#)). Although he died young, he made major contributions to the development of the kinetic theory of gases, to the understanding of color vision, and to the nature of Saturn's rings. He is probably best known for having combined existing knowledge of the laws of electricity and of magnetism with insights of his own into a complete overarching electromagnetic theory, represented by **Maxwell's equations**.



Figure 16.2 James Clerk Maxwell, a nineteenth-century physicist, developed a theory that explained the relationship between electricity and magnetism, and correctly predicted that visible light consists of electromagnetic waves.

Maxwell's Correction to the Laws of Electricity and Magnetism

The four basic laws of electricity and magnetism had been discovered experimentally through the work of physicists such as Oersted, Coulomb, Gauss, and Faraday. Maxwell discovered logical inconsistencies in these earlier results and identified the incompleteness of Ampère's law as their cause.

Recall that according to Ampère's law, the integral of the magnetic field around a closed loop C is proportional to the current I passing through any surface whose boundary is loop C itself:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I. \quad 16.1$$

There are infinitely many surfaces that can be attached to any loop, and Ampère's law stated in [Equation 16.1](#) is independent of the choice of surface.

Consider the set-up in [Figure 16.3](#). A source of emf is abruptly connected across a parallel-plate capacitor so that a time-dependent current I develops in the wire. Suppose we apply Ampère's law to loop C shown at a time before the capacitor is fully charged, so that $I \neq 0$. Surface S_1 gives a nonzero value for the enclosed current I , whereas surface S_2 gives zero for the enclosed current because no current passes through it:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \begin{cases} \mu_0 I & \text{if surface } S_1 \text{ is used} \\ 0 & \text{if surface } S_2 \text{ is used} \end{cases}.$$

Clearly, Ampère's law in its usual form does not work here. This is an internal contradiction in the theory which requires a modification to the theory, Ampère's law, itself.

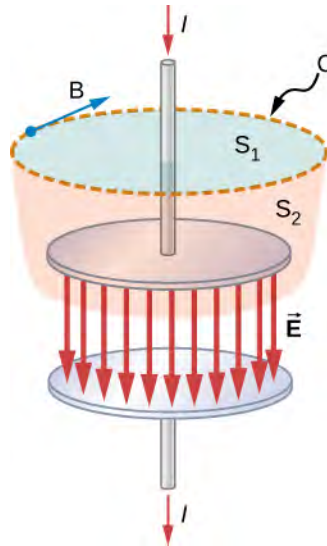


Figure 16.3 The currents through surface S_1 and surface S_2 are unequal, despite having the same boundary loop C .

How can Ampère's law be modified so that it works in all situations? Maxwell suggested including an additional contribution, called the displacement current I_d , to the real current I ,

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 (I + I_d) \quad 16.2$$

where the displacement current is defined to be

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}. \quad 16.3$$

Here ϵ_0 is the permittivity of free space and Φ_E is the electric flux, defined as

$$\Phi_E = \iint_{\text{Surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}.$$

The **displacement current** is analogous to a real current in Ampère's law, entering into Ampère's law in the same way. It is produced, however, by a changing electric field. It accounts for a changing electric field producing a magnetic field, just as a real current does, but the displacement current can produce a magnetic field even where no real current is present. When this extra term is included, the modified Ampère's law equation becomes

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad 16.4$$

and is independent of the surface S through which the current I is measured.

We can now examine this modified version of Ampère's law to confirm that it holds independent of whether the surface S_1 or the surface S_2 in [Figure 16.3](#) is chosen. The electric field $\vec{\mathbf{E}}$ corresponding to the flux Φ_E in [Equation 16.3](#) is between the capacitor plates. Therefore, the $\vec{\mathbf{E}}$ field and the displacement current through the surface S_1 are both zero, and [Equation 16.2](#) takes the form

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I. \quad 16.5$$

We must now show that for surface S_2 , through which no actual current flows, the displacement current leads to the same value $\mu_0 I$ for the right side of the Ampère's law equation. For surface S_2 , the equation becomes

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \frac{d}{dt} \left[\epsilon_0 \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \right]. \quad 16.6$$

Gauss's law for electric charge requires a closed surface and cannot ordinarily be applied to a surface like S_1 alone or S_2 alone. But the two surfaces S_1 and S_2 form a closed surface in [Figure 16.3](#) and can be used in Gauss's law. Because the electric field is zero on S_1 , the flux contribution through S_1 is zero. This gives us

$$\begin{aligned} \oiint_{\text{Surface } S_1 + S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} &= \iint_{\text{Surface } S_1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \\ &= 0 + \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \\ &= \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}. \end{aligned}$$

Therefore, we can replace the integral over S_2 in [Equation 16.6](#) with the closed Gaussian surface $S_1 + S_2$ and apply Gauss's law to obtain

$$\oint_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \frac{dQ_{\text{in}}}{dt} = \mu_0 I. \quad 16.7$$

Thus, the modified Ampère's law equation is the same using surface S_2 , where the right-hand side results from the displacement current, as it is for the surface S_1 , where the contribution comes from the actual flow of electric charge.



EXAMPLE 16.1

Displacement current in a charging capacitor

A parallel-plate capacitor with capacitance C whose plates have area A and separation distance d is connected to a resistor R and a battery of voltage V . The current starts to flow at $t = 0$. (a) Find the displacement current between the capacitor plates at time t . (b) From the properties of the capacitor, find the corresponding real current $I = \frac{dQ}{dt}$, and compare the answer to the expected current in the wires of the corresponding RC circuit.

Strategy

We can use the equations from the analysis of an RC circuit ([Alternating-Current Circuits](#)) plus Maxwell's version of Ampère's law.

Solution

- The voltage between the plates at time t is given by

$$V_C = \frac{1}{C}Q(t) = V_0(1 - e^{-t/RC}).$$

Let the z -axis point from the positive plate to the negative plate. Then the z -component of the electric field between the plates as a function of time t is

$$E_z(t) = \frac{V_0}{d}(1 - e^{-t/RC}).$$

Therefore, the z -component of the displacement current I_d between the plates is

$$I_d(t) = \epsilon_0 A \frac{\partial E_z(t)}{\partial t} = \epsilon_0 A \frac{V_0}{d} \times \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC},$$

where we have used $C = \epsilon_0 \frac{A}{d}$ for the capacitance.

- b. From the expression for V_C , the charge on the capacitor is

$$Q(t) = CV_C = CV_0(1 - e^{-t/RC}).$$

The current into the capacitor after the circuit is closed, is therefore

$$I = \frac{dQ}{dt} = \frac{V_0}{R} e^{-t/RC}.$$

This current is the same as I_d found in (a).

Maxwell's Equations

With the correction for the displacement current, Maxwell's equations take the form

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0} \quad \left(\text{Gauss's law} \right) \quad 16.8$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \quad \left(\text{Gauss's law for magnetism} \right) \quad 16.9$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_m}{dt} \quad \left(\text{Faraday's law} \right) \quad 16.10$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad \left(\text{Ampère-Maxwell law} \right). \quad 16.11$$

Once the fields have been calculated using these four equations, the Lorentz force equation

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \quad 16.12$$

gives the force that the fields exert on a particle with charge q moving with velocity $\vec{v}$. The Lorentz force equation combines the force of the electric field and of the magnetic field on the moving charge. The magnetic and electric forces have been examined in earlier modules. These four Maxwell's equations are, respectively,

Maxwell's Equations

1. Gauss's law

The electric flux through any closed surface is equal to the electric charge Q_{in} enclosed by the surface. Gauss's law [Equation 16.7] describes the relation between an electric charge and the electric field it produces. This is often pictured in terms of electric field lines originating from positive charges and terminating on negative charges, and indicating the direction of the electric field at each point in space.

2. Gauss's law for magnetism

The magnetic field flux through any closed surface is zero [Equation 16.8]. This is equivalent to the statement that magnetic field lines are continuous, having no beginning or end. Any magnetic field line entering the region enclosed by the surface must also leave it. No magnetic monopoles, where magnetic

field lines would terminate, are known to exist (see [Magnetic Fields and Lines](#)).

3. Faraday's law

A changing magnetic field induces an electromotive force (emf) and, hence, an electric field. The direction of the emf opposes the change. This third of Maxwell's equations, [Equation 16.9](#), is Faraday's law of induction and includes Lenz's law. The electric field from a changing magnetic field has field lines that form closed loops, without any beginning or end.

4. Ampère-Maxwell law

Magnetic fields are generated by moving charges or by changing electric fields. This fourth of Maxwell's equations, [Equation 16.10](#), encompasses Ampère's law and adds another source of magnetic fields, namely changing electric fields.

Maxwell's equations and the Lorentz force law together encompass all the laws of electricity and magnetism. The symmetry that Maxwell introduced into his mathematical framework may not be immediately apparent. Faraday's law describes how changing magnetic fields produce electric fields. The displacement current introduced by Maxwell results instead from a changing electric field and accounts for a changing electric field producing a magnetic field. The equations for the effects of both changing electric fields and changing magnetic fields differ in form only where the absence of magnetic monopoles leads to missing terms. This symmetry between the effects of changing magnetic and electric fields is essential in explaining the nature of electromagnetic waves.

Later application of Einstein's theory of relativity to Maxwell's complete and symmetric theory showed that electric and magnetic forces are not separate but are different manifestations of the same thing—the electromagnetic force. The electromagnetic force and weak nuclear force are similarly unified as the electroweak force. This unification of forces has been one motivation for attempts to unify all of the four basic forces in nature—the gravitational, electrical, strong, and weak nuclear forces (see [Particle Physics and Cosmology](#)).

The Mechanism of Electromagnetic Wave Propagation

To see how the symmetry introduced by Maxwell accounts for the existence of combined electric and magnetic waves that propagate through space, imagine a time-varying magnetic field $\vec{\mathbf{B}}_0(t)$ produced by the high-frequency alternating current seen in [Figure 16.4](#). We represent $\vec{\mathbf{B}}_0(t)$ in the diagram by one of its field lines. From Faraday's law, the changing magnetic field through a surface induces a time-varying electric field $\vec{\mathbf{E}}_0(t)$ at the boundary of that surface. The displacement current source for the electric field, like the Faraday's law source for the magnetic field, produces only closed loops of field lines, because of the mathematical symmetry involved in the equations for the induced electric and induced magnetic fields. A field line representation of $\vec{\mathbf{E}}_0(t)$ is shown. In turn, the changing electric field $\vec{\mathbf{E}}_0(t)$ creates a magnetic field $\vec{\mathbf{B}}_1(t)$ according to the modified Ampère's law. This changing field induces $\vec{\mathbf{E}}_1(t)$, which induces $\vec{\mathbf{B}}_2(t)$, and so on. We then have a self-continuing process that leads to the creation of time-varying electric and magnetic fields in regions farther and farther away from O . This process may be visualized as the propagation of an electromagnetic wave through space.

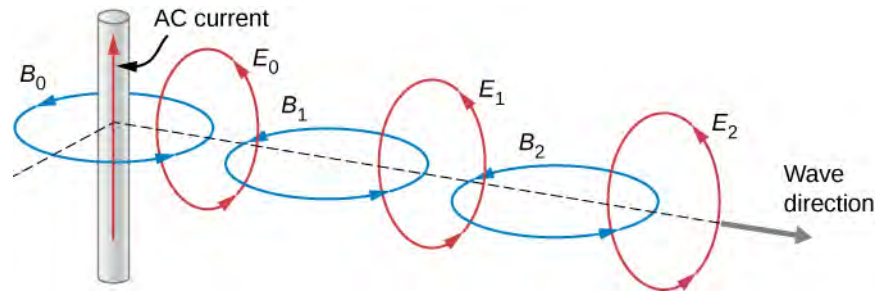


Figure 16.4 How changing $\vec{E}$ and $\vec{B}$ fields propagate through space.

In the next section, we show in more precise mathematical terms how Maxwell's equations lead to the prediction of electromagnetic waves that can travel through space without a material medium, implying a speed of electromagnetic waves equal to the speed of light.

Prior to Maxwell's work, experiments had already indicated that light was a wave phenomenon, although the nature of the waves was yet unknown. In 1801, Thomas Young (1773–1829) showed that when a light beam was separated by two narrow slits and then recombined, a pattern made up of bright and dark fringes was formed on a screen. Young explained this behavior by assuming that light was composed of waves that added constructively at some points and destructively at others (see [Interference](#)). Subsequently, Jean Foucault (1819–1868), with measurements of the speed of light in various media, and Augustin Fresnel (1788–1827), with detailed experiments involving interference and diffraction of light, provided further conclusive evidence that light was a wave. So, light was known to be a wave, and Maxwell had predicted the existence of electromagnetic waves that traveled at the speed of light. The conclusion seemed inescapable: Light must be a form of electromagnetic radiation. But Maxwell's theory showed that other wavelengths and frequencies than those of light were possible for electromagnetic waves. He showed that electromagnetic radiation with the same fundamental properties as visible light should exist at any frequency. It remained for others to test, and confirm, this prediction.

✓ CHECK YOUR UNDERSTANDING 16.1

When the emf across a capacitor is turned on and the capacitor is allowed to charge, when does the magnetic field induced by the displacement current have the greatest magnitude?

Hertz's Observations

The German physicist Heinrich Hertz (1857–1894) was the first to generate and detect certain types of electromagnetic waves in the laboratory. Starting in 1887, he performed a series of experiments that not only confirmed the existence of electromagnetic waves but also verified that they travel at the speed of light.

Hertz used an alternating-current *RLC* (resistor-inductor-capacitor) circuit that resonates at a known frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$ and connected it to a loop of wire, as shown in [Figure 16.5](#). High voltages induced

across the gap in the loop produced sparks that were visible evidence of the current in the circuit and helped generate electromagnetic waves.

Across the laboratory, Hertz placed another loop attached to another *RLC* circuit, which could be tuned (as the dial on a radio) to the same resonant frequency as the first and could thus be made to receive electromagnetic waves. This loop also had a gap across which sparks were generated, giving solid evidence that electromagnetic waves had been received.

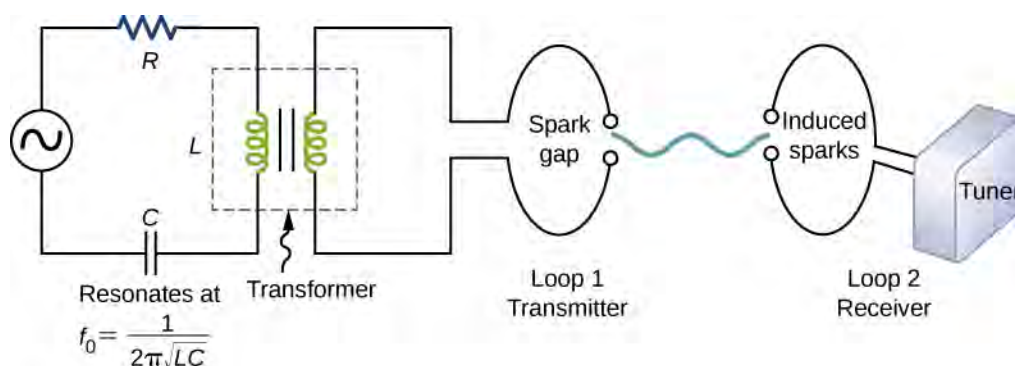


Figure 16.5 The apparatus used by Hertz in 1887 to generate and detect electromagnetic waves.

Hertz also studied the reflection, refraction, and interference patterns of the electromagnetic waves he generated, confirming their wave character. He was able to determine the wavelengths from the interference patterns, and knowing their frequencies, he could calculate the propagation speed using the equation $v = f\lambda$, where v is the speed of a wave, f is its frequency, and λ is its wavelength. Hertz was thus able to prove that electromagnetic waves travel at the speed of light. The SI unit for frequency, the hertz ($1 \text{ Hz} = 1 \text{ cycle/s}$), is named in his honor.

✓ CHECK YOUR UNDERSTANDING 16.2

Could a purely electric field propagate as a wave through a vacuum without a magnetic field? Justify your answer.

16.2 Plane Electromagnetic Waves

Learning Objectives

By the end of this section, you will be able to:

- Describe how Maxwell's equations predict the relative directions of the electric fields and magnetic fields, and the direction of propagation of plane electromagnetic waves
- Explain how Maxwell's equations predict that the speed of propagation of electromagnetic waves in free space is exactly the speed of light
- Calculate the relative magnitude of the electric and magnetic fields in an electromagnetic plane wave
- Describe how electromagnetic waves are produced and detected

Mechanical waves travel through a medium such as a string, water, or air. Perhaps the most significant prediction of Maxwell's equations is the existence of combined electric and magnetic (or electromagnetic) fields that propagate through space as electromagnetic waves. Because Maxwell's equations hold in free space, the predicted electromagnetic waves, unlike mechanical waves, do not require a medium for their propagation.

A general treatment of the physics of electromagnetic waves is beyond the scope of this textbook. We can, however, investigate the special case of an electromagnetic wave that propagates through free space along the x -axis of a given coordinate system.

Electromagnetic Waves in One Direction

An electromagnetic wave consists of an electric field, defined as usual in terms of the force per charge on a stationary charge, and a magnetic field, defined in terms of the force per charge on a moving charge. The electromagnetic field is assumed to be a function of only the x -coordinate and time. The y -component of the electric field is then written as $E_y(x, t)$, the z -component of the magnetic field as $B_z(x, t)$, etc. Because we are assuming free space, there are no free charges or currents, so we can set $Q_{\text{in}} = 0$ and $I = 0$ in Maxwell's equations.

The transverse nature of electromagnetic waves

We examine first what Gauss's law for electric fields implies about the relative directions of the electric field and the propagation direction in an electromagnetic wave. Assume the Gaussian surface to be the surface of a rectangular box whose cross-section is a square of side l and whose third side has length Δx , as shown in [Figure 16.6](#). Because the electric field is a function only of x and t , the y -component of the electric field is the same on both the top (labeled Side 2) and bottom (labeled Side 1) of the box, so that these two contributions to the flux cancel. The corresponding argument also holds for the net flux from the z -component of the electric field through Sides 3 and 4. Any net flux through the surface therefore comes entirely from the x -component of the electric field. Because the electric field has no y - or z -dependence, $E_x(x, t)$ is constant over the face of the box with area A and has a possibly different value $E_x(x + \Delta x, t)$ that is constant over the opposite face of the box. Applying Gauss's law gives

$$\text{Net flux} = -E_x(x, t) A + E_x(x + \Delta x, t) A = \frac{Q_{\text{in}}}{\epsilon_0} \quad 16.13$$

where $A = l \times l$ is the area of the front and back faces of the rectangular surface. But the charge enclosed is $Q_{\text{in}} = 0$, so this component's net flux is also zero, and [Equation 16.13](#) implies $E_x(x, t) = E_x(x + \Delta x, t)$ for any Δx . Therefore, if there is an x -component of the electric field, it cannot vary with x . A uniform field of that kind would merely be superposed artificially on the traveling wave, for example, by having a pair of parallel-charged plates. Such a component $E_x(x, t)$ would not be part of an electromagnetic wave propagating along the x -axis; so $E_x(x, t) = 0$ for this wave. Therefore, the only nonzero components of the electric field are $E_y(x, t)$ and $E_z(x, t)$, perpendicular to the direction of propagation of the wave.

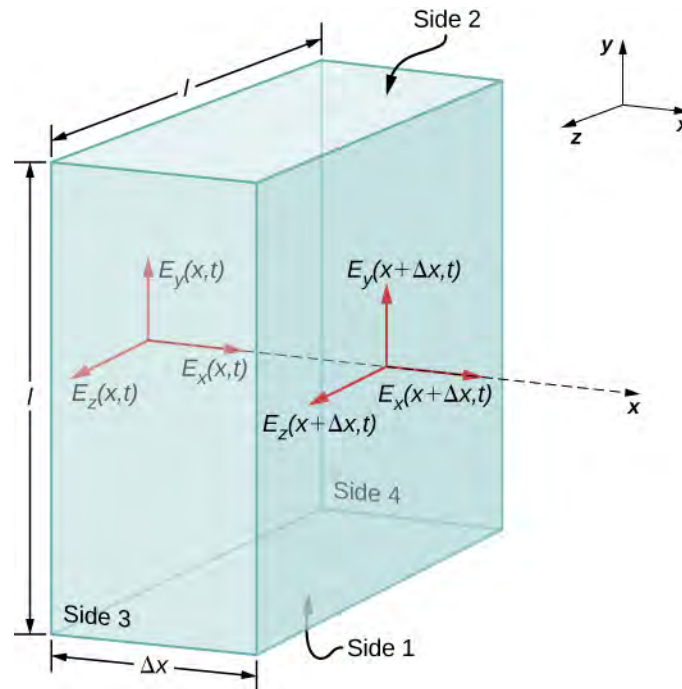


Figure 16.6 The surface of a rectangular box of dimensions $l \times l \times \Delta x$ is our Gaussian surface. The electric field shown is from an electromagnetic wave propagating along the x -axis.

A similar argument holds by substituting E for B and using Gauss's law for magnetism instead of Gauss's law for electric fields. This shows that the B field is also perpendicular to the direction of propagation of the wave. The electromagnetic wave is therefore a transverse wave, with its oscillating electric and magnetic fields perpendicular to its direction of propagation.

The speed of propagation of electromagnetic waves

We can next apply Maxwell's equations to the description given in connection with [Figure 16.4](#) in the previous section to obtain an equation for the E field from the changing B field, and for the B field from a changing E field. We then combine the two equations to show how the changing E and B fields propagate through space at

a speed precisely equal to the speed of light.

First, we apply Faraday's law over Side 3 of the Gaussian surface, using the path shown in [Figure 16.7](#). Because $E_x(x, t) = 0$, we have

$$\oint \vec{E} \cdot d\vec{s} = -E_y(x, t)l + E_y(x + \Delta x, t)l.$$

Assuming Δx is small and approximating $E_y(x + \Delta x, t)$ by

$$E_y(x + \Delta x, t) = E_y(x, t) + \frac{\partial E_y(x, t)}{\partial x} \Delta x,$$

we obtain

$$\oint \vec{E} \cdot d\vec{s} = \frac{\partial E_y(x, t)}{\partial x} (l\Delta x).$$

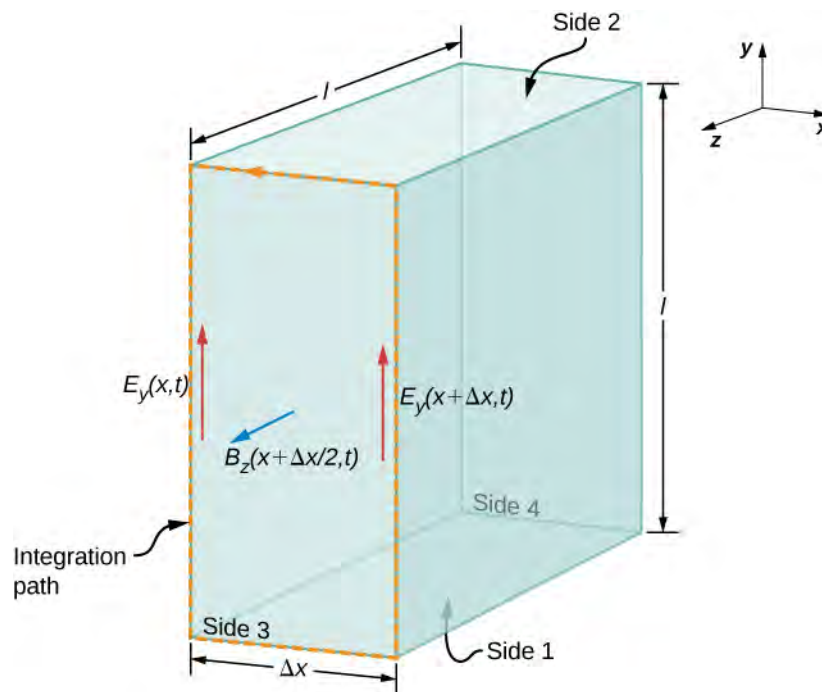


Figure 16.7 We apply Faraday's law to the front of the rectangle by evaluating $\oint \vec{E} \cdot d\vec{s}$ along the rectangular edge of Side 3 in the direction indicated, taking the B field crossing the face to be approximately its value in the middle of the area traversed.

Because Δx is small, the magnetic flux through the face can be approximated by its value in the center of the area traversed, namely $B_z\left(x + \frac{\Delta x}{2}, t\right)$. The flux of the B field through Face 3 is then the B field times the area,

$$\oint_S \vec{B} \cdot \vec{n} dA = B_z\left(x + \frac{\Delta x}{2}, t\right) (l\Delta x). \quad 16.14$$

From Faraday's law,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_S \vec{B} \cdot \vec{n} dA. \quad 16.15$$

Therefore, from [Equation 16.13](#) and [Equation 16.14](#),

$$\frac{\partial E_y(x, t)}{\partial x} (l\Delta x) = -\frac{\partial}{\partial t} \left[B_z\left(x + \frac{\Delta x}{2}, t\right) \right] (l\Delta x).$$

Canceling $l\Delta x$ and taking the limit as $\Delta x \rightarrow 0$, we are left with

$$\frac{\partial E_y(x, t)}{\partial x} = -\frac{\partial B_z(x, t)}{\partial t}. \quad 16.16$$

We could have applied Faraday's law instead to the top surface (numbered 2) in [Figure 16.7](#), to obtain the resulting equation

$$\frac{\partial E_z(x, t)}{\partial x} = -\frac{\partial B_y(x, t)}{\partial t}. \quad 16.17$$

This is the equation describing the spatially dependent E field produced by the time-dependent B field.

Next we apply the Ampère-Maxwell law (with $I = 0$) over the same two faces (Surface 3 and then Surface 2) of the rectangular box of [Figure 16.7](#). Applying [Equation 16.10](#),

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = \mu_0 \epsilon_0 (d/dt) \int_S \vec{\mathbf{E}} \cdot \mathbf{n} da$$

to Surface 3, and then to Surface 2, yields the two equations

$$\frac{\partial B_y(x, t)}{\partial x} = -\epsilon_0 \mu_0 \frac{\partial E_z(x, t)}{\partial t}, \text{ and} \quad 16.18$$

$$\frac{\partial B_z(x, t)}{\partial x} = -\epsilon_0 \mu_0 \frac{\partial E_y(x, t)}{\partial t}. \quad 16.19$$

These equations describe the spatially dependent B field produced by the time-dependent E field.

We next combine the equations showing the changing B field producing an E field with the equation showing the changing E field producing a B field. Taking the derivative of [Equation 16.16](#) with respect to x and using [Equation 16.26](#) gives

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial E_y}{\partial x} \right) = -\frac{\partial}{\partial x} \left(\frac{\partial B_z}{\partial t} \right) = -\frac{\partial}{\partial t} \left(\frac{\partial B_z}{\partial x} \right) = \frac{\partial}{\partial t} \left(\epsilon_0 \mu_0 \frac{\partial E_y}{\partial t} \right)$$

or

$$\frac{\partial^2 E_y}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}. \quad 16.20$$

This is the form taken by the general wave equation for our plane wave. Because the equations describe a wave traveling at some as-yet-unspecified speed c , we can assume the field components are each functions of $x - ct$ for the wave traveling in the $+x$ -direction, that is,

$$E_y(x, t) = f(\xi) \quad \text{where } \xi = x - ct. \quad 16.21$$

It is left as a mathematical exercise to show, using the chain rule for differentiation, that [Equation 16.17](#) and [Equation 16.18](#) imply

$$1 = \epsilon_0 \mu_0 c^2.$$

The speed of the electromagnetic wave in free space is therefore given in terms of the permeability and the permittivity of free space by

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}. \quad 16.22$$

We could just as easily have assumed an electromagnetic wave with field components $E_z(x, t)$ and $B_y(x, t)$. The same type of analysis with [Equation 16.25](#) and [Equation 16.24](#) would also show that the speed of an electromagnetic wave is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

The physics of traveling electromagnetic fields was worked out by Maxwell in 1873. He showed in a more general way than our derivation that electromagnetic waves always travel in free space with a speed given by

Equation 16.18. If we evaluate the speed $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$, we find that

$$c = \frac{1}{\sqrt{\left(8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right) \left(4\pi \times 10^{-7} \frac{\text{T}\cdot\text{m}}{\text{A}}\right)}} = 3.00 \times 10^8 \text{ m/s},$$

which is the speed of light. Imagine the excitement that Maxwell must have felt when he discovered this equation! He had found a fundamental connection between two seemingly unrelated phenomena: electromagnetic fields and light.

✓ CHECK YOUR UNDERSTANDING 16.3

The wave equation was obtained by (1) finding the E field produced by the changing B field, (2) finding the B field produced by the changing E field, and combining the two results. Which of Maxwell's equations was the basis of step (1) and which of step (2)?

How the E and B Fields Are Related

So far, we have seen that the rates of change of different components of the E and B fields are related, that the electromagnetic wave is transverse, and that the wave propagates at speed c . We next show what Maxwell's equations imply about the ratio of the E and B field magnitudes and the relative directions of the E and B fields.

We now consider solutions to [Equation 16.16](#) in the form of plane waves for the electric field:

$$E_y(x, t) = E_0 \cos(kx - \omega t). \quad 16.23$$

We have arbitrarily taken the wave to be traveling in the $+x$ -direction and chosen its phase so that the maximum field strength occurs at the origin at time $t = 0$. We are justified in considering only sines and cosines in this way, and generalizing the results, because Fourier's theorem implies we can express any wave, including even square step functions, as a superposition of sines and cosines.

At any one specific point in space, the E field oscillates sinusoidally at angular frequency ω between $+E_0$ and $-E_0$, and similarly, the B field oscillates between $+B_0$ and $-B_0$. The amplitude of the wave is the maximum value of $E_y(x, t)$. The period of oscillation T is the time required for a complete oscillation. The frequency f is the number of complete oscillations per unit of time, and is related to the angular frequency ω by $\omega = 2\pi f$. The wavelength λ is the distance covered by one complete cycle of the wave, and the wavenumber k is the number of wavelengths that fit into a distance of 2π in the units being used. These quantities are related in the same way as for a mechanical wave:

$$\omega = 2\pi f, \quad f = \frac{1}{T}, \quad k = \frac{2\pi}{\lambda}, \quad \text{and} \quad c = f\lambda = \omega/k.$$

Given that the solution of E_y has the form shown in [Equation 16.20](#), we need to determine the B field that accompanies it. From [Equation 16.24](#), the magnetic field component B_z must obey

$$\begin{aligned} \frac{\partial B_z}{\partial t} &= -\frac{\partial E_y}{\partial x} \\ \frac{\partial B_z}{\partial t} &= -\frac{\partial}{\partial x} E_0 \cos(kx - \omega t) = k E_0 \sin(kx - \omega t). \end{aligned} \quad 16.24$$

Because the solution for the B -field pattern of the wave propagates in the $+x$ -direction at the same speed c as the E -field pattern, it must be a function of $k(x - ct) = kx - \omega t$. Thus, we conclude from [Equation 16.21](#) that B_z is

$$B_z(x, t) = \frac{k}{\omega} E_0 \cos(kx - \omega t) = \frac{1}{c} E_0 \cos(kx - \omega t).$$

These results may be written as

$$\begin{aligned} E_y(x, t) &= E_0 \cos(kx - \omega t) \\ B_z(x, t) &= B_0 \cos(kx - \omega t) \end{aligned} \quad 16.25$$

$$\frac{E_y}{B_z} = \frac{E_0}{B_0} = c.$$

16.26

Therefore, the peaks of the E and B fields coincide, as do the troughs of the wave, and at each point, the E and B fields are in the same ratio equal to the speed of light c . The plane wave has the form shown in [Figure 16.8](#).

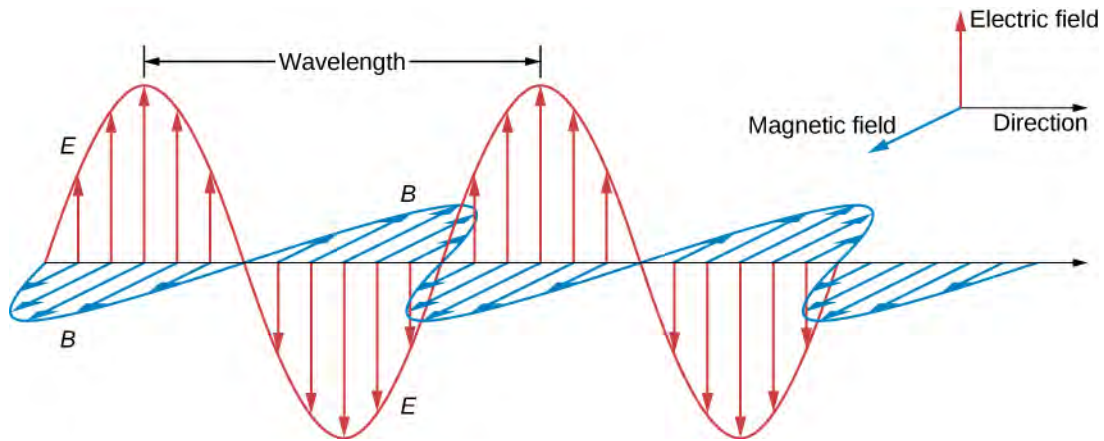


Figure 16.8 The plane wave solution of Maxwell's equations has the B field directly proportional to the E field at each point, with the relative directions shown.



EXAMPLE 16.2

Calculating B -Field Strength in an Electromagnetic Wave

What is the maximum strength of the B field in an electromagnetic wave that has a maximum E -field strength of 1000 V/m?

Strategy

To find the B -field strength, we rearrange [Equation 16.23](#) to solve for B , yielding

$$B = \frac{E}{c}.$$

Solution

We are given E , and c is the speed of light. Entering these into the expression for B yields

$$B = \frac{1000 \text{ V/m}}{3.00 \times 10^8 \text{ m/s}} = 3.33 \times 10^{-6} \text{ T}.$$

Significance

The B -field strength is less than a tenth of Earth's admittedly weak magnetic field. This means that a relatively strong electric field of 1000 V/m is accompanied by a relatively weak magnetic field.

Changing electric fields create relatively weak magnetic fields. The combined electric and magnetic fields can be detected in electromagnetic waves, however, by taking advantage of the phenomenon of resonance, as Hertz did. A system with the same natural frequency as the electromagnetic wave can be made to oscillate. All radio and TV receivers use this principle to pick up and then amplify weak electromagnetic waves, while rejecting all others not at their resonant frequency.

CHECK YOUR UNDERSTANDING 16.4

What conclusions did our analysis of Maxwell's equations lead to about these properties of a plane electromagnetic wave:

- (a) the relative directions of wave propagation, of the E field, and of B field,
- (b) the speed of travel of the wave and how the speed depends on frequency, and
- (c) the relative magnitudes of the E and B fields.

Production and Detection of Electromagnetic Waves

A steady electric current produces a magnetic field that is constant in time and which does not propagate as a wave. Accelerating charges, however, produce electromagnetic waves. An electric charge oscillating up and down, or an alternating current or flow of charge in a conductor, emit radiation at the frequencies of their oscillations. The electromagnetic field of a *dipole antenna* is shown in Figure 16.9. The positive and negative charges on the two conductors are made to reverse at the desired frequency by the output of a transmitter as the power source. The continually changing current accelerates charge in the antenna, and this results in an oscillating electric field a distance away from the antenna. The changing electric fields produce changing magnetic fields that in turn produce changing electric fields, which thereby propagate as electromagnetic waves. The frequency of this radiation is the same as the frequency of the ac source that is accelerating the electrons in the antenna. The two conducting elements of the dipole antenna are commonly straight wires. The total length of the two wires is typically about one-half of the desired wavelength (hence, the alternative name *half-wave antenna*), because this allows standing waves to be set up and enhances the effectiveness of the radiation.

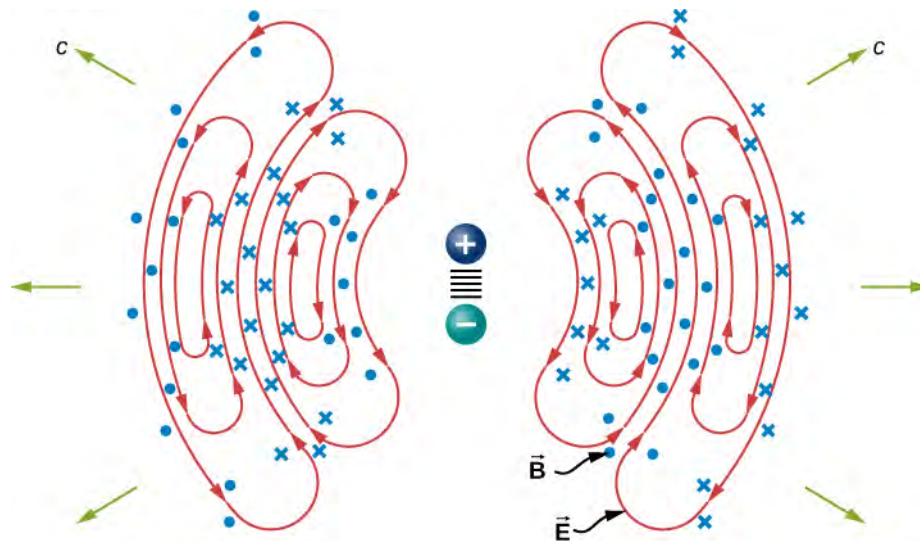


Figure 16.9 The oscillatory motion of the charges in a dipole antenna produces electromagnetic radiation.

The electric field lines in one plane are shown. The magnetic field is perpendicular to this plane. This radiation field has cylindrical symmetry around the axis of the dipole. Field lines near the dipole are not shown. The pattern is not at all uniform in all directions. The strongest signal is in directions perpendicular to the axis of the antenna, which would be horizontal if the antenna is mounted vertically. There is zero intensity along the axis of the antenna. The fields detected far from the antenna are from the changing electric and magnetic fields inducing each other and traveling as electromagnetic waves. Far from the antenna, the wave fronts, or surfaces of equal phase for the electromagnetic wave, are almost spherical. Even farther from the antenna, the radiation propagates like electromagnetic plane waves.

The electromagnetic waves carry energy away from their source, similar to a sound wave carrying energy away from a standing wave on a guitar string. An antenna for receiving electromagnetic signals works in reverse. Incoming electromagnetic waves induce oscillating currents in the antenna, each at its own frequency. The radio receiver includes a tuner circuit, whose resonant frequency can be adjusted. The tuner responds strongly to the desired frequency but not others, allowing the user to tune to the desired broadcast. Electrical components amplify the signal formed by the moving electrons. The signal is then converted into an audio and/or video format.

INTERACTIVE

Use this [simulation \(https://openstax.org/l/21radwavsim\)](https://openstax.org/l/21radwavsim) to broadcast radio waves. Wiggle the transmitter electron manually or have it oscillate automatically. Display the field as a curve or vectors. The strip chart shows the electron positions at the transmitter and at the receiver.

16.3 Energy Carried by Electromagnetic Waves

Learning Objectives

By the end of this section, you will be able to:

- Express the time-averaged energy density of electromagnetic waves in terms of their electric and magnetic field amplitudes
- Calculate the Poynting vector and the energy intensity of electromagnetic waves
- Explain how the energy of an electromagnetic wave depends on its amplitude, whereas the energy of a photon is proportional to its frequency

Anyone who has used a microwave oven knows there is energy in electromagnetic waves. Sometimes this energy is obvious, such as in the warmth of the summer Sun. Other times, it is subtle, such as the unfelt energy of gamma rays, which can destroy living cells.

Electromagnetic waves bring energy into a system by virtue of their electric and magnetic fields. These fields can exert forces and move charges in the system and, thus, do work on them. However, there is energy in an electromagnetic wave itself, whether it is absorbed or not. Once created, the fields carry energy away from a source. If some energy is later absorbed, the field strengths are diminished and anything left travels on.

Clearly, the larger the strength of the electric and magnetic fields, the more work they can do and the greater the energy the electromagnetic wave carries. In electromagnetic waves, the amplitude is the maximum field strength of the electric and magnetic fields (Figure 16.10). The wave energy is determined by the wave amplitude.

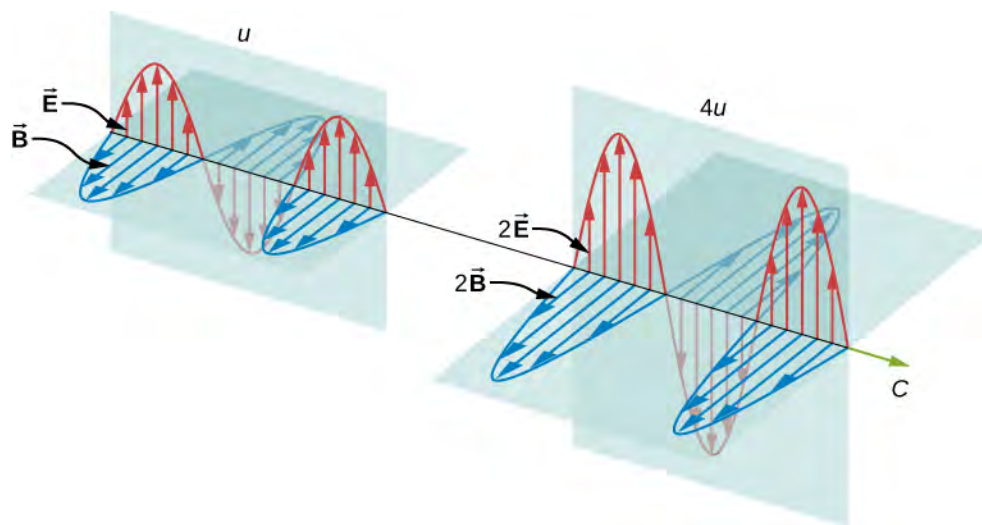


Figure 16.10 Energy carried by a wave depends on its amplitude. With electromagnetic waves, doubling the E fields and B fields quadruples the energy density u and the energy flux uc .

For a plane wave traveling in the direction of the positive x -axis with the phase of the wave chosen so that the wave maximum is at the origin at $t = 0$, the electric and magnetic fields obey the equations

$$E_y(x, t) = E_0 \cos(kx - \omega t)$$

$$B_z(x, t) = B_0 \cos(kx - \omega t).$$

The energy in any part of the electromagnetic wave is the sum of the energies of the electric and magnetic fields. This energy per unit volume, or energy density u , is the sum of the energy density from the electric field

and the energy density from the magnetic field. Expressions for both field energy densities were discussed earlier (u_E in [Capacitance](#) and u_B in [Inductance](#)). Combining these the contributions, we obtain

$$u(x, t) = u_E + u_B = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2.$$

The expression $E = cB = \frac{1}{\sqrt{\epsilon_0\mu_0}} B$ then shows that the magnetic energy density u_B and electric energy density u_E are equal, despite the fact that changing electric fields generally produce only small magnetic fields. The equality of the electric and magnetic energy densities leads to

$$u(x, t) = \epsilon_0 E^2 = \frac{B^2}{\mu_0}. \quad 16.27$$

The energy density moves with the electric and magnetic fields in a similar manner to the waves themselves.

We can find the rate of transport of energy by considering a small time interval Δt . As shown in [Figure 16.11](#), the energy contained in a cylinder of length $c\Delta t$ and cross-sectional area A passes through the cross-sectional plane in the interval Δt .

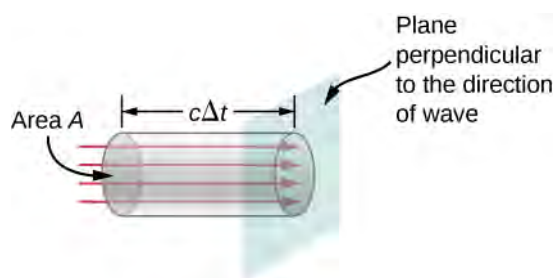


Figure 16.11 The energy $uAc\Delta t$ contained in the electric and magnetic fields of the electromagnetic wave in the volume $Ac\Delta t$ passes through the area A in time Δt .

The energy passing through area A in time Δt is

$$u \times \text{volume} = uAc\Delta t.$$

The energy per unit area per unit time passing through a plane perpendicular to the wave, called the energy flux and denoted by S , can be calculated by dividing the energy by the area A and the time interval Δt .

$$S = \frac{\text{Energy passing area } A \text{ in time } \Delta t}{A\Delta t} = uc = \epsilon_0 c E^2 = \frac{1}{\mu_0} EB.$$

More generally, the flux of energy through any surface also depends on the orientation of the surface. To take the direction into account, we introduce a vector $\vec{S}$, called the **Poynting vector**, with the following definition:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}. \quad 16.28$$

The cross-product of $\vec{E}$ and $\vec{B}$ points in the direction perpendicular to both vectors. To confirm that the direction of $\vec{S}$ is that of wave propagation, and not its negative, return to [Figure 16.7](#). Note that Lenz's and Faraday's laws imply that when the magnetic field shown is increasing in time, the electric field is greater at x than at $x + \Delta x$. The electric field is decreasing with increasing x at the given time and location. The proportionality between electric and magnetic fields requires the electric field to increase in time along with the magnetic field. This is possible only if the wave is propagating to the right in the diagram, in which case, the relative orientations show that $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ is specifically in the direction of propagation of the electromagnetic wave.

The energy flux at any place also varies in time, as can be seen by substituting u from [Equation 16.23](#) into [Equation 16.27](#).

$$S(x, t) = c\epsilon_0 E_0^2 \cos^2(kx - \omega t) \quad 16.29$$

Because the frequency of visible light is very high, of the order of 10^{14} Hz, the energy flux for visible light through any area is an extremely rapidly varying quantity. Most measuring devices, including our eyes, detect only an average over many cycles. The time average of the energy flux is the intensity I of the electromagnetic wave and is the power per unit area. It can be expressed by averaging the cosine function in [Equation 16.29](#) over one complete cycle, which is the same as time-averaging over many cycles (here, T is one period):

$$I = S_{\text{avg}} = c\epsilon_0 E_0^2 \frac{1}{T} \int_0^T \cos^2\left(2\pi \frac{t}{T}\right) dt. \quad 16.30$$

We can either evaluate the integral, or else note that because the sine and cosine differ merely in phase, the average over a complete cycle for $\cos^2(\xi)$ is the same as for $\sin^2(\xi)$, to obtain

$$\langle \cos^2 \xi \rangle = \frac{1}{2} [\langle \cos^2 \xi \rangle + \langle \sin^2 \xi \rangle] = \frac{1}{2} \langle 1 \rangle = \frac{1}{2}.$$

where the angle brackets $\langle \dots \rangle$ stand for the time-averaging operation. The intensity of light moving at speed c in vacuum is then found to be

$$I = S_{\text{avg}} = \frac{1}{2} c\epsilon_0 E_0^2 \quad 16.31$$

in terms of the maximum electric field strength E_0 , which is also the electric field amplitude. Algebraic manipulation produces the relationship

$$I = \frac{cB_0^2}{2\mu_0} \quad 16.32$$

where B_0 is the magnetic field amplitude, which is the same as the maximum magnetic field strength. One more expression for I_{avg} in terms of both electric and magnetic field strengths is useful. Substituting the fact that $cB_0 = E_0$, the previous expression becomes

$$I = \frac{E_0 B_0}{2\mu_0}. \quad 16.33$$

We can use whichever of the three preceding equations is most convenient, because the three equations are really just different versions of the same result: The energy in a wave is related to amplitude squared. Furthermore, because these equations are based on the assumption that the electromagnetic waves are sinusoidal, the peak intensity is twice the average intensity; that is, $I_0 = 2I$.



EXAMPLE 16.3

A Laser Beam

The beam from a small laboratory laser typically has an intensity of about $1.0 \times 10^{-3} \text{ W/m}^2$. Assuming that the beam is composed of plane waves, calculate the amplitudes of the electric and magnetic fields in the beam.

Strategy

Use the equation expressing intensity in terms of electric field to calculate the electric field from the intensity.

Solution

From [Equation 16.31](#), the intensity of the laser beam is

$$I = \frac{1}{2} c\epsilon_0 E_0^2.$$

The amplitude of the electric field is therefore

$$E_0 = \sqrt{\frac{2}{c\epsilon_0} I} = \sqrt{\frac{2}{(3.00 \times 10^8 \text{ m/s})(8.85 \times 10^{-12} \text{ F/m})} (1.0 \times 10^{-3} \text{ W/m}^2)} = 0.87 \text{ V/m}.$$

The amplitude of the magnetic field can be obtained from [Equation 16.20](#):

$$B_0 = \frac{E_0}{c} = 2.9 \times 10^{-9} \text{ T}.$$



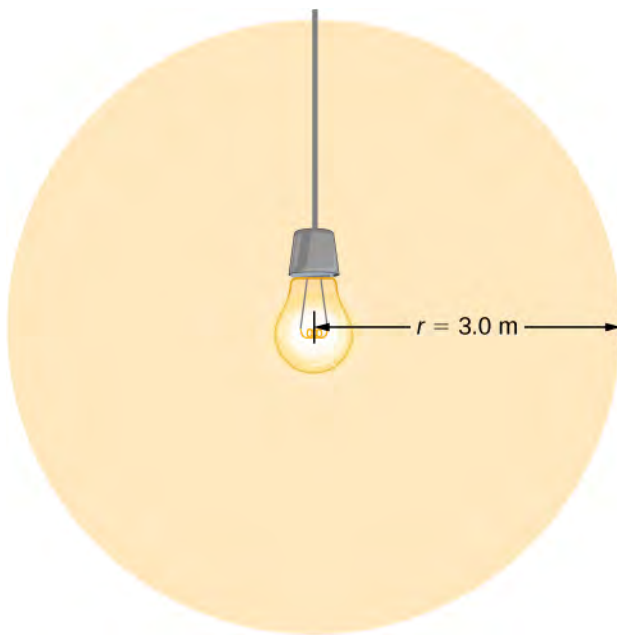
EXAMPLE 16.4

Light Bulb Fields

A light bulb emits 5.00 W of power as visible light. What are the average electric and magnetic fields from the light at a distance of 3.0 m?

Strategy

Assume the bulb's power output P is distributed uniformly over a sphere of radius 3.0 m to calculate the intensity, and from it, the electric field.



Solution

The power radiated as visible light is then

$$\begin{aligned} I &= \frac{P}{4\pi r^2} = \frac{c\epsilon_0 E_0^2}{2}, \\ E_0 &= \sqrt{\frac{2P}{4\pi r^2 c\epsilon_0}} = \sqrt{\frac{2(5.00 \text{ W})}{4\pi(3.0 \text{ m})^2 (3.00 \times 10^8 \text{ m/s})(8.85 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)}} = 5.77 \text{ N/C}, \\ B_0 &= E_0/c = 1.92 \times 10^{-8} \text{ T}. \end{aligned}$$

Significance

The intensity I falls off as the distance squared if the radiation is dispersed uniformly in all directions.

EXAMPLE 16.5

Radio Range

A 60-kW radio transmitter on Earth sends its signal to a satellite 100 km away (Figure 16.12). At what distance in the same direction would the signal have the same maximum field strength if the transmitter's output power were increased to 90 kW?

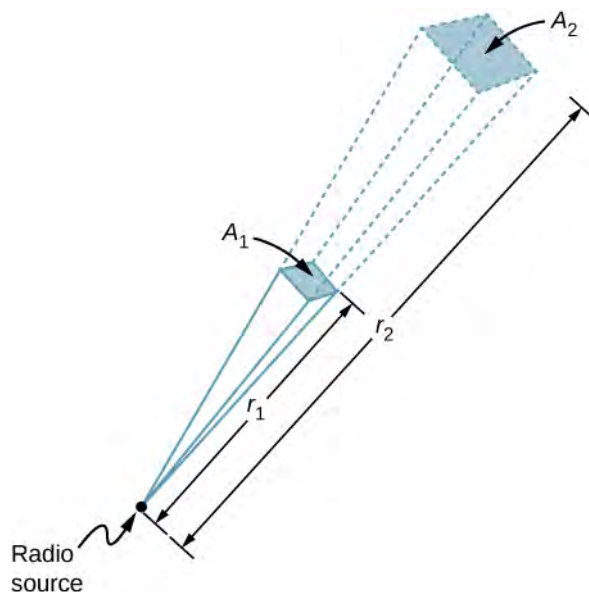


Figure 16.12 In three dimensions, a signal spreads over a solid angle as it travels outward from its source.

Strategy

The area over which the power in a particular direction is dispersed increases as distance squared, as illustrated in the figure. Change the power output P by a factor of $(90 \text{ kW}/60 \text{ kW})$ and change the area by the same factor to keep $I = \frac{P}{A} = \frac{c\epsilon_0 E_0^2}{2}$ the same. Then use the proportion of area A in the diagram to distance squared to find the distance that produces the calculated change in area.

Solution

Using the proportionality of the areas to the squares of the distances, and solving, we obtain from the diagram

$$\frac{r_2^2}{r_1^2} = \frac{A_2}{A_1} = \frac{90 \text{ W}}{60 \text{ W}},$$

$$r_2 = \sqrt{\frac{90}{60}} (100 \text{ km}) = 122 \text{ km}.$$

Significance

The range of a radio signal is the maximum distance between the transmitter and receiver that allows for normal operation. In the absence of complications such as reflections from obstacles, the intensity follows an inverse square law, and doubling the range would require multiplying the power by four.

16.4 Momentum and Radiation Pressure

Learning Objectives

By the end of this section, you will be able to:

- Describe the relationship of the radiation pressure and the energy density of an electromagnetic wave
- Explain how the radiation pressure of light, while small, can produce observable astronomical effects

Material objects consist of charged particles. An electromagnetic wave incident on the object exerts forces on the charged particles, in accordance with the Lorentz force, [Equation 16.11](#). These forces do work on the particles of the object, increasing its energy, as discussed in the previous section. The energy that sunlight carries is a familiar part of every warm sunny day. A much less familiar feature of electromagnetic radiation is the extremely weak pressure that electromagnetic radiation produces by exerting a force in the direction of the wave. This force occurs because electromagnetic waves contain and transport momentum.

To understand the direction of the force for a very specific case, consider a plane electromagnetic wave incident on a metal in which electron motion, as part of a current, is damped by the resistance of the metal, so that the average electron motion is in phase with the force causing it. This is comparable to an object moving against friction and stopping as soon as the force pushing it stops ([Figure 16.13](#)). When the electric field is in the direction of the positive y -axis, electrons move in the negative y -direction, with the magnetic field in the direction of the positive z -axis. By applying the right-hand rule, and accounting for the negative charge of the electron, we can see that the force on the electron from the magnetic field is in the direction of the positive x -axis, which is the direction of wave propagation. When the E field reverses, the B field does too, and the force is again in the same direction. Maxwell's equations together with the Lorentz force equation imply the existence of radiation pressure much more generally than this specific example, however.

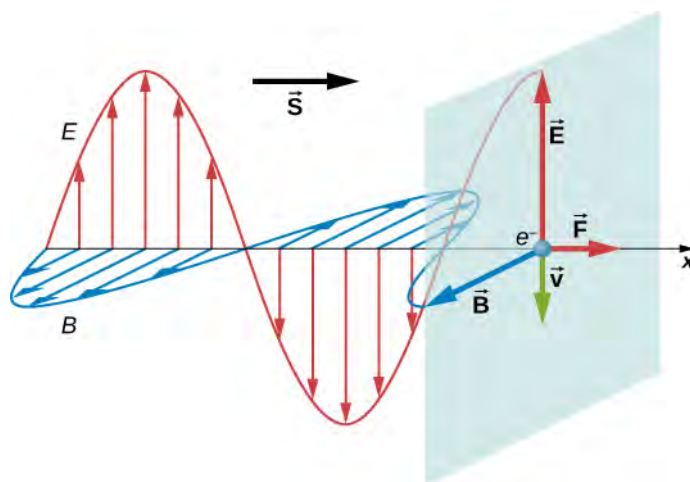


Figure 16.13 Electric and magnetic fields of an electromagnetic wave can combine to produce a force in the direction of propagation, as illustrated for the special case of electrons whose motion is highly damped by the resistance of a metal.

Maxwell predicted that an electromagnetic wave carries momentum. An object absorbing an electromagnetic wave would experience a force in the direction of propagation of the wave. The force corresponds to radiation pressure exerted on the object by the wave. The force would be twice as great if the radiation were reflected rather than absorbed.

Maxwell's prediction was confirmed in 1903 by Nichols and Hull by precisely measuring radiation pressures with a torsion balance. The schematic arrangement is shown in [Figure 16.14](#). The mirrors suspended from a fiber were housed inside a glass container. Nichols and Hull were able to obtain a small measurable deflection of the mirrors from shining light on one of them. From the measured deflection, they could calculate the unbalanced force on the mirror, and obtained agreement with the predicted value of the force.

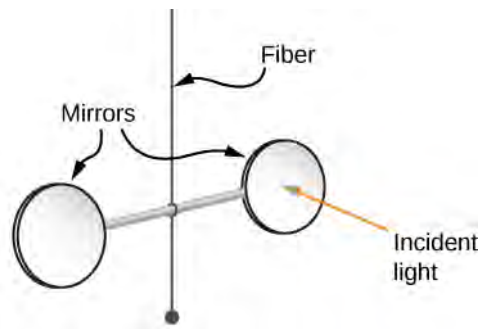


Figure 16.14 Simplified diagram of the central part of the apparatus Nichols and Hull used to precisely measure radiation pressure and confirm Maxwell's prediction.

The **radiation pressure** p_{rad} applied by an electromagnetic wave on a perfectly absorbing surface turns out to be equal to the energy density of the wave:

$$p_{\text{rad}} = u \text{ (Perfect absorber).} \quad 16.34$$

If the material is perfectly reflecting, such as a metal surface, and if the incidence is along the normal to the surface, then the pressure exerted is twice as much because the momentum direction reverses upon reflection:

$$p_{\text{rad}} = 2u \text{ (Perfect reflector).} \quad 16.35$$

We can confirm that the units are right:

$$[u] = \frac{\text{J}}{\text{m}^3} = \frac{\text{N} \cdot \text{m}}{\text{m}^3} = \frac{\text{N}}{\text{m}^2} = \text{units of pressure.}$$

[Equation 16.34](#) and [Equation 16.35](#) give the instantaneous pressure, but because the energy density oscillates rapidly, we are usually interested in the time-averaged radiation pressure, which can be written in terms of intensity:

$$p = \langle p_{\text{rad}} \rangle = \begin{cases} I/c & \text{Perfect absorber} \\ 2I/c & \text{Perfect reflector.} \end{cases} \quad 16.36$$

Radiation pressure plays a role in explaining many observed astronomical phenomena, including the appearance of comets. Comets are basically chunks of icy material in which frozen gases and particles of rock and dust are embedded. When a comet approaches the Sun, it warms up and its surface begins to evaporate. The *coma* of the comet is the hazy area around it from the gases and dust. Some of the gases and dust form tails when they leave the comet. Notice in [Figure 16.15](#) that a comet has *two* tails. The *ion tail* (or *gas tail* in [Figure 16.15](#)) is composed mainly of ionized gases. These ions interact electromagnetically with the solar wind, which is a continuous stream of charged particles emitted by the Sun. The force of the solar wind on the ionized gases is strong enough that the ion tail almost always points directly away from the Sun. The second tail is composed of dust particles. Because the *dust tail* is electrically neutral, it does not interact with the solar wind. However, this tail is affected by the radiation pressure produced by the light from the Sun. Although quite small, this pressure is strong enough to cause the dust tail to be displaced from the path of the comet.

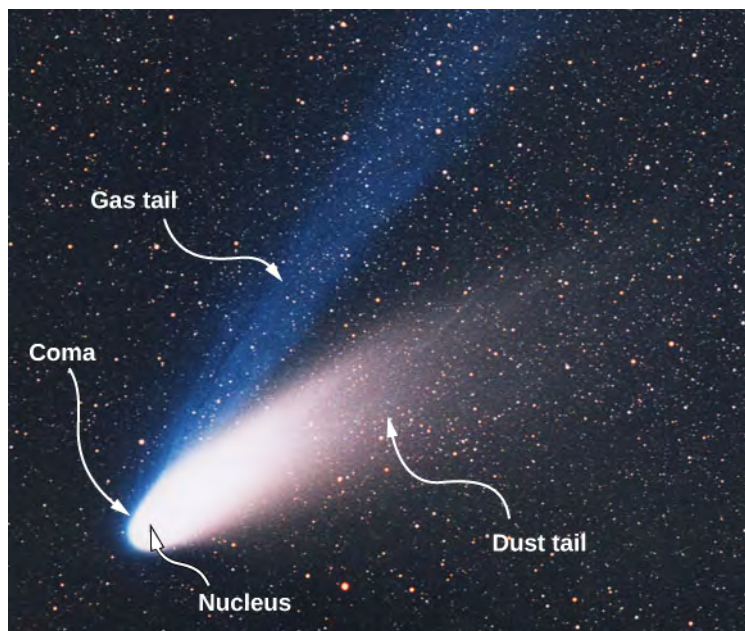


Figure 16.15 Evaporation of material being warmed by the Sun forms two tails, as shown in this photo of Comet Ison. (credit: modification of work by E. Slawik—ESO)

EXAMPLE 16.6

Halley's Comet

On February 9, 1986, Comet Halley was at its closest point to the Sun, about 9.0×10^{10} m from the center of the Sun. The average power output of the Sun is 3.8×10^{26} W.

- Calculate the radiation pressure on the comet at this point in its orbit. Assume that the comet reflects all the incident light.
- Suppose that a 10-kg chunk of material of cross-sectional area 4.0×10^{-2} m² breaks loose from the comet. Calculate the force on this chunk due to the solar radiation. Compare this force with the gravitational force of the Sun.

Strategy

Calculate the intensity of solar radiation at the given distance from the Sun and use that to calculate the radiation pressure. From the pressure and area, calculate the force.

Solution

- The intensity of the solar radiation is the average solar power per unit area. Hence, at 9.0×10^{10} m from the center of the Sun, we have

$$I = S_{\text{avg}} = \frac{3.8 \times 10^{26} \text{ W}}{4\pi(9.0 \times 10^{10} \text{ m})^2} = 3.7 \times 10^3 \text{ W/m}^2.$$

Assuming the comet reflects all the incident radiation, we obtain from [Equation 16.36](#)

$$p = \frac{2I}{c} = \frac{2(3.7 \times 10^3 \text{ W/m}^2)}{3.00 \times 10^8 \text{ m/s}} = 2.5 \times 10^{-5} \text{ N/m}^2.$$

- The force on the chunk due to the radiation is

$$\begin{aligned} F &= pA = (2.5 \times 10^{-5} \text{ N/m}^2)(4.0 \times 10^{-2} \text{ m}^2) \\ &= 1.0 \times 10^{-6} \text{ N}, \end{aligned}$$

whereas the gravitational force of the Sun is

$$F_g = \frac{GMm}{r^2} = \frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2) (2.0 \times 10^{30} \text{ kg}) (10 \text{ kg})}{(9.0 \times 10^{10} \text{ m})^2} = 0.16 \text{ N}.$$

Significance

The gravitational force of the Sun on the chunk is therefore much greater than the force of the radiation.

After Maxwell showed that light carried momentum as well as energy, a novel idea eventually emerged, initially only as science fiction. Perhaps a spacecraft with a large reflecting light sail could use radiation pressure for propulsion. Such a vehicle would not have to carry fuel. It would experience a constant but small force from solar radiation, instead of the short bursts from rocket propulsion. It would accelerate slowly, but by being accelerated continuously, it would eventually reach great speeds. A spacecraft with small total mass and a sail with a large area would be necessary to obtain a usable acceleration.

When the space program began in the 1960s, the idea started to receive serious attention from NASA. The most recent development in light propelled spacecraft has come from a citizen-funded group, the Planetary Society. It is currently testing the use of light sails to propel a small vehicle built from *CubeSats*, tiny satellites that NASA places in orbit for various research projects during space launches intended mainly for other purposes.

The *LightSail* spacecraft shown below (Figure 16.16) consists of three *CubeSats* bundled together. It has a total mass of only about 5 kg and is about the size as a loaf of bread. Its sails are made of very thin Mylar and open after launch to have a surface area of 32 m².

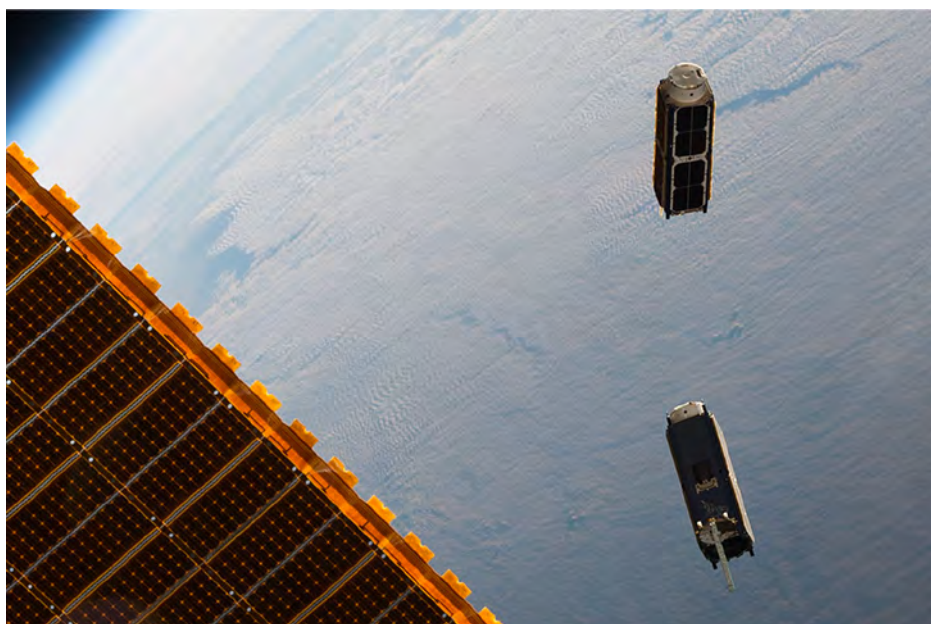


Figure 16.16 Two small *CubeSat* satellites deployed from the International Space Station in May, 2016. The solar sails open out when the *CubeSats* are far enough away from the Station. (credit: modification of work by NASA)

INTERACTIVE

The first *LightSail* spacecraft was launched in 2015 to test the sail deployment system. It was placed in low-earth orbit in 2015 by hitching a ride on an Atlas 5 rocket launched for an unrelated mission. The test was successful, but the low-earth orbit allowed too much drag on the spacecraft to accelerate it by sunlight. Eventually, it burned in the atmosphere, as expected. The next Planetary Society's *LightSail* solar sailing spacecraft is scheduled for 2016. An [illustration \(https://openstax.org/l/21lightsail\)](https://openstax.org/l/21lightsail) of the spacecraft, as it is expected to appear in flight, can be seen on the Planetary Society's website.

EXAMPLE 16.7

LightSail Acceleration

The intensity of energy from sunlight at a distance of 1 AU from the Sun is 1370 W/m^2 . The *LightSail* spacecraft has sails with total area of 32 m^2 and a total mass of 5.0 kg . Calculate the maximum acceleration *LightSail* spacecraft could achieve from radiation pressure when it is about 1 AU from the Sun.

Strategy

The maximum acceleration can be expected when the sail is opened directly facing the Sun. Use the light intensity to calculate the radiation pressure and from it, the force on the sails. Then use Newton's second law to calculate the acceleration.

Solution

The radiation pressure is

$$F = pA = 2uA = \frac{2I}{c}A = \frac{2(1370 \text{ W/m}^2)(32 \text{ m}^2)}{(3.00 \times 10^8 \text{ m/s})} = 2.92 \times 10^{-4} \text{ N}.$$

The resulting acceleration is

$$a = \frac{F}{m} = \frac{2.92 \times 10^{-4} \text{ N}}{5.0 \text{ kg}} = 5.8 \times 10^{-5} \text{ m/s}^2.$$

Significance

If this small acceleration continued for a year, the craft would attain a speed of 1829 m/s , or 6600 km/h .

CHECK YOUR UNDERSTANDING 16.5

How would the speed and acceleration of a radiation-propelled spacecraft be affected as it moved farther from the Sun on an interplanetary space flight?

16.5 The Electromagnetic Spectrum

Learning Objectives

By the end of this section, you will be able to:

- Explain how electromagnetic waves are divided into different ranges, depending on wavelength and corresponding frequency
- Describe how electromagnetic waves in different categories are produced
- Describe some of the many practical everyday applications of electromagnetic waves

Electromagnetic waves have a vast range of practical everyday applications that includes such diverse uses as communication by cell phone and radio broadcasting, WiFi, cooking, vision, medical imaging, and treating cancer. In this module, we discuss how electromagnetic waves are classified into categories such as radio, infrared, ultraviolet, and so on. We also summarize some of the main applications for each range.

The different categories of electromagnetic waves differ in their wavelength range, or equivalently, in their corresponding frequency ranges. Their properties change smoothly from one frequency range to the next, with different applications in each range. A brief overview of the production and utilization of electromagnetic waves is found in [Table 16.1](#).

Type of wave	Production	Applications	Issues
Radio	Accelerating charges	Communications Remote controls MRI	Requires control for band use
Microwaves	Accelerating charges and thermal agitation	Communications Ovens Radar Cell phone use	
Infrared	Thermal agitation and electronic transitions	Thermal imaging Heating	Absorbed by atmosphere Greenhouse effect
Visible light	Thermal agitation and electronic transitions	Photosynthesis Human vision	
Ultraviolet	Thermal agitation and electronic transitions	Sterilization Vitamin D production	Ozone depletion Cancer causing
X-rays	Inner electronic transitions and fast collisions	Security Medical diagnosis Cancer therapy	Cancer causing
Gamma rays	Nuclear decay	Nuclear medicine Security Medical diagnosis Cancer therapy	Cancer causing Radiation damage

Table 16.1 Electromagnetic Waves

The relationship $c = f\lambda$ between frequency f and wavelength λ applies to all waves and ensures that greater frequency means smaller wavelength. [Figure 16.17](#) shows how the various types of electromagnetic waves are categorized according to their wavelengths and frequencies—that is, it shows the electromagnetic spectrum.

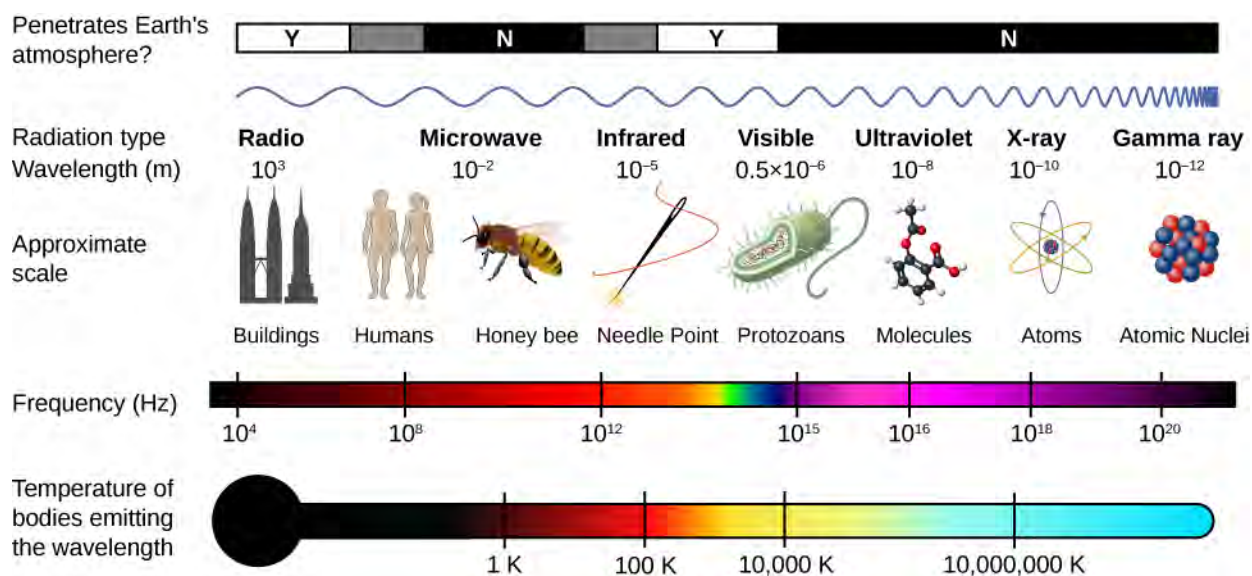


Figure 16.17 The electromagnetic spectrum, showing the major categories of electromagnetic waves.

Radio Waves

The term **radio waves** refers to electromagnetic radiation with wavelengths greater than about 0.1 m. Radio waves are commonly used for audio communications (i.e., for radios), but the term is used for electromagnetic waves in this range regardless of their application. Radio waves typically result from an alternating current in the wires of a broadcast antenna. They cover a very broad wavelength range and are divided into many subranges, including microwaves, electromagnetic waves used for AM and FM radio, cellular telephones, and TV signals.

There is no lowest frequency of radio waves, but ELF waves, or “extremely low frequency” are among the lowest frequencies commonly encountered, from 3 Hz to 3 kHz. The accelerating charge in the ac currents of electrical power lines produce electromagnetic waves in this range. ELF waves are able to penetrate sea water, which strongly absorbs electromagnetic waves of higher frequency, and therefore are useful for submarine communications.

In order to use an electromagnetic wave to transmit information, the amplitude, frequency, or phase of the wave is *modulated*, or varied in a controlled way that encodes the intended information into the wave. In AM radio transmission, the amplitude of the wave is modulated to mimic the vibrations of the sound being conveyed. Fourier’s theorem implies that the modulated AM wave amounts to a superposition of waves covering some narrow frequency range. Each AM station is assigned a specific carrier frequency that, by international agreement, is allowed to vary by ± 5 kHz. In FM radio transmission, the frequency of the wave is modulated to carry this information, as illustrated in Figure 16.18, and the frequency of each station is allowed to use 100 kHz on each side of its carrier frequency. The electromagnetic wave produces a current in a receiving antenna, and the radio or television processes the signal to produce the sound and any image. The higher the frequency of the radio wave used to carry the data, the greater the detailed variation of the wave that can be carried by modulating it over each time unit, and the more data that can be transmitted per unit of time. The assigned frequencies for AM broadcasting are 540 to 1600 kHz, and for FM are 88 MHz to 108 MHz.

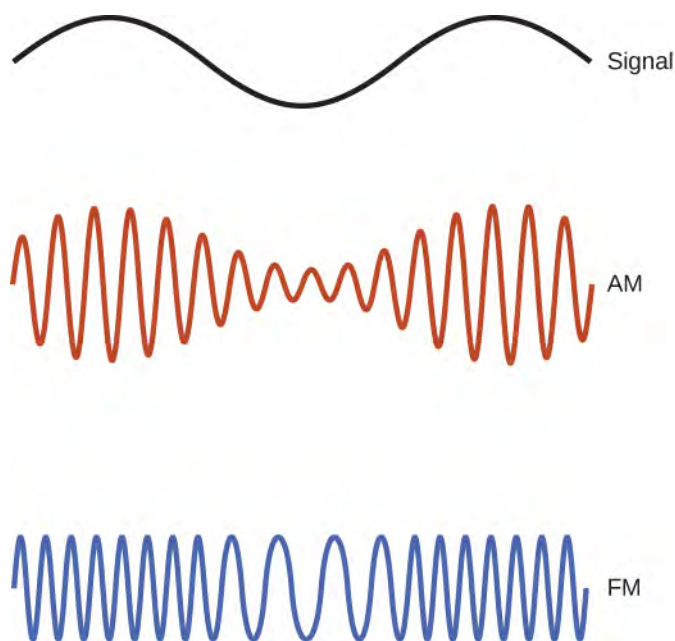


Figure 16.18 Electromagnetic waves are used to carry communications signals by varying the wave's amplitude (AM), its frequency (FM), or its phase.

Cell phone conversations, and television voice and video images are commonly transmitted as digital data, by converting the signal into a sequence of binary ones and zeros. This allows clearer data transmission when the signal is weak, and allows using computer algorithms to compress the digital data to transmit more data in each frequency range. Computer data as well is transmitted as a sequence of binary ones and zeros, each one or zero constituting one bit of data.

Microwaves

Microwaves are the highest-frequency electromagnetic waves that can be produced by currents in macroscopic circuits and devices. Microwave frequencies range from about 10^9 Hz to nearly 10^{12} Hz. Their high frequencies correspond to short wavelengths compared with other radio waves—hence the name “microwave.” Microwaves also occur naturally as the cosmic background radiation left over from the origin of the universe. Along with other ranges of electromagnetic waves, they are part of the radiation that any object above absolute zero emits and absorbs because of **thermal agitation**, that is, from the thermal motion of its atoms and molecules.

Most satellite-transmitted information is carried on microwaves. **Radar** is a common application of microwaves. By detecting and timing microwave echoes, radar systems can determine the distance to objects as diverse as clouds, aircraft, or even the surface of Venus.

Microwaves of 2.45 GHz are commonly used in microwave ovens. The electrons in a water molecule tend to remain closer to the oxygen nucleus than the hydrogen nuclei ([Figure 16.19](#)). This creates two separated centers of equal and opposite charges, giving the molecule a dipole moment (see [Electric Field](#)). The oscillating electric field of the microwaves inside the oven exerts a torque that tends to align each molecule first in one direction and then in the other, with the motion of each molecule coupled to others around it. This pumps energy into the continual thermal motion of the water to heat the food. The plate under the food contains no water, and remains relatively unheated.

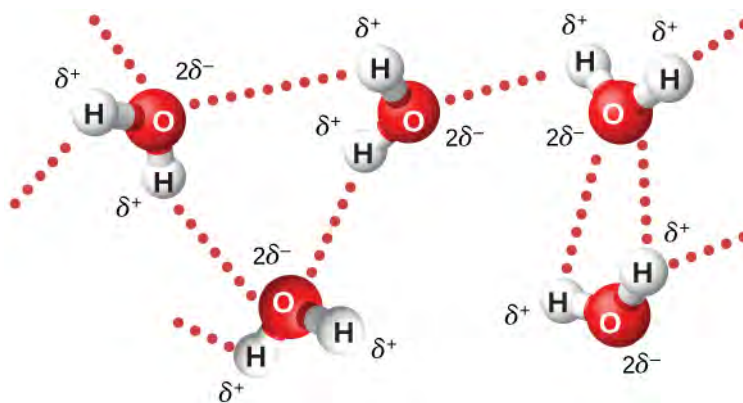


Figure 16.19 The oscillating electric field in a microwave oven exerts a torque on water molecules because of their dipole moment, and the torque reverses direction 4.90×10^9 times per second. Interactions between the molecules distributes the energy being pumped into them. The δ^+ and δ^- denote the charge distribution on the molecules.

The microwaves in a microwave oven reflect off the walls of the oven, so that the superposition of waves produces standing waves, similar to the standing waves of a vibrating guitar or violin string (see [Normal Modes of a Standing Sound Wave](#)). A rotating fan acts as a stirrer by reflecting the microwaves in different directions, and food turntables, help spread out the hot spots.



EXAMPLE 16.8

Why Microwave Ovens Heat Unevenly

How far apart are the hotspots in a 2.45-GHz microwave oven?

Strategy

Consider the waves along one direction in the oven, being reflected at the opposite wall from where they are generated.

Solution

The antinodes, where maximum intensity occurs, are half the wavelength apart, with separation

$$d = \frac{1}{2} \lambda = \frac{1}{2} \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{2 (2.45 \times 10^9 \text{ Hz})} = 6.02 \text{ cm}.$$

Significance

The distance between the hot spots in a microwave oven are determined by the wavelength of the microwaves.

A cell phone has a radio receiver and a weak radio transmitter, both of which can quickly tune to hundreds of specifically assigned microwave frequencies. The low intensity of the transmitted signal gives it an intentionally limited range. A ground-based system links the phone to only to the broadcast tower assigned to the specific small area, or cell, and smoothly transitions its connection to the next cell when the signal reception there is the stronger one. This enables a cell phone to be used while changing location.

Microwaves also provide the WiFi that enables owners of cell phones, laptop computers, and similar devices to connect wirelessly to the Internet at home and at coffee shops and airports. A wireless WiFi router is a device that exchanges data over the Internet through the cable or another connection, and uses microwaves to exchange the data wirelessly with devices such as cell phones and computers. The term WiFi itself refers to the standards followed in modulating and analyzing the microwaves so that wireless routers and devices from different manufacturers work compatibly with one another. The computer data in each direction consist of sequences of binary zeros and ones, each corresponding to a binary bit. The microwaves are in the range of 2.4 GHz to 5.0 GHz range.

Other wireless technologies also use microwaves to provide everyday communications between devices. Bluetooth developed alongside WiFi as a standard for radio communication in the 2.4-GHz range between nearby devices, for example, to link to headphones and audio earpieces to devices such as radios, or a driver's cell phone to a hands-free device to allow answering phone calls without fumbling directly with the cell phone.

Microwaves find use also in radio tagging, using RFID (radio frequency identification) technology. Examples are RFID tags attached to store merchandise, transponder for toll booths use attached to the windshield of a car, or even a chip embedded into a pet's skin. The device responds to a microwave signal by emitting a signal of its own with encoded information, allowing stores to quickly ring up items at their cash registers, drivers to charge tolls to their account without stopping, and lost pets to be reunited with their owners. NFC (near field communication) works similarly, except it is much shorter range. Its mechanism of interaction is the induced magnetic field at microwave frequencies between two coils. Cell phones that have NFC capability and the right software can supply information for purchases using the cell phone instead of a physical credit card. The very short range of the data transfer is a desired security feature in this case.

Infrared Radiation

The boundary between the microwave and infrared regions of the electromagnetic spectrum is not well defined (see [Figure 16.17](#)). **Infrared radiation** is generally produced by thermal motion, and the vibration and rotation of atoms and molecules. Electronic transitions in atoms and molecules can also produce infrared radiation. About half of the solar energy arriving at Earth is in the infrared region, with most of the rest in the visible part of the spectrum. About 23% of the solar energy is absorbed in the atmosphere, about 48% is absorbed at Earth's surface, and about 29% is reflected back into space.¹

The range of infrared frequencies extends up to the lower limit of visible light, just below red. In fact, infrared means “below red.” Water molecules rotate and vibrate particularly well at infrared frequencies.

Reconnaissance satellites can detect buildings, vehicles, and even individual humans by their infrared emissions, whose power radiation is proportional to the fourth power of the absolute temperature. More mundanely, we use infrared lamps, including those called *quartz heaters*, to preferentially warm us because we absorb infrared better than our surroundings.

The familiar handheld “remotes” for changing channels and settings on television sets often transmit their signal by modulating an infrared beam. If you try to use a TV remote without the infrared emitter being in direct line of sight with the infrared detector, you may find the television not responding. Some remotes use Bluetooth instead and reduce this annoyance.

Visible Light

Visible light is the narrow segment of the electromagnetic spectrum between about 400 nm and about 750 nm to which the normal human eye responds. Visible light is produced by vibrations and rotations of atoms and molecules, as well as by electronic transitions within atoms and molecules. The receivers or detectors of light largely utilize electronic transitions.

Red light has the lowest frequencies and longest wavelengths, whereas violet has the highest frequencies and shortest wavelengths ([Figure 16.20](#)). Blackbody radiation from the Sun peaks in the visible part of the spectrum but is more intense in the red than in the violet, making the sun yellowish in appearance.

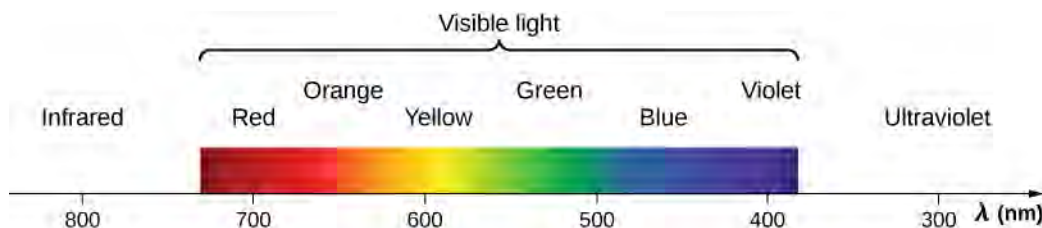


Figure 16.20 A small part of the electromagnetic spectrum that includes its visible components. The divisions between infrared, visible, and ultraviolet are not perfectly distinct, nor are those between the seven rainbow colors.

1 <http://earthobservatory.nasa.gov/Features/EnergyBalance/page4.php>

Living things—plants and animals—have evolved to utilize and respond to parts of the electromagnetic spectrum in which they are embedded. We enjoy the beauty of nature through visible light. Plants are more selective. Photosynthesis uses parts of the visible spectrum to make sugars.

Ultraviolet Radiation

Ultraviolet means “above violet.” The electromagnetic frequencies of **ultraviolet radiation (UV)** extend upward from violet, the highest-frequency visible light. The highest-frequency ultraviolet overlaps with the lowest-frequency X-rays. The wavelengths of ultraviolet extend from 400 nm down to about 10 nm at its highest frequencies. Ultraviolet is produced by atomic and molecular motions and electronic transitions.

UV radiation from the Sun is broadly subdivided into three wavelength ranges: UV-A (320–400 nm) is the lowest frequency, then UV-B (290–320 nm) and UV-C (220–290 nm). Most UV-B and all UV-C are absorbed by ozone (O₃) molecules in the upper atmosphere. Consequently, 99% of the solar UV radiation reaching Earth's surface is UV-A.

Sunburn is caused by large exposures to UV-B and UV-C, and repeated exposure can increase the likelihood of skin cancer. The tanning response is a defense mechanism in which the body produces pigments in inert skin layers to reduce exposure of the living cells below.

As examined in a later chapter, the shorter the wavelength of light, the greater the energy change of an atom or molecule that absorbs the light in an electronic transition. This makes short-wavelength ultraviolet light damaging to living cells. It also explains why ultraviolet radiation is better able than visible light to cause some materials to glow, or *fluoresce*.

Besides the adverse effects of ultraviolet radiation, there are also benefits of exposure in nature and uses in technology. Vitamin D production in the skin results from exposure to UV-B radiation, generally from sunlight. Several studies suggest vitamin D deficiency is associated with the development of a range of cancers (prostate, breast, colon), as well as osteoporosis. Low-intensity ultraviolet has applications such as providing the energy to cause certain dyes to fluoresce and emit visible light, for example, in printed money to display hidden watermarks as counterfeit protection.

X-Rays

X-rays have wavelengths from about 10^{-8} m to 10^{-12} m. They have shorter wavelengths, and higher frequencies, than ultraviolet, so that the energy they transfer at an atomic level is greater. As a result, X-rays have adverse effects on living cells similar to those of ultraviolet radiation, but they are more penetrating. Cancer and genetic defects can be induced by X-rays. Because of their effect on rapidly dividing cells, X-rays can also be used to treat and even cure cancer.

The widest use of X-rays is for imaging objects that are opaque to visible light, such as the human body or aircraft parts. In humans, the risk of cell damage is weighed carefully against the benefit of the diagnostic information obtained.

Gamma Rays

Soon after nuclear radioactivity was first detected in 1896, it was found that at least three distinct types of radiation were being emitted, and these were designated as alpha, beta, and gamma rays. The most penetrating nuclear radiation, the **gamma ray (γ ray)**, was later found to be an extremely high-frequency electromagnetic wave.

The lower end of the γ -ray frequency range overlaps the upper end of the X-ray range. Gamma rays have characteristics identical to X-rays of the same frequency—they differ only in source. The name “gamma rays” is generally used for electromagnetic radiation emitted by a nucleus, while X-rays are generally produced by bombarding a target with energetic electrons in an X-ray tube. At higher frequencies, γ rays are more penetrating and more damaging to living tissue. They have many of the same uses as X-rays, including cancer therapy. Gamma radiation from radioactive materials is used in nuclear medicine.

 INTERACTIVE

Use this [simulation \(https://openstax.org/l/21simlightmol\)](https://openstax.org/l/21simlightmol) to explore how light interacts with molecules in our atmosphere.

Explore how light interacts with molecules in our atmosphere.

Identify that absorption of light depends on the molecule and the type of light.

Relate the energy of the light to the resulting motion.

Identify that energy increases from microwave to ultraviolet.

Predict the motion of a molecule based on the type of light it absorbs.

 CHECK YOUR UNDERSTANDING 16.6

How do the electromagnetic waves for the different kinds of electromagnetic radiation differ?

CHAPTER REVIEW

Key Terms

displacement current extra term in Maxwell's equations that is analogous to a real current but accounts for a changing electric field producing a magnetic field, even when the real current is present

gamma ray (γ ray) extremely high frequency electromagnetic radiation emitted by the nucleus of an atom, either from natural nuclear decay or induced nuclear processes in nuclear reactors and weapons; the lower end of the γ -ray frequency range overlaps the upper end of the X-ray range, but γ rays can have the highest frequency of any electromagnetic radiation

infrared radiation region of the electromagnetic spectrum with a frequency range that extends from just below the red region of the visible light spectrum up to the microwave region, or from $0.74 \mu\text{m}$ to $300 \mu\text{m}$

Maxwell's equations set of four equations that comprise a complete, overarching theory of electromagnetism

microwaves electromagnetic waves with wavelengths in the range from 1 mm to 1 m; they can be produced by currents in macroscopic circuits and devices

Poynting vector vector equal to the cross product of the electric- and magnetic fields, that describes

the flow of electromagnetic energy through a surface

radar common application of microwaves; radar can determine the distance to objects as diverse as clouds and aircraft, as well as determine the speed of a car or the intensity of a rainstorm

radiation pressure force divided by area applied by an electromagnetic wave on a surface

radio waves electromagnetic waves with wavelengths in the range from 1 mm to 100 km; they are produced by currents in wires and circuits and by astronomical phenomena

thermal agitation thermal motion of atoms and molecules in any object at a temperature above absolute zero, which causes them to emit and absorb radiation

ultraviolet radiation electromagnetic radiation in the range extending upward in frequency from violet light and overlapping with the lowest X-ray frequencies, with wavelengths from 400 nm down to about 10 nm

visible light narrow segment of the electromagnetic spectrum to which the normal human eye responds, from about 400 to 750 nm

X-ray invisible, penetrating form of very high frequency electromagnetic radiation, overlapping both the ultraviolet range and the γ -ray range

Key Equations

Displacement current

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

Gauss's law

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

Gauss's law for magnetism

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Faraday's law

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_m}{dt}$$

Ampère-Maxwell law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Wave equation for plane EM wave

$$\frac{\partial^2 E_y}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}$$

Speed of EM waves

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Ratio of E field to B field in electromagnetic wave	$c = \frac{E}{B}$
Energy flux (Poynting) vector	$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$
Average intensity of an electromagnetic wave	$I = S_{\text{avg}} = \frac{c\epsilon_0 E_0^2}{2} = \frac{cB_0^2}{2\mu_0} = \frac{E_0 B_0}{2\mu_0}$
Radiation pressure	$p = \begin{cases} I/c & \text{Perfect absorber} \\ 2I/c & \text{Perfect reflector} \end{cases}$

Summary

16.1 Maxwell's Equations and Electromagnetic Waves

- Maxwell's prediction of electromagnetic waves resulted from his formulation of a complete and symmetric theory of electricity and magnetism, known as Maxwell's equations.
- The four Maxwell's equations together with the Lorentz force law encompass the major laws of electricity and magnetism. The first of these is Gauss's law for electricity; the second is Gauss's law for magnetism; the third is Faraday's law of induction (including Lenz's law); and the fourth is Ampère's law in a symmetric formulation that adds another source of magnetism, namely changing electric fields.
- The symmetry introduced between electric and magnetic fields through Maxwell's displacement current explains the mechanism of electromagnetic wave propagation, in which changing magnetic fields produce changing electric fields and vice versa.
- Although light was already known to be a wave, the nature of the wave was not understood before Maxwell. Maxwell's equations also predicted electromagnetic waves with wavelengths and frequencies outside the range of light. These theoretical predictions were first confirmed experimentally by Heinrich Hertz.

16.2 Plane Electromagnetic Waves

- Maxwell's equations predict that the directions of the electric and magnetic fields of the wave, and the wave's direction of propagation, are all mutually perpendicular. The electromagnetic wave is a transverse wave.
- The strengths of the electric and magnetic parts of the wave are related by $c = E/B$, which implies that the magnetic field B is very weak

relative to the electric field E .

- Accelerating charges create electromagnetic waves (for example, an oscillating current in a wire produces electromagnetic waves with the same frequency as the oscillation).

16.3 Energy Carried by Electromagnetic Waves

- The energy carried by any wave is proportional to its amplitude squared. For electromagnetic waves, this means intensity can be expressed as

$$I = \frac{c\epsilon_0 E_0^2}{2}$$

where I is the average intensity in W/m^2 and E_0 is the maximum electric field strength of a continuous sinusoidal wave. This can also be expressed in terms of the maximum magnetic field strength B_0 as

$$I = \frac{cB_0^2}{2\mu_0}$$

and in terms of both electric and magnetic fields as

$$I = \frac{E_0 B_0}{2\mu_0}.$$

The three expressions for I_{avg} are all equivalent.

16.4 Momentum and Radiation Pressure

- Electromagnetic waves carry momentum and exert radiation pressure.
- The radiation pressure of an electromagnetic wave is directly proportional to its energy density.
- The pressure is equal to twice the electromagnetic energy intensity if the wave is reflected and equal to the incident energy intensity if the wave is absorbed.

16.5 The Electromagnetic Spectrum

- The relationship among the speed of propagation, wavelength, and frequency for any wave is given by $v = f\lambda$, so that for electromagnetic waves, $c = f\lambda$, where f is the frequency, λ is the wavelength, and c is the

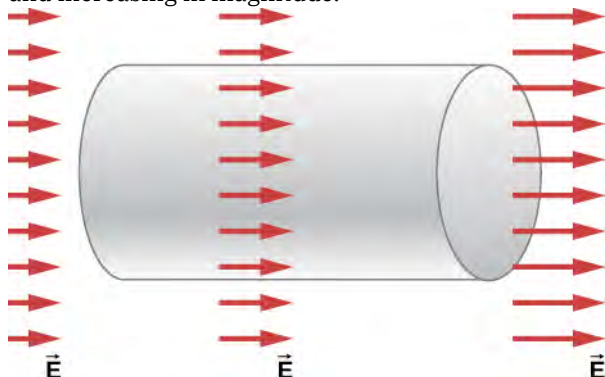
speed of light.

- The electromagnetic spectrum is separated into many categories and subcategories, based on the frequency and wavelength, source, and uses of the electromagnetic waves.

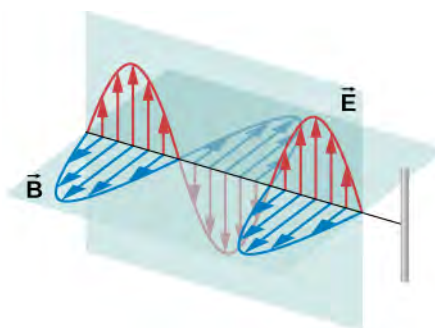
Conceptual Questions

16.1 Maxwell's Equations and Electromagnetic Waves

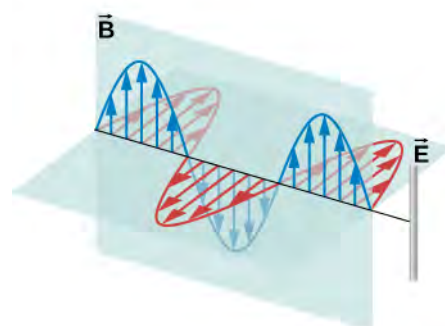
- Explain how the displacement current maintains the continuity of current in a circuit containing a capacitor.
- Describe the field lines of the induced magnetic field along the edge of the imaginary horizontal cylinder shown below if the cylinder is in a spatially uniform electric field that is horizontal, pointing to the right, and increasing in magnitude.



- Why is it much easier to demonstrate in a student lab that a changing magnetic field induces an electric field than it is to demonstrate that a changing electric field produces a magnetic field?



(a)

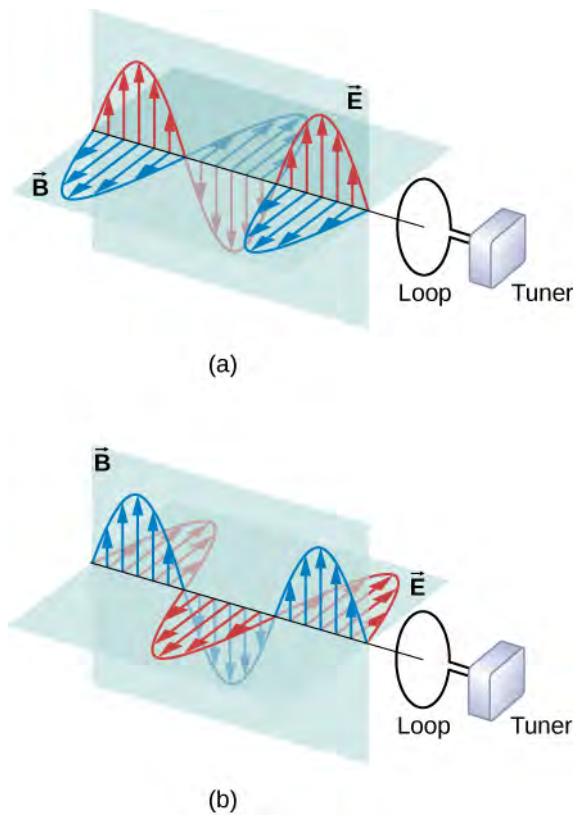


(b)

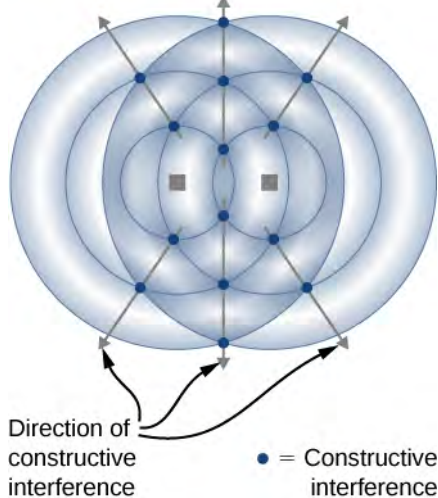
- In which situation shown below will the electromagnetic wave be more successful in inducing a current in the loop? Explain.

16.2 Plane Electromagnetic Waves

- If the electric field of an electromagnetic wave is oscillating along the z -axis and the magnetic field is oscillating along the x -axis, in what possible direction is the wave traveling?
- In which situation shown below will the electromagnetic wave be more successful in inducing a current in the wire? Explain.



7. Under what conditions might wires in a circuit where the current flows in only one direction emit electromagnetic waves?
8. Shown below is the interference pattern of two radio antennas broadcasting the same signal. Explain how this is analogous to the interference pattern for sound produced by two speakers. Could this be used to make a directional antenna system that broadcasts preferentially in certain directions? Explain.



16.3 Energy Carried by Electromagnetic Waves

9. When you stand outdoors in the sunlight, why can you feel the energy that the sunlight carries, but not the momentum it carries?
10. How does the intensity of an electromagnetic wave depend on its electric field? How does it depend on its magnetic field?
11. What is the physical significance of the Poynting vector?
12. A 2.0-mW helium-neon laser transmits a continuous beam of red light of cross-sectional area 0.25 cm^2 . If the beam does not diverge appreciably, how would its rms electric field vary with distance from the laser? Explain.

16.4 Momentum and Radiation Pressure

13. Why is the radiation pressure of an electromagnetic wave on a perfectly reflecting surface twice as large as the pressure on a perfectly absorbing surface?
14. Why did the early Hubble Telescope photos of Comet Ison approaching Earth show it to have merely a fuzzy coma around it, and not the pronounced double tail that developed later (see below)?



Figure 16.21 (credit: modification of work by NASA, ESA, J.-Y. Li (Planetary Science Institute), and the Hubble Comet ISON Imaging Science Team)

15. (a) If the electric field and magnetic field in a sinusoidal plane wave were interchanged, in which direction relative to before would the energy propagate?
(b) What if the electric and the magnetic fields were both changed to their negatives?

16.5 The Electromagnetic Spectrum

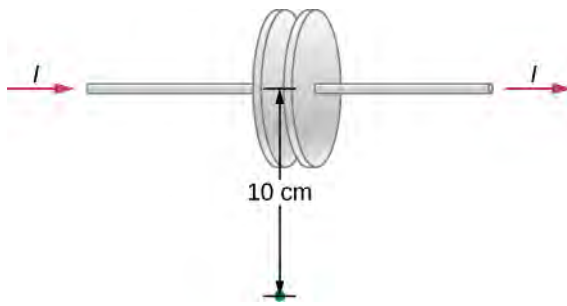
16. Compare the speed, wavelength, and frequency of radio waves and X-rays traveling in a vacuum.
17. Accelerating electric charge emits electromagnetic radiation. How does this apply in each case: (a) radio waves, (b) infrared radiation.
18. Compare and contrast the meaning of the prefix “micro” in the names of SI units in the term *microwaves*.
19. Part of the light passing through the air is scattered in all directions by the molecules comprising the atmosphere. The wavelengths of visible light are larger than molecular sizes, and the scattering is strongest for wavelengths of light closest to sizes of molecules.
(a) Which of the main colors of light is scattered the most? (b) Explain why this would give the sky its familiar background color at midday.
20. When a bowl of soup is removed from a microwave oven, the soup is found to be steaming hot, whereas the bowl is only warm to the touch. Discuss the temperature changes that have occurred in terms of energy transfer.
21. Certain orientations of a broadcast television antenna give better reception than others for a particular station. Explain.
22. What property of light corresponds to loudness in sound?
23. Is the visible region a major portion of the electromagnetic spectrum?
24. Can the human body detect electromagnetic radiation that is outside the visible region of the spectrum?
25. Radio waves normally have their E and B fields in specific directions, whereas visible light usually has its E and B fields in random and rapidly changing directions that are perpendicular to each other and to the propagation direction. Can you explain why?
26. Give an example of resonance in the reception of electromagnetic waves.
27. Illustrate that the size of details of an object that can be detected with electromagnetic waves is related to their wavelength, by comparing details observable with two different types (for example, radar and visible light).
28. In which part of the electromagnetic spectrum are each of these waves:
(a) $f = 10.0$ kHz, (b) $f = \lambda = 750$ nm,
(c) $f = 1.25 \times 10^8$ Hz, (d) 0.30 nm
29. In what range of electromagnetic radiation are the electromagnetic waves emitted by power lines in a country that uses 50-Hz ac current?
30. If a microwave oven could be modified to merely tune the waves generated to be in the infrared range instead of using microwaves, how would this affect the uneven heating of the oven?
31. A leaky microwave oven in a home can sometimes cause interference with the homeowner’s WiFi system. Why?
32. When a television news anchor in a studio speaks to a reporter in a distant country, there is sometimes a noticeable lag between when the anchor speaks in the studio and when the remote reporter hears it and replies. Explain what causes this delay.

Problems

16.1 Maxwell’s Equations and Electromagnetic Waves

33. Show that the magnetic field at a distance r from the axis of two circular parallel plates, produced by placing charge $Q(t)$ on the plates is

$$B_{\text{ind}} = \frac{\mu_0}{2\pi r} \frac{dQ(t)}{dt}.$$
34. Express the displacement current in a capacitor in terms of the capacitance and the rate of change of the voltage across the capacitor.
35. A potential difference $V(t) = V_0 \sin \omega t$ is maintained across a parallel-plate capacitor with capacitance C consisting of two circular parallel plates. A thin wire with resistance R connects the centers of the two plates, allowing charge to leak between plates while they are charging.
(a) Obtain expressions for the leakage current $I_{\text{res}}(t)$ in the thin wire. Use these results to obtain an expression for the current $I_{\text{real}}(t)$ in the wires connected to the capacitor.
(b) Find the displacement current in the space between the plates from the changing electric field between the plates.
(c) Compare $I_{\text{real}}(t)$ with the sum of the displacement current $I_d(t)$ and resistor current $I_{\text{res}}(t)$ between the plates, and explain why the relationship you observe would be expected.
36. Suppose the parallel-plate capacitor shown below is accumulating charge at a rate of 0.010 C/s. What is the induced magnetic field at a distance of 10 cm from the capacitor?



37. The potential difference $V(t)$ between parallel plates shown above is instantaneously increasing at a rate of 10^7 V/s. What is the displacement current between the plates if the separation of the plates is 1.00 cm and they have an area of 0.200 m^2 ?
38. A parallel-plate capacitor has a plate area of $A = 0.250 \text{ m}^2$ and a separation of 0.0100 m . What must be the angular frequency ω for a voltage $V(t) = V_0 \sin \omega t$ with $V_0 = 100 \text{ V}$ to produce a maximum displacement induced current of 1.00 A between the plates?
39. The voltage across a parallel-plate capacitor with area $A = 800 \text{ cm}^2$ and separation $d = 2 \text{ mm}$ varies sinusoidally as $V = (15 \text{ mV}) \cos(150t)$, where t is in seconds. Find the displacement current between the plates.
40. The voltage across a parallel-plate capacitor with area A and separation d varies with time t as $V = at^2$, where a is a constant. Find the displacement current between the plates.

16.2 Plane Electromagnetic Waves

41. If the Sun suddenly turned off, we would not know it until its light stopped coming. How long would that be, given that the Sun is $1.496 \times 10^{11} \text{ m}$ away?
42. What is the maximum electric field strength in an electromagnetic wave that has a maximum magnetic field strength of $5.00 \times 10^{-4} \text{ T}$ (about 10 times Earth's magnetic field)?
43. An electromagnetic wave has a frequency of 12 MHz. What is its wavelength in vacuum?
44. If electric and magnetic field strengths vary sinusoidally in time at frequency 1.00 GHz , being zero at $t = 0$, then $E = E_0 \sin 2\pi ft$ and $B = B_0 \sin 2\pi ft$. (a) When are the field strengths next equal to zero? (b) When do they reach their most negative value? (c) How much time is needed for them to complete one cycle?
45. The electric field of an electromagnetic wave traveling in vacuum is described by the following wave function:

$\vec{E} = (5.00 \text{ V/m}) \cos [kx - (6.00 \times 10^9 \text{ s}^{-1})t + 0.40] \hat{j}$
 where k is the wavenumber in rad/m , x is in m , t is in s .
 Find the following quantities:

- (a) amplitude
 (b) frequency
 (c) wavelength
 (d) the direction of the travel of the wave
 (e) the associated magnetic field wave
46. A plane electromagnetic wave of frequency 20 GHz moves in the positive y -axis direction such that its electric field is pointed along the z -axis. The amplitude of the electric field is 10 V/m . The start of time is chosen so that at $t = 0$, the electric field has a value 10 V/m at the origin. (a) Write the wave function that will describe the electric field wave. (b) Find the wave function that will describe the associated magnetic field wave.
47. The following represents an electromagnetic wave traveling in the direction of the positive y -axis:
 $E_x = 0; E_y = E_0 \cos(kx - \omega t); E_z = 0$
 $B_x = 0; B_y = 0; B_z = B_0 \cos(kx - \omega t)$
 The wave is passing through a wide tube of circular cross-section of radius R whose axis is along the y -axis. Find the expression for the displacement current through the tube.

16.3 Energy Carried by Electromagnetic Waves

48. While outdoors on a sunny day, a student holds a large convex lens of radius 4.0 cm above a sheet of paper to produce a bright spot on the paper that is 1.0 cm in radius, rather than a sharp focus. By what factor is the electric field in the bright spot of light related to the electric field of sunlight leaving the side of the lens facing the paper?
49. A plane electromagnetic wave travels northward. At one instant, its electric field has a magnitude of 6.0 V/m and points eastward. What are the magnitude and direction of the magnetic field at this instant?
50. The electric field of an electromagnetic wave is given by $E = (6.0 \times 10^{-3} \text{ V/m}) \sin \left[2\pi \left(\frac{x}{18 \text{ m}} - \frac{t}{6.0 \times 10^{-8} \text{ s}} \right) \right] \hat{j}$. Write the equations for the associated magnetic field and Poynting vector.
51. A radio station broadcasts at a frequency of 760 kHz . At a receiver some distance from the antenna, the maximum magnetic field of the

electromagnetic wave detected is
 $2.15 \times 10^{-11} \text{ T}$.

- (a) What is the maximum electric field? (b) What is the wavelength of the electromagnetic wave?
52. The filament in a clear incandescent light bulb radiates visible light at a power of 5.00 W. Model the glass part of the bulb as a sphere of radius $r_0 = 3.00 \text{ cm}$ and calculate the amount of electromagnetic energy from visible light inside the bulb.
53. At what distance does a 100-W lightbulb produce the same intensity of light as a 75-W lightbulb produces 10 m away? (Assume both have the same efficiency for converting electrical energy in the circuit into emitted electromagnetic energy.)
54. An incandescent light bulb emits only 2.6 W of its power as visible light. What is the rms electric field of the emitted light at a distance of 3.0 m from the bulb?
55. A 150-W lightbulb emits 5% of its energy as electromagnetic radiation. What is the magnitude of the average Poynting vector 10 m from the bulb?
56. A small helium-neon laser has a power output of 2.5 mW. What is the electromagnetic energy in a 1.0-m length of the beam?
57. At the top of Earth's atmosphere, the time-averaged Poynting vector associated with sunlight has a magnitude of about 1.4 kW/m^2 .
 (a) What are the maximum values of the electric and magnetic fields for a wave of this intensity?
 (b) What is the total power radiated by the sun? Assume that the Earth is $1.5 \times 10^{11} \text{ m}$ from the Sun and that sunlight is composed of electromagnetic plane waves.
58. The magnetic field of a plane electromagnetic wave moving along the z axis is given by $\vec{B} = B_0 (\cos kz + \omega t) \hat{j}$, where $B_0 = 5.00 \times 10^{-10} \text{ T}$ and $k = 3.14 \times 10^{-2} \text{ m}^{-1}$.
 (a) Write an expression for the electric field associated with the wave. (b) What are the frequency and the wavelength of the wave? (c) What is its average Poynting vector?
59. What is the intensity of an electromagnetic wave with a peak electric field strength of 125 V/m?
60. Assume the helium-neon lasers commonly used in student physics laboratories have power outputs of 0.500 mW. (a) If such a laser beam is projected onto a circular spot 1.00 mm in diameter, what is its intensity? (b) Find the peak magnetic field strength. (c) Find the peak electric field strength.
61. An AM radio transmitter broadcasts 50.0 kW of power uniformly in all directions. (a) Assuming all of the radio waves that strike the ground are completely absorbed, and that there is no absorption by the atmosphere or other objects, what is the intensity 30.0 km away? (*Hint:* Half the power will be spread over the area of a hemisphere.) (b) What is the maximum electric field strength at this distance?
62. Suppose the maximum safe intensity of microwaves for human exposure is taken to be 1.00 W/m^2 . (a) If a radar unit leaks 10.0 W of microwaves (other than those sent by its antenna) uniformly in all directions, how far away must you be to be exposed to an intensity considered to be safe? Assume that the power spreads uniformly over the area of a sphere with no complications from absorption or reflection. (b) What is the maximum electric field strength at the safe intensity? (Note that early radar units leaked more than modern ones do. This caused identifiable health problems, such as cataracts, for people who worked near them.)
63. A 2.50-m-diameter university communications satellite dish receives TV signals that have a maximum electric field strength (for one channel) of $7.50 \text{ } \mu\text{V/m}$ (see below). (a) What is the intensity of this wave? (b) What is the power received by the antenna? (c) If the orbiting satellite broadcasts uniformly over an area of $1.50 \times 10^{13} \text{ m}^2$ (a large fraction of North America), how much power does it radiate?



64. Lasers can be constructed that produce an extremely high intensity electromagnetic wave for a brief time—called pulsed lasers. They are used to initiate nuclear fusion, for example. Such a laser may produce an electromagnetic wave with a maximum electric field strength of 1.00×10^{11} V/m for a time of 1.00 ns. (a) What is the maximum magnetic field strength in the wave? (b) What is the intensity of the beam? (c) What energy does it deliver on an 1.00-mm^2 area?

16.4 Momentum and Radiation Pressure

65. A 150-W lightbulb emits 5% of its energy as electromagnetic radiation. What is the radiation pressure on an absorbing sphere of radius 10 m that surrounds the bulb?
66. What pressure does light emitted uniformly in all directions from a 100-W incandescent light bulb exert on a mirror at a distance of 3.0 m, if 2.6 W of the power is emitted as visible light?
67. A microscopic spherical dust particle of radius $2\text{ }\mu\text{m}$ and mass $10\text{ }\mu\text{g}$ is moving in outer space at a constant speed of 30 cm/sec. A wave of light strikes it from the opposite direction of its motion and gets absorbed. Assuming the particle accelerates opposite to the motion uniformly to zero speed in one second, what is the average electric field amplitude in the light?
68. A Styrofoam spherical ball of radius 2 mm and mass $20\text{ }\mu\text{g}$ is to be suspended by the radiation pressure in a vacuum tube in a lab. How much intensity will be required if the light is completely absorbed the ball?
69. Suppose that $\vec{S}_{\text{avg}}$ for sunlight at a point on the surface of Earth is 900 W/m^2 . (a) If sunlight falls perpendicularly on a kite with a reflecting surface of area 0.75 m^2 , what is the average force on the kite due to radiation pressure? (b) How is your answer affected if the kite material is black and absorbs all sunlight?
70. Sunlight reaches the ground with an intensity of about 1.0 kW/m^2 . A sunbather has a body surface area of 0.8 m^2 facing the sun while reclining on a beach chair on a clear day. (a) how much energy from direct sunlight reaches the sunbather's skin per second? (b) What pressure does the sunlight exert if it is absorbed?
71. Suppose a spherical particle of mass m and radius R in space absorbs light of intensity I for time t . (a) How much work does the radiation pressure do to accelerate the particle from rest in the given time it absorbs the light? (b) How much energy carried by the electromagnetic waves is absorbed by the particle over this time based on the radiant energy incident on the particle?

16.5 The Electromagnetic Spectrum

72. How many helium atoms, each with a radius of about 31 pm, must be placed end to end to have a length equal to one wavelength of 470 nm blue light?
73. If you wish to detect details of the size of atoms (about 0.2 nm) with electromagnetic radiation, it must have a wavelength of about this size. (a) What is its frequency? (b) What type of electromagnetic radiation might this be?
74. Find the frequency range of visible light, given that it encompasses wavelengths from 380 to 760 nm.
75. (a) Calculate the wavelength range for AM radio given its frequency range is 540 to 1600 kHz. (b) Do the same for the FM frequency range of 88.0 to 108 MHz.
76. Radio station WWVB, operated by the National Institute of Standards and Technology (NIST) from Fort Collins, Colorado, at a low frequency of 60 kHz, broadcasts a time synchronization signal whose range covers the entire continental US. The timing of the synchronization signal is controlled by a set of atomic clocks to an accuracy of 1×10^{-12} s, and repeats every 1 minute. The signal is used

- for devices, such as radio-controlled watches, that automatically synchronize with it at preset local times. WWVB's long wavelength signal tends to propagate close to the ground.
- (a) Calculate the wavelength of the radio waves from WWVB.
- (b) Estimate the error that the travel time of the signal causes in synchronizing a radio controlled watch in Norfolk, Virginia, which is 1570 mi (2527 km) from Fort Collins, Colorado.
- 77.** An outdoor WiFi unit for a picnic area has a 100-mW output and a range of about 30 m. What output power would reduce its range to 12 m for use with the same devices as before? Assume there are no obstacles in the way and that microwaves into the ground are simply absorbed.
- 78.** The prefix “mega” (M) and “kilo” (k), when referring to amounts of computer data, refer to factors of 1024 or 2^{10} rather than 1000 for the prefix *kilo*, and $1024^2 = 2^{20}$ rather than $1,000,000$ for the prefix *Mega* (M). If a wireless (WiFi) router transfers 150 Mbps of data, how many bits per second is that in decimal arithmetic?
- 79.** A computer user finds that his wireless router transmits data at a rate of 75 Mbps (megabits per second). Compare the average time to transmit one bit of data with the time difference between the wifi signal reaching an observer's cell phone directly and by bouncing back to the observer from a wall 8.00 m past the observer.
- 80.** (a) The ideal size (most efficient) for a broadcast antenna with one end on the ground is one-fourth the wavelength ($\lambda/4$) of the electromagnetic radiation being sent out. If a new radio station has such an antenna that is 50.0 m high, what frequency does it broadcast most efficiently? Is this in the AM or FM band? (b) Discuss the analogy of the fundamental resonant mode of an air column closed at one end to the resonance of currents on an antenna that is one-fourth their wavelength.
- 81.** What are the wavelengths of (a) X-rays of frequency 2.0×10^{17} Hz? (b) Yellow light of frequency 5.1×10^{14} Hz? (c) Gamma rays of frequency 1.0×10^{23} Hz?
- 82.** For red light of $\lambda = 660$ nm, what are f , ω , and k ?
- 83.** A radio transmitter broadcasts plane electromagnetic waves whose maximum electric field at a particular location is 1.55×10^{-3} V/m. What is the maximum magnitude of the oscillating magnetic field at that location? How does it compare with Earth's magnetic field?
- 84.** (a) Two microwave frequencies authorized for use in microwave ovens are: 915 and 2450 MHz. Calculate the wavelength of each. (b) Which frequency would produce smaller hot spots in foods due to interference effects?
- 85.** During normal beating, the heart creates a maximum 4.00-mV potential across 0.300 m of a person's chest, creating a 1.00-Hz electromagnetic wave. (a) What is the maximum electric field strength created? (b) What is the corresponding maximum magnetic field strength in the electromagnetic wave? (c) What is the wavelength of the electromagnetic wave?
- 86.** Distances in space are often quoted in units of light-years, the distance light travels in 1 year. (a) How many meters is a light-year? (b) How many meters is it to Andromeda, the nearest large galaxy, given that it is 2.54×10^6 ly away? (c) The most distant galaxy yet discovered is 13.4×10^9 ly away. How far is this in meters?
- 87.** A certain 60.0-Hz ac power line radiates an electromagnetic wave having a maximum electric field strength of 13.0 kV/m. (a) What is the wavelength of this very-low-frequency electromagnetic wave? (b) What type of electromagnetic radiation is this wave (b) What is its maximum magnetic field strength?
- 88.** (a) What is the frequency of the 193-nm ultraviolet radiation used in laser eye surgery? (b) Assuming the accuracy with which this electromagnetic radiation can ablate (reshape) the cornea is directly proportional to wavelength, how much more accurate can this UV radiation be than the shortest visible wavelength of light?

Additional Problems

- 89.** In a region of space, the electric field is pointed along the x -axis, but its magnitude changes as described by

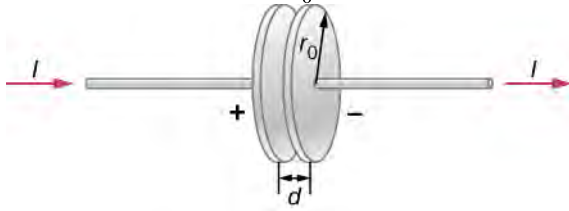
$$E_x = (10 \text{ N/C}) \sin (20x - 500t)$$

$$E_y = E_z = 0$$
 where t is in nanoseconds and x is in cm. Find the displacement current through a circle of radius 3 cm in the $x = 0$ plane at $t = 0$.
- 90.** A microwave oven uses electromagnetic waves of frequency $f = 2.45 \times 10^9$ Hz to heat foods. The waves reflect from the inside walls of the oven to produce an interference pattern of standing waves whose antinodes are hot spots that can leave observable pit marks in some foods. The pit marks are measured to be 6.0 cm apart. Use the method employed by Heinrich Hertz to calculate the speed of electromagnetic waves this implies.
- Use the [Appendix D](#) for the next two exercises
- 91.** Galileo proposed measuring the speed of light by uncovering a lantern and having an assistant a known distance away uncover his lantern when he saw the light from Galileo's lantern, and timing the delay. How far away must the assistant be for the delay to equal the human reaction time of about 0.25 s?
- 92.** Show that the wave equation in one dimension $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$ is satisfied by any doubly differentiable function of either the form $f(x - vt)$ or $f(x + vt)$.
- 93.** On its highest power setting, a microwave oven increases the temperature of 0.400 kg of spaghetti by 45.0°C in 120 s. (a) What was the rate of energy absorption by the spaghetti, given that its specific heat is $3.76 \times 10^3 \text{ J/kg} \cdot ^\circ\text{C}$? Assume the spaghetti is perfectly absorbing. (b) Find the average intensity of the microwaves, given that they are absorbed over a circular area 20.0 cm in diameter. (c) What is the peak electric field strength of the microwave? (d) What is its peak magnetic field strength?
- 94.** A certain microwave oven projects 1.00 kW of microwaves onto a 30-cm-by-40-cm area. (a) What is its intensity in W/m^2 ? (b) Calculate the maximum electric field strength E_0 in these waves. (c) What is the maximum magnetic field strength B_0 ?
- 95.** Electromagnetic radiation from a 5.00-mW laser is concentrated on a 1.00-mm^2 area. (a) What is the intensity in W/m^2 ? (b) Suppose a 2.00-nC electric charge is in the beam. What is the maximum electric force it experiences? (c) If the electric charge moves at 400 m/s, what maximum magnetic force can it feel?
- 96.** A 200-turn flat coil of wire 30.0 cm in diameter acts as an antenna for FM radio at a frequency of 100 MHz. The magnetic field of the incoming electromagnetic wave is perpendicular to the coil and has a maximum strength of $1.00 \times 10^{-12} \text{ T}$. (a) What power is incident on the coil? (b) What average emf is induced in the coil over one-fourth of a cycle? (c) If the radio receiver has an inductance of $2.50 \mu\text{H}$, what capacitance must it have to resonate at 100 MHz?
- 97.** Suppose a source of electromagnetic waves radiates uniformly in all directions in empty space where there are no absorption or interference effects. (a) Show that the intensity is inversely proportional to r^2 , the distance from the source squared. (b) Show that the magnitudes of the electric and magnetic fields are inversely proportional to r .
- 98.** A radio station broadcasts its radio waves with a power of 50,000 W. What would be the intensity of this signal if it is received on a planet orbiting Proxima Centuri, the closest star to our Sun, at 4.243 ly away?
- 99.** The Poynting vector describes a flow of energy whenever electric and magnetic fields are present. Consider a long cylindrical wire of radius r with a current I in the wire, with resistance R and voltage V . From the expressions for the electric field along the wire and the magnetic field around the wire, obtain the magnitude and direction of the Poynting vector at the surface. Show that it accounts for an energy flow into the wire from the fields around it that accounts for the Ohmic heating of the wire.

- 100.** The Sun's energy strikes Earth at an intensity of 1.37 kW/m^2 . Assume as a model approximation that all of the light is absorbed. (Actually, about 30% of the light intensity is reflected out into space.)
- (a) Calculate the total force that the Sun's radiation exerts on Earth.
 - (b) Compare this to the force of gravity between the Sun and Earth.
- Earth's mass is $5.972 \times 10^{24} \text{ kg}$.
- 101.** If a *Lightsail* spacecraft were sent on a Mars mission, by what ratio of the final force to the initial force would its propulsion be reduced when it reached Mars?
- 102.** Lunar astronauts placed a reflector on the Moon's surface, off which a laser beam is periodically reflected. The distance to the Moon is calculated from the round-trip time.
- (a) To what accuracy in meters can the distance to the Moon be determined, if this time can be measured to 0.100 ns ?
 - (b) What percent accuracy is this, given the average distance to the Moon is $384,400 \text{ km}$?
- 103.** Radar is used to determine distances to various objects by measuring the round-trip time for an echo from the object. (a) How far away is the planet Venus if the echo time is 1000 s ? (b) What is the echo time for a car 75.0 m from a highway police radar unit? (c) How accurately (in nanoseconds) must you be able to measure the echo time to an airplane 12.0 km away to determine its distance within 10.0 m ?
- 104.** Calculate the ratio of the highest to lowest frequencies of electromagnetic waves the eye can see, given the wavelength range of visible light is from 380 to 760 nm . (Note that the ratio of highest to lowest frequencies the ear can hear is 1000 .)
- 105.** How does the wavelength of radio waves for an AM radio station broadcasting at 1030 KHz compare with the wavelength of the lowest audible sound waves (of 20 Hz). The speed of sound in air at 20°C is about 343 m/s .

Challenge Problems

- 106.** A parallel-plate capacitor with plate separation d is connected to a source of emf that places a time-dependent voltage $V(t)$ across its circular plates of radius r_0 and area $A = \pi r_0^2$ (see below).



- Write an expression for the time rate of change of energy inside the capacitor in terms of $V(t)$ and $dV(t)/dt$.
- Assuming that $V(t)$ is increasing with time, identify the directions of the electric field lines inside the capacitor and of the magnetic field lines at the edge of the region between the plates, and then the direction of the Poynting vector $\vec{S}$ at this location.
- Obtain expressions for the time dependence of $E(t)$, for $B(t)$ from the displacement current, and for the magnitude of the Poynting vector at the edge of the region between the plates.
- From $\vec{S}$, obtain an expression in terms of $V(t)$ and $dV(t)/dt$ for the rate at which electromagnetic field energy enters the region between the plates.
- Compare the results of parts (a) and (d) and explain the relationship between them.

- 107.** A particle of cosmic dust has a density $\rho = 2.0 \text{ g/cm}^3$. (a) Assuming the dust particles are spherical and light absorbing, and are at the same distance as Earth from the Sun, determine the particle size for which radiation pressure from sunlight is equal to the Sun's force of gravity on the dust particle. (b) Explain how the forces compare if the particle radius is smaller. (c) Explain what this implies about the sizes of dust particle likely to be present in the inner solar system compared with outside the Oort cloud.

ANSWER KEY

Chapter 1

Check Your Understanding

- 1.1 The actual amount (mass) of gasoline left in the tank when the gauge hits “empty” is less in the summer than in the winter. The gasoline has the same volume as it does in the winter when the “add fuel” light goes on, but because the gasoline has expanded, there is less mass.
- 1.2 Not necessarily, as the thermal stress is also proportional to Young’s modulus.
- 1.3 To a good approximation, the heat transfer depends only on the temperature difference. Since the temperature differences are the same in both cases, the same 25 kJ is necessary in the second case. (As we will see in the next section, the answer would have been different if the object had been made of some substance that changes phase anywhere between 30 °C and 50 °C.)
- 1.4 The ice and liquid water are in thermal equilibrium, so that the temperature stays at the freezing temperature as long as ice remains in the liquid. (Once all of the ice melts, the water temperature will start to rise.)
- 1.5 Snow is formed from ice crystals and thus is the solid phase of water. Because enormous heat is necessary for phase changes, it takes a certain amount of time for this heat to be transferred from the air, even if the air is above 0 °C.
- 1.6 Conduction: Heat transfers into your hands as you hold a hot cup of coffee. Convection: Heat transfers as the barista “steams” cold milk to make hot cocoa. Radiation: Heat transfers from the Sun to a jar of water with tea leaves in it to make “Sun tea.” A great many other answers are possible.
- 1.7 Because area is the product of two spatial dimensions, it increases by a factor of four when each dimension is doubled ($A_{\text{final}} = (2d)^2 = 4d^2 = 4A_{\text{initial}}$). The distance, however, simply doubles. Because the temperature difference and the coefficient of thermal conductivity are independent of the spatial dimensions, the rate of heat transfer by conduction increases by a factor of four divided by two, or two:

$$P_{\text{final}} = \frac{kA_{\text{final}}(T_h - T_c)}{d_{\text{final}}} = \frac{k(4A_{\text{final}})(T_h - T_c)}{2d_{\text{initial}}} = 2\frac{kA_{\text{final}}(T_h - T_c)}{d_{\text{initial}}} = 2P_{\text{initial}}.$$
- 1.8 Using a fan increases the flow of air: Warm air near your body is replaced by cooler air from elsewhere. Convection increases the rate of heat transfer so that moving air “feels” cooler than still air.
- 1.9 The radiated heat is proportional to the fourth power of the *absolute temperature*. Because $T_1 = 293$ K and $T_2 = 313$ K, the rate of heat transfer increases by about 30% of the original rate.

Conceptual Questions

1. They are at the same temperature, and if they are placed in contact, no net heat flows between them.
3. The reading will change.
5. The cold water cools part of the inner surface, making it contract, while the rest remains expanded. The strain is too great for the strength of the material. Pyrex contracts less, so it experiences less strain.
7. In principle, the lid expands more than the jar because metals have higher coefficients of expansion than glass. That should make unscrewing the lid easier. (In practice, getting the lid and jar wet may make gripping them more difficult.)
9. After being heated, the length is $(1 + 300\alpha)(1 \text{ m})$. After being cooled, the length is $(1 - 300\alpha)(1 + 300\alpha)(1 \text{ m})$. That answer is not 1 m, but it should be. The explanation is that even if α is exactly constant, the relation $\Delta L = \alpha L \Delta T$ is strictly true only in the limit of small ΔT . Since α values are small, the discrepancy is unimportant in practice.
11. Temperature differences cause heat transfer.
13. No, it is stored as thermal energy. A thermodynamic system does not have a well-defined quantity of heat.
15. It raises the boiling point, so the water, which the food gains heat from, is at a higher temperature.
17. Yes, by raising the pressure above 56 atm.
19. work
21. 0 °C (at or near atmospheric pressure)

23. Condensation releases heat, so it speeds up the melting.
25. Because of water's high specific heat, it changes temperature less than land. Also, evaporation reduces temperature rises. The air tends to stay close to equilibrium with the water, so its temperature does not change much where there's a lot of water around, as in San Francisco but not Sacramento.
27. The liquid is oxygen, whose boiling point is above that of nitrogen but whose melting point is below the boiling point of liquid nitrogen. The crystals that sublime are carbon dioxide, which has no liquid phase at atmospheric pressure. The crystals that melt are water, whose melting point is above carbon dioxide's sublimation point. The water came from the instructor's breath.
29. Increasing circulation to the surface will warm the person, as the temperature of the water is warmer than human body temperature. Sweating will cause no evaporative cooling under water or in the humid air immediately above the tub.
31. It spread the heat over the area above the heating elements, evening the temperature there, but does not spread the heat much beyond the heating elements.
33. Heat is conducted from the fire through the fire box to the circulating air and then convected by the air into the room (forced convection).
35. The tent is heated by the Sun and transfers heat to you by all three processes, especially radiation.
37. If shielded, it measures the air temperature. If not, it measures the combined effect of air temperature and net radiative heat gain from the Sun.
39. Turn the thermostat down. To have the house at the normal temperature, the heating system must replace all the heat that was lost. For all three mechanisms of heat transfer, the greater the temperature difference between inside and outside, the more heat is lost and must be replaced. So the house should be at the lowest temperature that does not allow freezing damage.
41. Air is a good insulator, so there is little conduction, and the heated air rises, so there is little convection downward.

Problems

43. That must be Celsius. Your Fahrenheit temperature is 102 °F. Yes, it is time to get treatment.
45. a. $\Delta T_C = 22.2^\circ\text{C}$; b. We know that $\Delta T_F = T_{F2} - T_{F1}$. We also know that $T_{F2} = \frac{9}{5}T_{C2} + 32$ and $T_{F1} = \frac{9}{5}T_{C1} + 32$. So, substituting, we have $\Delta T_F = \left(\frac{9}{5}T_{C2} + 32\right) - \left(\frac{9}{5}T_{C1} + 32\right)$. Partially solving and rearranging the equation, we have $\Delta T_F = \frac{9}{5}(T_{C2} - T_{C1})$. Therefore, $\Delta T_F = \frac{9}{5}\Delta T_C$.
47. a. -40° ; b. 575 K
49. Using Table 1.2 to find the coefficient of thermal expansion of marble:

$$L = L_0 + \Delta L = L_0 (1 + \alpha \Delta T) = 170 \text{ m} [1 + (2.5 \times 10^{-6}/^\circ\text{C})(-45.0^\circ\text{C})] = 169.98 \text{ m}.$$
 (Answer rounded to five significant figures to show the slight difference in height.)
51. We use β instead of α since this is a volume expansion with constant surface area. Therefore:

$$\Delta L = \alpha L \Delta T = (6.0 \times 10^{-5}/^\circ\text{C})(0.0300 \text{ m})(3.00^\circ\text{C}) = 5.4 \times 10^{-6} \text{ m}.$$
53. On the warmer day, our tape measure will expand linearly. Therefore, each measured dimension will be smaller than the actual dimension of the land. Calling these measured dimensions l' and w' , we will find a new area, A . Let's calculate these measured dimensions:

$$l' = l_0 - \Delta l = (20 \text{ m}) - (20^\circ\text{C})(20 \text{ m})\left(\frac{1.2 \times 10^{-5}}{^\circ\text{C}}\right) = 19.9952 \text{ m};$$

$$A' = l \times w' = (29.9928 \text{ m})(19.9952 \text{ m}) = 599.71 \text{ m}^2;$$

$$\text{Cost change} = (A - A')\left(\frac{\$60,000}{\text{m}^2}\right) = ((600 - 599.71) \text{ m}^2)\left(\frac{\$60,000}{\text{m}^2}\right) = \$17,000.$$
 Because the area gets smaller, the price of the land *decreases* by about \$17,000.
55. a. Use Table 1.2 to find the coefficients of thermal expansion of steel and aluminum. Then

$$\Delta L_{\text{Al}} - \Delta L_{\text{steel}} = (\alpha_{\text{Al}} - \alpha_{\text{steel}})L_0\Delta T = \left(\frac{2.5 \times 10^{-5}}{^\circ\text{C}} - \frac{1.2 \times 10^{-5}}{^\circ\text{C}}\right)(1.00 \text{ m})(22^\circ\text{C}) = 2.9 \times 10^{-4} \text{ m}.$$
 b. By the same method with $L_0 = 30.0 \text{ m}$, we have $\Delta L = 8.6 \times 10^{-3} \text{ m}$.
57. $\Delta V = 0.475 \text{ L}$
59. If we start with the freezing of water, then it would expand to

$$(1 \text{ m}^3) \left(\frac{1000 \text{ kg/m}^3}{917 \text{ kg/m}^3} \right) = 1.09 \text{ m}^3 = 1.98 \times 10^8 \text{ N/m}^2 \text{ of ice.}$$

61. $m = 5.20 \times 10^8 \text{ J}$

63. $Q = mc\Delta T \Rightarrow \Delta T = \frac{Q}{mc}$; a. 21.0°C ; b. 25.0°C ; c. 29.3°C ; d. 50.0°C

65. $Q = mc\Delta T \Rightarrow c = \frac{Q}{m\Delta T} = \frac{1.04 \text{ kcal}}{(0.250 \text{ kg})(45.0^\circ\text{C})} = 0.0924 \text{ kcal/kg} \cdot ^\circ\text{C}$. It is copper.

67. a. $Q = m_w c_w \Delta T + m_{A1} c_{A1} \Delta T = (m_w c_w + m_{A1} c_{A1}) \Delta T$;

$$Q = \left[(0.500 \text{ kg})(1.00 \text{ kcal/kg} \cdot ^\circ\text{C}) + (0.100 \text{ kg})(0.215 \text{ kcal/kg} \cdot ^\circ\text{C}) \right] (54.9^\circ\text{C}) = 28.63 \text{ kcal};$$

$$\frac{Q}{m_p} = \frac{28.63 \text{ kcal}}{5.00 \text{ g}} = 5.73 \text{ kcal/g}; \text{ b. } \frac{Q}{m_p} = \frac{200 \text{ kcal}}{33 \text{ g}} = 6 \text{ kcal/g, which is consistent with our results to part (a), to one significant figure.}$$

69. 0.139°C

71. It should be lower. The beaker will not make much difference: 16.3°C

73. a. $1.00 \times 10^5 \text{ J}$; b. $3.68 \times 10^5 \text{ J}$; c. The ice is much more effective in absorbing heat because it first must be melted, which requires a lot of energy, and then it gains the same amount of heat as the bag that started with water. The first $2.67 \times 10^5 \text{ J}$ of heat is used to melt the ice, then it absorbs the $1.00 \times 10^5 \text{ J}$ of heat as water.

75. 58.1 g

77. Let M be the mass of pool water and m be the mass of pool water that evaporates.

$$Mc\Delta T = mL_V(37^\circ\text{C}) \Rightarrow \frac{m}{M} = \frac{c\Delta T}{L_V(37^\circ\text{C})} = \frac{(1.00 \text{ kcal/kg} \cdot ^\circ\text{C})(1.50^\circ\text{C})}{580 \text{ kcal/kg}} = 2.59 \times 10^{-3};$$

(Note that L_V for water at 37°C is used here as a better approximation than L_V for 100°C water.)

79. a. $1.47 \times 10^{15} \text{ kg}$; b. $4.90 \times 10^{20} \text{ J}$; c. 48.5 y

81. a. 9.35 L ; b. Crude oil is less dense than water, so it floats on top of the water, thereby exposing it to the oxygen in the air, which it uses to burn. Also, if the water is under the oil, it is less able to absorb the heat generated by the oil.

83. a. 319 kcal ; b. 2.00°C

85. First bring the ice up to 0°C and melt it with heat Q_1 : 4.74 kcal . This lowers the temperature of water by ΔT_2 : 23.15°C . Now, the heat lost by the hot water equals that gained by the cold water (T_f is the final temperature): 20.6°C

87. Let the subscripts r, e, v, and w represent rock, equilibrium, vapor, and water, respectively.

$$m_r c_r (T_1 - T_e) = m_v L_V + m_w c_w (T_e - T_2);$$

$$\begin{aligned} m_r &= \frac{m_v L_V + m_w c_w (T_e - T_2)}{c_r (T_1 - T_e)} \\ &= \frac{(0.0250 \text{ kg})(2256 \times 10^3 \text{ J/kg}) + (3.975 \text{ kg})(4186 \times 10^3 \text{ J/kg} \cdot ^\circ\text{C})(100^\circ\text{C} - 15^\circ\text{C})}{(840 \text{ J/kg} \cdot ^\circ\text{C})(500^\circ\text{C} - 100^\circ\text{C})} \\ &= 4.38 \text{ kg} \end{aligned}$$

89. a. $1.01 \times 10^3 \text{ W}$; b. One 1-kilowatt room heater is needed.

91. 84.0 W

93. 2.59 kg

95. a. 39.7 W ; b. 820 kcal

97. $\frac{Q}{t} = \frac{kA(T_2 - T_1)}{d}$, so that

$$\frac{(Q/t)_{\text{wall}}}{(Q/t)_{\text{window}}} = \frac{k_{\text{wall}} A_{\text{wall}} d_{\text{window}}}{k_{\text{window}} A_{\text{window}} d_{\text{wall}}} = \frac{(2 \times 0.042 \text{ J/s} \cdot \text{m} \cdot ^\circ\text{C})(10.0 \text{ m}^2)(0.750 \times 10^{-2} \text{ m})}{(0.84 \text{ J/s} \cdot \text{m} \cdot ^\circ\text{C})(2.00 \text{ m}^2)(13.0 \times 10^{-2} \text{ m})}$$

This gives 0.0288 wall: window, or 35:1 window: wall

99. $\frac{Q}{t} = \frac{kA(T_2 - T_1)}{d} = \frac{kA\Delta T}{d} \Rightarrow$

$$\Delta T = \frac{d(Q/t)}{kA} = \frac{(6.00 \times 10^{-3} \text{ m})(2256 \text{ W})}{(0.84 \text{ J/s} \cdot \text{m} \cdot ^\circ\text{C})(1.54 \times 10^{-2} \text{ m}^2)} = 1046^\circ\text{C} = 1.05 \times 10^3 \text{ K}$$

- 101.** We found in the preceding problem that $P = 126\Delta T \text{ W} \cdot ^\circ\text{C}$ as baseline energy use. So the total heat loss during this period is $Q = (126 \text{ J/s} \cdot ^\circ\text{C})(15.0^\circ\text{C})(120 \text{ days})(86.4 \times 10^3 \text{ s/day}) = 1960 \times 10^6 \text{ J}$. At the cost of \$1/MJ, the cost is \$1960. From an earlier problem, the savings is 12% or \$235/y. We need 150 m^2 of insulation in the attic. At $\$4/\text{m}^2$, this is a \$500 cost. So the payback period is $\$600/(\$235/\text{y}) = 2.6 \text{ years}$ (excluding labor costs).

Additional Problems

- 103.** 7.39%
105. $\frac{F}{A} = (210 \times 10^9 \text{ Pa})(12 \times 10^{-6}/^\circ\text{C})(40^\circ\text{C} - (-15^\circ\text{C})) = 1.4 \times 10^8 \text{ N/m}^2$.
107. a. 1.06 cm; b. 1.11 cm
109. 1.7 kJ/(kg $\cdot$ $^\circ\text{C}$)
111. a. $1.57 \times 10^4 \text{ kcal}$; b. $18.3 \text{ kW} \cdot \text{h}$; c. $1.29 \times 10^4 \text{ kcal}$
113. 6.3°C . All of the ice melted.
115. 63.9°C , all the ice melted
117. a. 83 W; b. $1.97 \times 10^3 \text{ W}$; The single-pane window has a rate of heat conduction equal to 1969/83, or 24 times that of a double-pane window.
119. The rate of heat transfer by conduction is 20.0 W. On a daily basis, this is 1,728 kJ/day. Daily food intake is $2400 \text{ kcal/d} \times 4186 \text{ J/kcal} = 10,050 \text{ kJ/day}$. So only 17.2% of energy intake goes as heat transfer by conduction to the environment at this ΔT .
121. 620 K

Challenge Problems

- 123.** Denoting the period by P , we know $P = 2\pi\sqrt{L/g}$. When the temperature increases by dT , the length increases by αLdT . Then the new length is a.

$$P = 2\pi\sqrt{\frac{L+\alpha LdT}{g}} = 2\pi\sqrt{\frac{L}{g}(1+\alpha dT)} = 2\pi\sqrt{\frac{L}{g}}(1+\frac{1}{2}\alpha dT) = P(1+\frac{1}{2}\alpha dT)$$
 by the binomial expansion. b. The clock runs slower, as its new period is 1.00019 s. It loses 16.4 s per day.
125. The amount of heat to melt the ice and raise it to 100°C is not enough to condense the steam, but it is more than enough to lower the steam's temperature by 50°C , so the final state will consist of steam and liquid water in equilibrium, and the final temperature is 100°C ; 9.5 g of steam condenses, so the final state contains 49.5 g of steam and 40.5 g of liquid water.
127. a. $dL/dT = kT/\rho L$; b. $L = \sqrt{2kTt/\rho L_f}$; c. yes
129. a. $4(\pi R^2)T_s^4$; b. $4e\sigma\pi R^2T_s^4$; c. $8e\sigma\pi R^2T_e^4$; d. $T_s^4 = 2T_e^4$; e. $e\sigma T_s^4 + \frac{1}{4}(1-A)S = \sigma T_s^4$; f. 288 K

Chapter 2

Check Your Understanding

- 2.1** We first need to calculate the molar mass (the mass of one mole) of niacin. To do this, we must multiply the number of atoms of each element in the molecule by the element's molar mass.
 $(6 \text{ mol of carbon})(12.0 \text{ g/mol}) + (5 \text{ mol hydrogen})(1.0 \text{ g/mol})$
 $+ (1 \text{ mol nitrogen})(14 \text{ g/mol}) + (2 \text{ mol oxygen})(16.0 \text{ g/mol}) = 123 \text{ g/mol}$
 Then we need to calculate the number of moles in 14 mg.

$$\left(\frac{14 \text{ mg}}{123 \text{ g/mol}}\right)\left(\frac{1 \text{ g}}{1000 \text{ mg}}\right) = 1.14 \times 10^{-4} \text{ mol}.$$

 Then, we use Avogadro's number to calculate the number of molecules:
 $N = nN_A = (1.14 \times 10^{-4} \text{ mol})(6.02 \times 10^{23} \text{ molecules/mol}) = 6.85 \times 10^{19} \text{ molecules}.$
2.2 The density of a gas is equal to a constant, the average molecular mass, times the number density N/V . From the ideal gas law, $pV = Nk_B T$, we see that $N/V = p/k_B T$. Therefore, at constant temperature, if the density and, consequently, the number density are reduced by half, the pressure must also be reduced by half, and $p_f = 0.500 \text{ atm}$.

Chapter 16

Check Your Understanding

- 16.1** It is greatest immediately after the current is switched on. The displacement current and the magnetic field from it are proportional to the rate of change of electric field between the plates, which is greatest when the plates first begin to charge.
- 16.2** No. The changing electric field according to the modified version of Ampère's law would necessarily induce a changing magnetic field.
- 16.3** (1) Faraday's law, (2) the Ampère-Maxwell law
- 16.4** a. The directions of wave propagation, of the E field, and of B field are all mutually perpendicular. b. The speed of the electromagnetic wave is the speed of light $c = 1/\sqrt{\epsilon_0\mu_0}$ independent of frequency. c. The ratio of electric and magnetic field amplitudes is $E/B = c$.
- 16.5** Its acceleration would decrease because the radiation force is proportional to the intensity of light from the Sun, which decreases with distance. Its speed, however, would not change except for the effects of gravity from the Sun and planets.
- 16.6** They fall into different ranges of wavelength, and therefore also different corresponding ranges of frequency.

Conceptual Questions

- The current into the capacitor to change the electric field between the plates is equal to the displacement current between the plates.
- The first demonstration requires simply observing the current produced in a wire that experiences a changing magnetic field. The second demonstration requires moving electric charge from one location to another, and therefore involves electric currents that generate a changing electric field. The magnetic fields from these currents are not easily separated from the magnetic field that the displacement current produces.
- in (a), because the electric field is parallel to the wire, accelerating the electrons
- A steady current in a dc circuit will not produce electromagnetic waves. If the magnitude of the current varies while remaining in the same direction, the wires will emit electromagnetic waves, for example, if the current is turned on or off.
- The amount of energy (about 100 W/m^2) is can quickly produce a considerable change in temperature, but the light pressure (about $3.00 \times 10^{-7} \text{ N/m}^2$) is much too small to notice.
- It has the magnitude of the energy flux and points in the direction of wave propagation. It gives the direction of energy flow and the amount of energy per area transported per second.
- The force on a surface acting over time Δt is the momentum that the force would impart to the object. The momentum change of the light is doubled if the light is reflected back compared with when it is absorbed, so the force acting on the object is twice as great.
- a. According to the right hand rule, the direction of energy propagation would reverse. b. This would leave the vector $\vec{S}$, and therefore the propagation direction, the same.
- a. Radio waves are generally produced by alternating current in a wire or an oscillating electric field between two plates; b. Infrared radiation is commonly produced by heated bodies whose atoms and the charges in them vibrate at about the right frequency.
- a. blue; b. Light of longer wavelengths than blue passes through the air with less scattering, whereas more of the blue light is scattered in different directions in the sky to give it is blue color.
- A typical antenna has a stronger response when the wires forming it are orientated parallel to the electric field of the radio wave.
- No, it is very narrow and just a small portion of the overall electromagnetic spectrum.
- Visible light is typically produced by changes of energies of electrons in randomly oriented atoms and molecules. Radio waves are typically emitted by an ac current flowing along a wire, that has fixed orientation and produces electric fields pointed in particular directions.
- Radar can observe objects the size of an airplane and uses radio waves of about 0.5 cm in wavelength. Visible light can be used to view single biological cells and has wavelengths of about 10^{-7} m .

29. ELF radio waves
 31. The frequency of 2.45 GHz of a microwave oven is close to the specific frequencies in the 2.4 GHz band used for WiFi.

Problems

33.
$$B_{\text{ind}} = \frac{\mu_0}{2\pi r} I_{\text{ind}} = \frac{\mu_0}{2\pi r} \epsilon_0 \frac{\partial \Phi_E}{\partial t} = \frac{\mu_0}{2\pi r} \epsilon_0 \left(A \frac{\partial E}{\partial t} \right) = \frac{\mu_0}{2\pi r} \epsilon_0 A \left(\frac{1}{d} \frac{dV(t)}{dt} \right)$$
35. a. $I_{\text{res}} = \frac{V_0 \sin \omega t}{R}$; b. $I_d = CV_0 \omega \cos \omega t$;
 c. $I_{\text{real}} = I_{\text{res}} + \frac{dQ}{dt} = \frac{V_0 \sin \omega t}{R} + CV_0 \frac{d}{dt} \sin \omega t = \frac{V_0 \sin \omega t}{R} + CV_0 \omega \cos \omega t$; which is the sum of I_{res} and I_{real} , consistent with how the displacement current maintaining the continuity of current.
37. $1.77 \times 10^{-3} \text{ A}$
 39. $I_d = (7.97 \times 10^{-10} \text{ A}) \sin(150t)$
 41. 499 s
 43. 25 m
 45. a. 5.00 V/m; b. $9.55 \times 10^8 \text{ Hz}$; c. 31.4 cm; d. toward the +x-axis;
 e. $B = (1.67 \times 10^{-8} \text{ T}) \cos[kx - (6 \times 10^9 \text{ s}^{-1})t + 0.40] \hat{\mathbf{k}}$
 47. $I_d = \pi \epsilon_0 \omega R^2 E_0 \sin(kx - \omega t)$
 49. The magnetic field is downward, and it has magnitude $2.00 \times 10^{-8} \text{ T}$.
 51. a. $6.45 \times 10^{-3} \text{ V/m}$; b. 394 m
 53. 11.5 m
 55. $5.97 \times 10^{-3} \text{ W/m}^2$
 57. a. $E_0 = 1027 \text{ V/m}$, $B_0 = 3.42 \times 10^{-6} \text{ T}$; b. $3.96 \times 10^{26} \text{ W}$
 59. 20.8 W/m^2
 61. a. $4.42 \times 10^{-6} \text{ W/m}^2$; b. $5.77 \times 10^{-2} \text{ V/m}$
 63. a. $7.47 \times 10^{-14} \text{ W/m}^2$; b. $3.66 \times 10^{-13} \text{ W}$; c. 1.12 W
 65. $1.99 \times 10^{-11} \text{ N/m}^2$

$$F = ma = (p)(\pi r^2), p = \frac{ma}{\pi r^2} = \frac{\epsilon_0}{2} E_0^2$$

 67.
$$E_0 = \sqrt{\frac{2ma}{\epsilon_0 \pi r^2}} = \sqrt{\frac{2(10^{-8} \text{ kg})(0.30 \text{ m/s}^2)}{(8.854 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)(\pi)(2 \times 10^{-6} \text{ m})^2}}$$

$$E_0 = 7.34 \times 10^6 \text{ V/m}$$

 69. a. $4.50 \times 10^{-6} \text{ N}$; b. it is reduced to half the pressure, $2.25 \times 10^{-6} \text{ N}$
 71. a. $W = \frac{1}{2} \frac{\pi^2 r^4}{mc^2} I^2 t^2$; b. $E = \pi r^2 I t$
 73. a. $1.5 \times 10^{18} \text{ Hz}$; b. X-rays
 75. a. The wavelength range is 187 m to 556 m. b. The wavelength range is 2.78 m to 3.41 m.
 77. $P' = \left(\frac{12 \text{ m}}{30 \text{ m}}\right)^2 (100 \text{ mW}) = 16 \text{ mW}$
 79. time for 1 bit = $1.27 \times 10^{-8} \text{ s}$, difference in travel time is $5.34 \times 10^{-8} \text{ s}$
 81. a. $1.5 \times 10^{-9} \text{ m}$; b. $5.9 \times 10^{-7} \text{ m}$; c. $3.0 \times 10^{-15} \text{ m}$
 83. $5.17 \times 10^{-12} \text{ T}$, the non-oscillating geomagnetic field of 25–65 μT is much larger
 85. a. $1.33 \times 10^{-2} \text{ V/m}$; b. $4.44 \times 10^{-11} \text{ T}$; c. $3.00 \times 10^8 \text{ m}$
 87. a. $5.00 \times 10^6 \text{ m}$; b. radio wave; c. $4.33 \times 10^{-5} \text{ T}$

Additional Problems

89. $I_d = (10 \text{ N/C}) (8.845 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2) \pi (0.03 \text{ m})^2 (5000 \frac{1}{\text{s}}) = 1.25 \times 10^{-9} \text{ A}$
 91. $3.75 \times 10^7 \text{ km}$, which is much greater than Earth's circumference
 93. a. 564 W; b. $1.80 \times 10^4 \text{ W/m}^2$; c. $3.68 \times 10^3 \text{ V/m}$; d. $1.23 \times 10^{-5} \text{ T}$
 95. a. $5.00 \times 10^3 \text{ W/m}^2$; b. $3.88 \times 10^{-6} \text{ N}$; c. $5.18 \times 10^{-12} \text{ N}$

97. a. $I = \frac{P}{A} = \frac{P}{4\pi r^2} \propto \frac{1}{r^2}$; b. $I \propto E_0^2$, $B_0^2 \Rightarrow E_0^2$, $B_0^2 \propto \frac{1}{r^2} \Rightarrow E_0, B_0 \propto \frac{1}{r}$

Power into the wire $= \int \vec{S} \cdot d\vec{A} = \left(\frac{1}{\mu_0} EB \right) (2\pi r L)$

99.
$$= \frac{1}{\mu_0} \left(\frac{V}{L} \right) \left(\frac{\mu_0 i}{2\pi r} \right) (2\pi r L) = iV = i^2 R$$

101. 0.431

103. a. 1.5×10^{11} m; b. 5.0×10^{-7} s; c. 33 ns

sound: $\lambda_{\text{sound}} = \frac{v_s}{f} = \frac{343 \text{ m/s}}{20.0 \text{ Hz}} = 17.2 \text{ m}$

105. radio: $\lambda_{\text{radio}} = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{1030 \times 10^3 \text{ Hz}} = 291 \text{ m}$; or $17.1 \lambda_{\text{sound}}$

Challenge Problems

107. a. $0.29 \mu\text{m}$; b. The radiation pressure is greater than the Sun's gravity if the particle size is smaller, because the gravitational force varies as the radius cubed while the radiation pressure varies as the radius squared. c. The radiation force outward implies that particles smaller than this are less likely to be near the Sun than outside the range of the Sun's radiation pressure.

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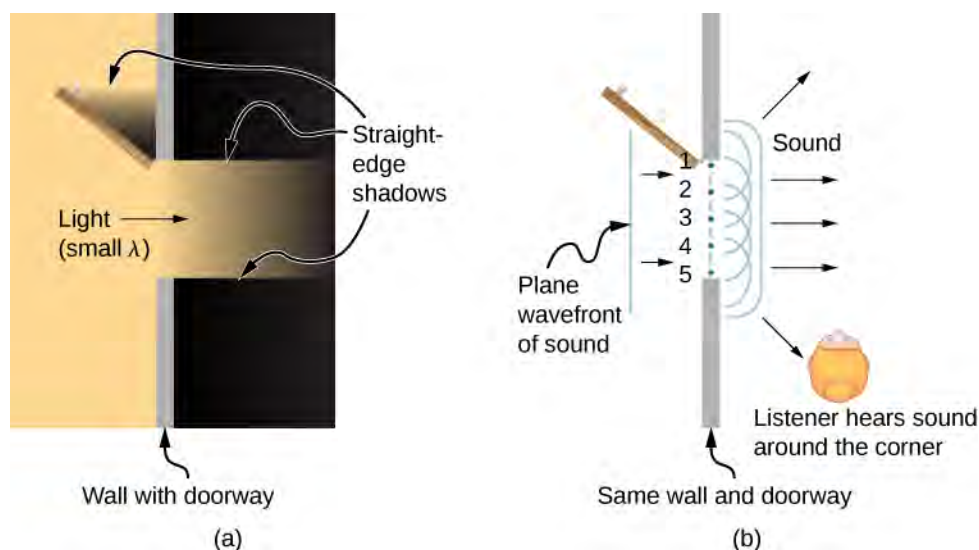


Figure 1.30 (a) Light passing through a doorway makes a sharp outline on the floor. Since light's wavelength is very small compared with the size of the door, it acts like a ray. (b) Sound waves bend into all parts of the room, a wave effect, because their wavelength is similar to the size of the door.

If we pass light through smaller openings such as slits, we can use Huygens's principle to see that light bends as sound does (Figure 1.31). The bending of a wave around the edges of an opening or an obstacle is called diffraction. Diffraction is a wave characteristic and occurs for all types of waves. If diffraction is observed for some phenomenon, it is evidence that the phenomenon is a wave. Thus, the horizontal diffraction of the laser beam after it passes through the slits in Figure 1.31 is evidence that light is a wave. You will learn about diffraction in much more detail in the chapter on [Diffraction](#).

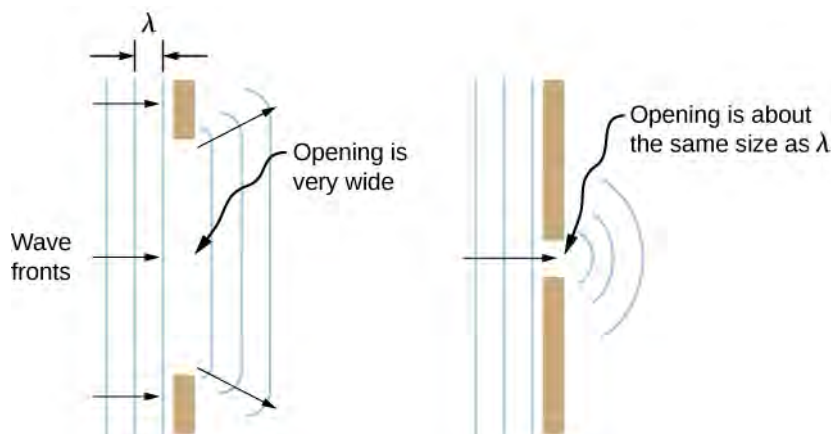


Figure 1.31 Huygens's principle applied to a plane wave front striking an opening. The edges of the wave front bend after passing through the opening, a process called diffraction. The amount of bending is more extreme for a small opening, consistent with the fact that wave characteristics are most noticeable for interactions with objects about the same size as the wavelength.

1.7 Polarization

Learning Objectives

By the end of this section, you will be able to:

- Explain the change in intensity as polarized light passes through a polarizing filter
- Calculate the effect of polarization by reflection and Brewster's angle
- Describe the effect of polarization by scattering
- Explain the use of polarizing materials in devices such as LCDs

Polarizing sunglasses are familiar to most of us. They have a special ability to cut the glare of light reflected

from water or glass (Figure 1.32). They have this ability because of a wave characteristic of light called polarization. What is polarization? How is it produced? What are some of its uses? The answers to these questions are related to the wave character of light.



Figure 1.32 These two photographs of a river show the effect of a polarizing filter in reducing glare in light reflected from the surface of water. Part (b) of this figure was taken with a polarizing filter and part (a) was not. As a result, the reflection of clouds and sky observed in part (a) is not observed in part (b). Polarizing sunglasses are particularly useful on snow and water. (credit a and credit b: modifications of work by “Amithshs”/Wikimedia Commons)

Malus's Law

Light is one type of electromagnetic (EM) wave. As noted in the previous chapter on [Electromagnetic Waves](#), EM waves are *transverse waves* consisting of varying electric and magnetic fields that oscillate perpendicular to the direction of propagation (Figure 1.33). However, in general, there are no specific directions for the oscillations of the electric and magnetic fields; they vibrate in any randomly oriented plane perpendicular to the direction of propagation. **Polarization** is the attribute that a wave's oscillations do have a definite direction relative to the direction of propagation of the wave. (This is not the same type of polarization as that discussed for the separation of charges.) Waves having such a direction are said to be **polarized**. For an EM wave, we define the **direction of polarization** to be the direction parallel to the electric field. Thus, we can think of the electric field arrows as showing the direction of polarization, as in Figure 1.33.

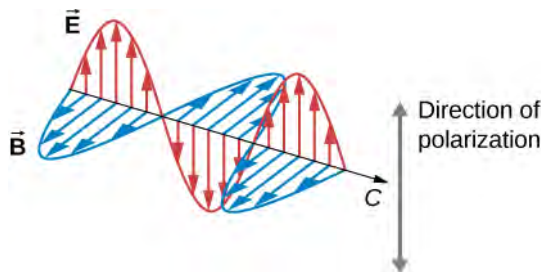


Figure 1.33 An EM wave, such as light, is a transverse wave. The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields are perpendicular to the direction of propagation. The direction of polarization of the wave is the direction of the electric field.

To examine this further, consider the transverse waves in the ropes shown in Figure 1.34. The oscillations in one rope are in a vertical plane and are said to be **vertically polarized**. Those in the other rope are in a horizontal plane and are **horizontally polarized**. If a vertical slit is placed on the first rope, the waves pass through. However, a vertical slit blocks the horizontally polarized waves. For EM waves, the direction of the electric field is analogous to the disturbances on the ropes.

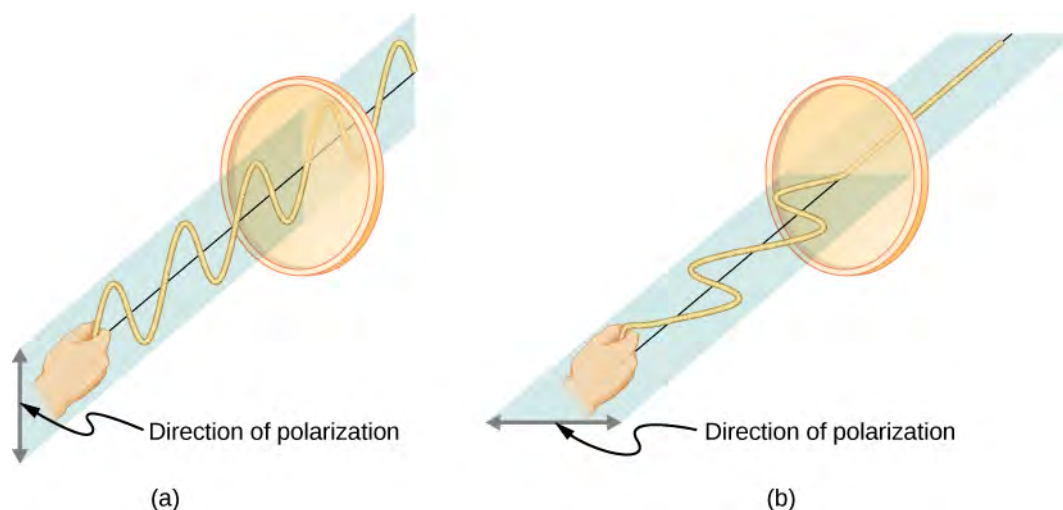


Figure 1.34 The transverse oscillations in one rope (a) are in a vertical plane, and those in the other rope (b) are in a horizontal plane. The first is said to be vertically polarized, and the other is said to be horizontally polarized. Vertical slits pass vertically polarized waves and block horizontally polarized waves.

The Sun and many other light sources produce waves that have the electric fields in random directions ([Figure 1.35\(a\)](#)). Such light is said to be **unpolarized**, because it is composed of many waves with all possible directions of polarization. Polaroid materials—which were invented by the founder of the Polaroid Corporation, Edwin Land—act as a polarizing slit for light, allowing only polarization in one direction to pass through. Polarizing filters are composed of long molecules aligned in one direction. If we think of the molecules as many slits, analogous to those for the oscillating ropes, we can understand why only light with a specific polarization can get through. The axis of a polarizing filter is the direction along which the filter passes the electric field of an EM wave.

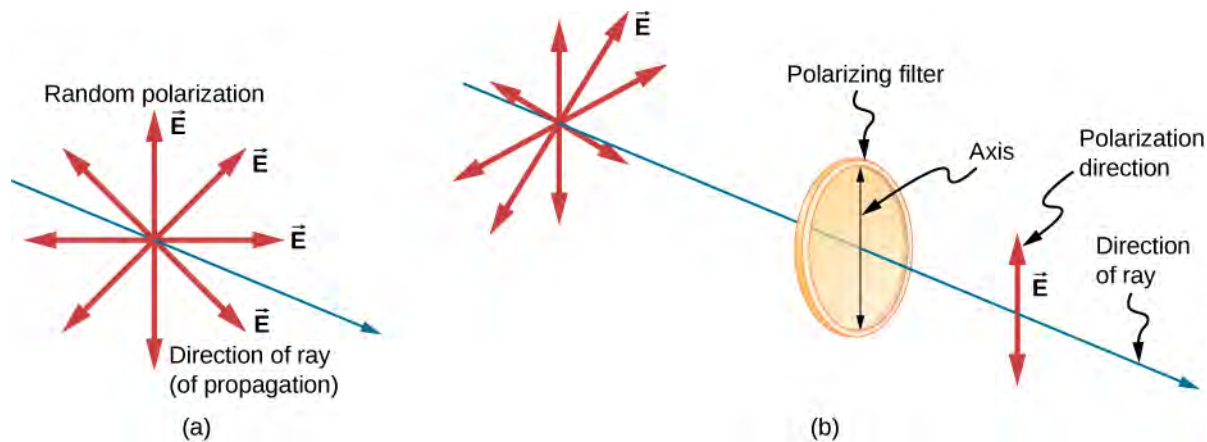


Figure 1.35 The slender arrow represents a ray of unpolarized light. The bold arrows represent the direction of polarization of the individual waves composing the ray. (a) If the light is unpolarized, the arrows point in all directions. (b) A polarizing filter has a polarization axis that acts as a slit passing through electric fields parallel to its direction. The direction of polarization of an EM wave is defined to be the direction of its electric field.

[Figure 1.36](#) shows the effect of two polarizing filters on originally unpolarized light. The first filter polarizes the light along its axis. When the axes of the first and second filters are aligned (parallel), then all of the polarized light passed by the first filter is also passed by the second filter. If the second polarizing filter is rotated, only the component of the light parallel to the second filter's axis is passed. When the axes are perpendicular, no light is passed by the second filter.

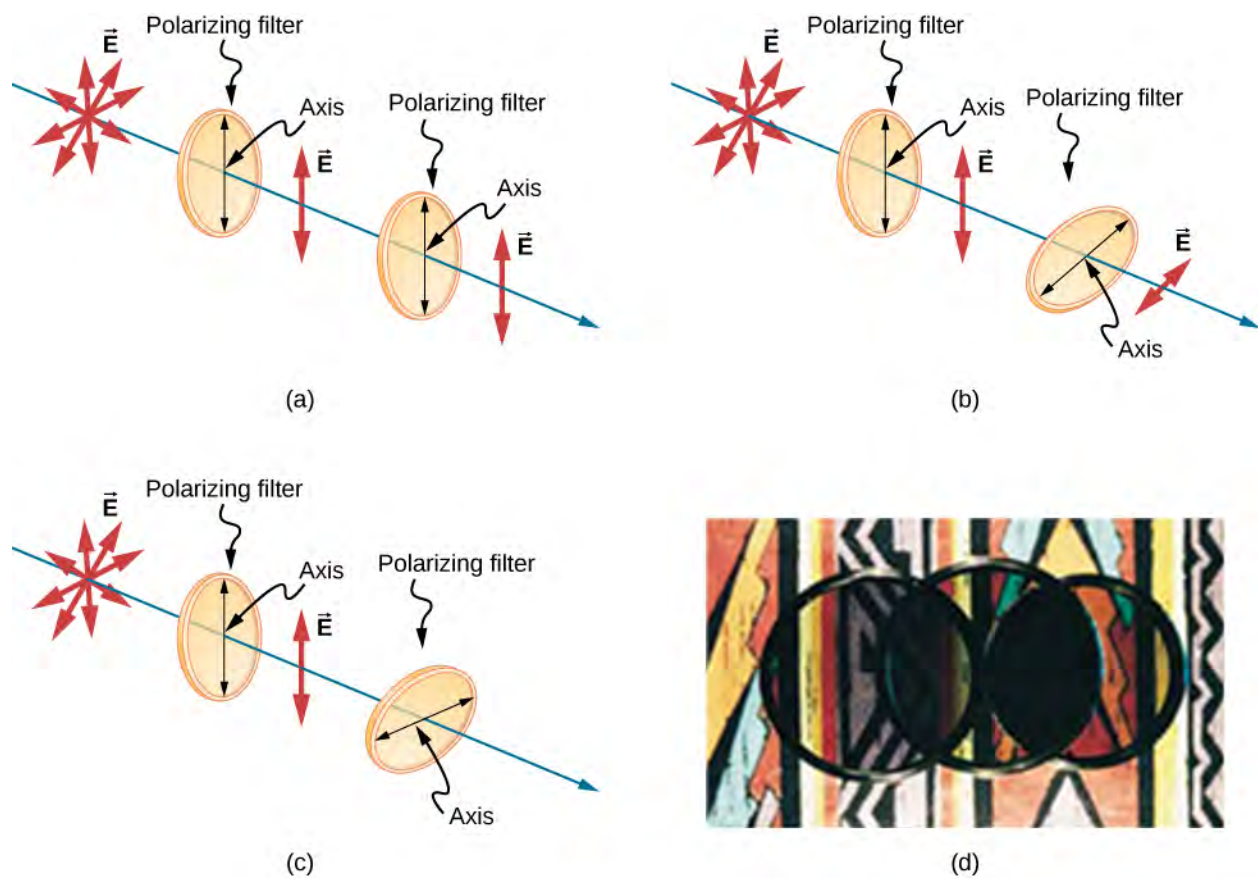


Figure 1.36 The effect of rotating two polarizing filters, where the first polarizes the light. (a) All of the polarized light is passed by the second polarizing filter, because its axis is parallel to the first. (b) As the second filter is rotated, only part of the light is passed. (c) When the second filter is perpendicular to the first, no light is passed. (d) In this photograph, a polarizing filter is placed above two others. Its axis is perpendicular to the filter on the right (dark area) and parallel to the filter on the left (lighter area). (credit d: modification of work by P.P. Urone)

Only the component of the EM wave parallel to the axis of a filter is passed. Let us call the angle between the direction of polarization and the axis of a filter θ . If the electric field has an amplitude E , then the transmitted part of the wave has an amplitude $E \cos \theta$ (Figure 1.37). Since the intensity of a wave is proportional to its amplitude squared, the intensity I of the transmitted wave is related to the incident wave by

$$I = I_0 \cos^2 \theta \quad 1.6$$

where I_0 is the intensity of the polarized wave before passing through the filter. This equation is known as **Malus's law**.

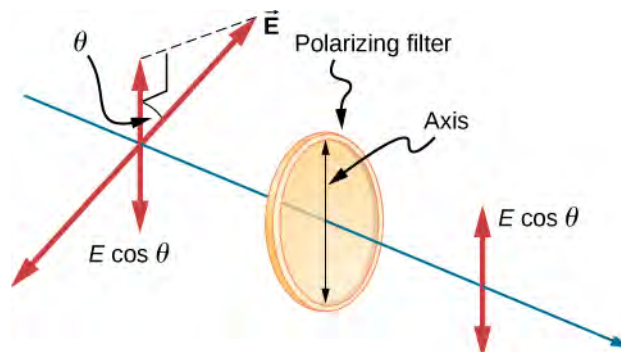


Figure 1.37 A polarizing filter transmits only the component of the wave parallel to its axis, reducing the intensity of any light not

polarized parallel to its axis.

INTERACTIVE

This [Open Source Physics animation \(https://openstax.org/l/21phyanielefie\)](https://openstax.org/l/21phyanielefie) helps you visualize the electric field vectors as light encounters a polarizing filter. You can rotate the filter—note that the angle displayed is in radians. You can also rotate the animation for 3D visualization.



EXAMPLE 1.7

Calculating Intensity Reduction by a Polarizing Filter

What angle is needed between the direction of polarized light and the axis of a polarizing filter to reduce its intensity by 90.0%?

Strategy

When the intensity is reduced by 90.0%, it is 10.0% or 0.100 times its original value. That is, $I = 0.100 I_0$. Using this information, the equation $I = I_0 \cos^2 \theta$ can be used to solve for the needed angle.

Solution

Solving the equation $I = I_0 \cos^2 \theta$ for $\cos \theta$ and substituting with the relationship between I and I_0 gives

$$\cos \theta = \sqrt{\frac{I}{I_0}} = \sqrt{\frac{0.100 I_0}{I_0}} = 0.3162.$$

Solving for θ yields

$$\theta = \cos^{-1} 0.3162 = 71.6^\circ.$$

Significance

A fairly large angle between the direction of polarization and the filter axis is needed to reduce the intensity to 10.0% of its original value. This seems reasonable based on experimenting with polarizing films. It is interesting that at an angle of 45° , the intensity is reduced to 50% of its original value. Note that 71.6° is 18.4° from reducing the intensity to zero, and that at an angle of 18.4° , the intensity is reduced to 90.0% of its original value, giving evidence of symmetry.

CHECK YOUR UNDERSTANDING 1.6

Although we did not specify the direction in [Example 1.7](#), let's say the polarizing filter was rotated clockwise by 71.6° to reduce the light intensity by 90.0%. What would be the intensity reduction if the polarizing filter were rotated counterclockwise by 71.6° ?

Polarization by Reflection

By now, you can probably guess that polarizing sunglasses cut the glare in reflected light, because that light is polarized. You can check this for yourself by holding polarizing sunglasses in front of you and rotating them while looking at light reflected from water or glass. As you rotate the sunglasses, you will notice the light gets bright and dim, but not completely black. This implies the reflected light is partially polarized and cannot be completely blocked by a polarizing filter.

[Figure 1.38](#) illustrates what happens when unpolarized light is reflected from a surface. Vertically polarized light is preferentially refracted at the surface, so the reflected light is left more horizontally polarized. The reasons for this phenomenon are beyond the scope of this text, but a convenient mnemonic for remembering this is to imagine the polarization direction to be like an arrow. Vertical polarization is like an arrow perpendicular to the surface and is more likely to stick and not be reflected. Horizontal polarization is like an

arrow bouncing on its side and is more likely to be reflected. Sunglasses with vertical axes thus block more reflected light than unpolarized light from other sources.

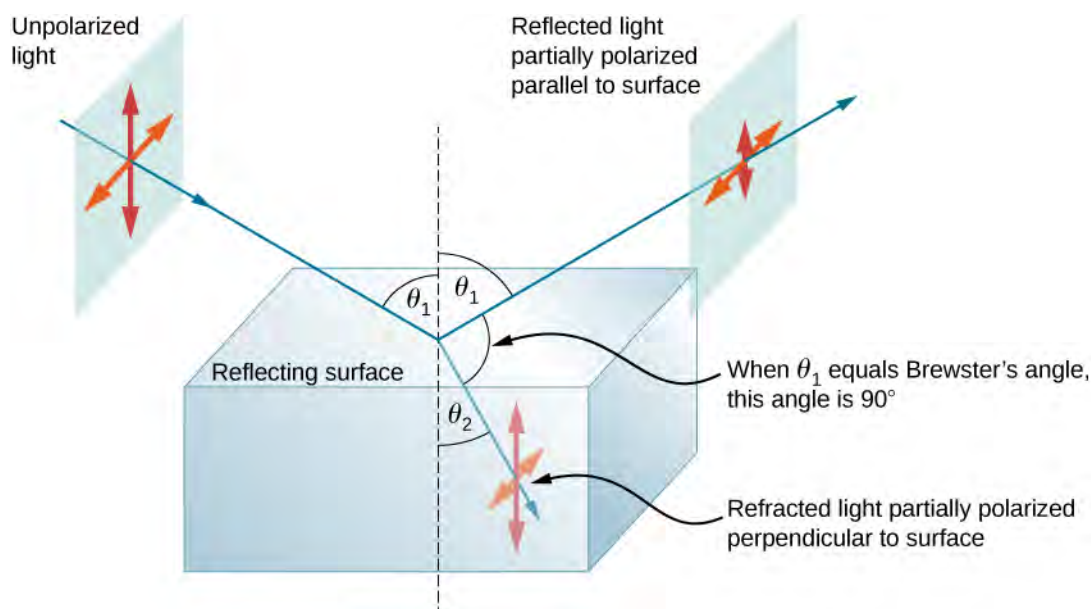


Figure 1.38 Polarization by reflection. Unpolarized light has equal amounts of vertical and horizontal polarization. After interaction with a surface, the vertical components are preferentially absorbed or refracted, leaving the reflected light more horizontally polarized. This is akin to arrows striking on their sides and bouncing off, whereas arrows striking on their tips go into the surface.

Since the part of the light that is not reflected is refracted, the amount of polarization depends on the indices of refraction of the media involved. It can be shown that reflected light is completely polarized at an angle of reflection θ_b given by

$$\tan \theta_b = \frac{n_2}{n_1} \quad 1.7$$

where n_1 is the medium in which the incident and reflected light travel and n_2 is the index of refraction of the medium that forms the interface that reflects the light. This equation is known as **Brewster's law** and θ_b is known as **Brewster's angle**, named after the nineteenth-century Scottish physicist who discovered them.

INTERACTIVE

This [Open Source Physics animation \(https://openstax.org/l/21phyaniinoref\)](https://openstax.org/l/21phyaniinoref) shows incident, reflected, and refracted light as rays and EM waves. Try rotating the animation for 3D visualization and also change the angle of incidence. Near Brewster's angle, the reflected light becomes highly polarized.



EXAMPLE 1.8

Calculating Polarization by Reflection

(a) At what angle will light traveling in air be completely polarized horizontally when reflected from water? (b) From glass?

Strategy

All we need to solve these problems are the indices of refraction. Air has $n_1 = 1.00$, water has $n_2 = 1.333$, and crown glass has $n_2 = 1.520$. The equation $\tan \theta_b = \frac{n_2}{n_1}$ can be directly applied to find θ_b in each case.

Solution

- a. Putting the known quantities into the equation

$$\tan \theta_b = \frac{n_2}{n_1}$$

gives

$$\tan \theta_b = \frac{n_2}{n_1} = \frac{1.333}{1.00} = 1.333.$$

Solving for the angle θ_b yields

$$\theta_b = \tan^{-1} 1.333 = 53.1^\circ.$$

- b. Similarly, for crown glass and air,

$$\tan \theta'_b = \frac{n'_2}{n_1} = \frac{1.520}{1.00} = 1.52.$$

Thus,

$$\theta'_b = \tan^{-1} 1.52 = 56.7^\circ.$$

Significance

Light reflected at these angles could be completely blocked by a good polarizing filter held with its axis vertical. Brewster's angle for water and air are similar to those for glass and air, so that sunglasses are equally effective for light reflected from either water or glass under similar circumstances. Light that is not reflected is refracted into these media. Therefore, at an incident angle equal to Brewster's angle, the refracted light is slightly polarized vertically. It is not completely polarized vertically, because only a small fraction of the incident light is reflected, so a significant amount of horizontally polarized light is refracted.

CHECK YOUR UNDERSTANDING 1.7

What happens at Brewster's angle if the original incident light is already 100% vertically polarized?

Atomic Explanation of Polarizing Filters

Polarizing filters have a polarization axis that acts as a slit. This slit passes EM waves (often visible light) that have an electric field parallel to the axis. This is accomplished with long molecules aligned perpendicular to the axis, as shown in [Figure 1.39](#).

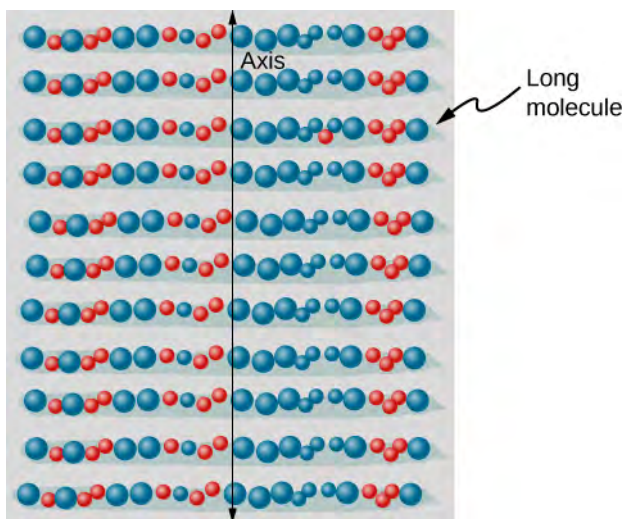


Figure 1.39 Long molecules are aligned perpendicular to the axis of a polarizing filter. In an EM wave, the component of the electric field perpendicular to these molecules passes through the filter, whereas the component parallel to the molecules is absorbed.

[Figure 1.40](#) illustrates how the component of the electric field parallel to the long molecules is absorbed. An EM wave is composed of oscillating electric and magnetic fields. The electric field is strong compared with the magnetic field and is more effective in exerting force on charges in the molecules. The most affected charged particles are the electrons, since electron masses are small. If an electron is forced to oscillate, it can absorb energy from the EM wave. This reduces the field in the wave and, hence, reduces its intensity. In long molecules, electrons can more easily oscillate parallel to the molecule than in the perpendicular direction. The electrons are bound to the molecule and are more restricted in their movement perpendicular to the molecule. Thus, the electrons can absorb EM waves that have a component of their electric field parallel to the molecule. The electrons are much less responsive to electric fields perpendicular to the molecule and allow these fields to pass. Thus, the axis of the polarizing filter is perpendicular to the length of the molecule.

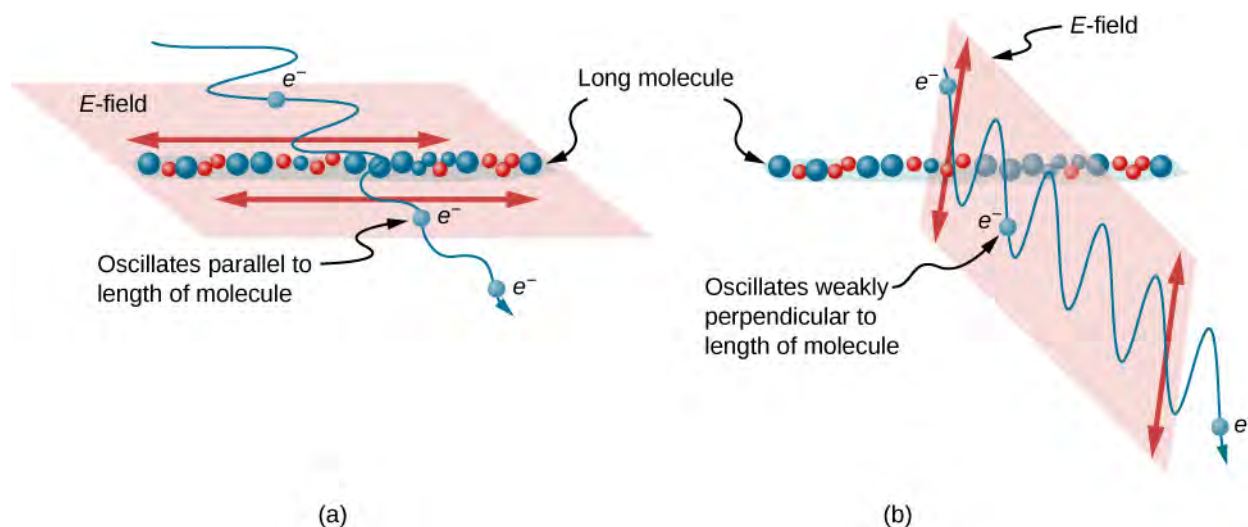


Figure 1.40 Diagram of an electron in a long molecule oscillating parallel to the molecule. The oscillation of the electron absorbs energy and reduces the intensity of the component of the EM wave that is parallel to the molecule.

Polarization by Scattering

If you hold your polarizing sunglasses in front of you and rotate them while looking at blue sky, you will see the sky get bright and dim. This is a clear indication that light scattered by air is partially polarized. [Figure 1.41](#) helps illustrate how this happens. Since light is a transverse EM wave, it vibrates the electrons of air molecules perpendicular to the direction that it is traveling. The electrons then radiate like small antennae. Since they are oscillating perpendicular to the direction of the light ray, they produce EM radiation that is polarized perpendicular to the direction of the ray. When viewing the light along a line perpendicular to the original ray, as in the figure, there can be no polarization in the scattered light parallel to the original ray, because that would require the original ray to be a longitudinal wave. Along other directions, a component of the other polarization can be projected along the line of sight, and the scattered light is only partially polarized. Furthermore, multiple scattering can bring light to your eyes from other directions and can contain different polarizations.

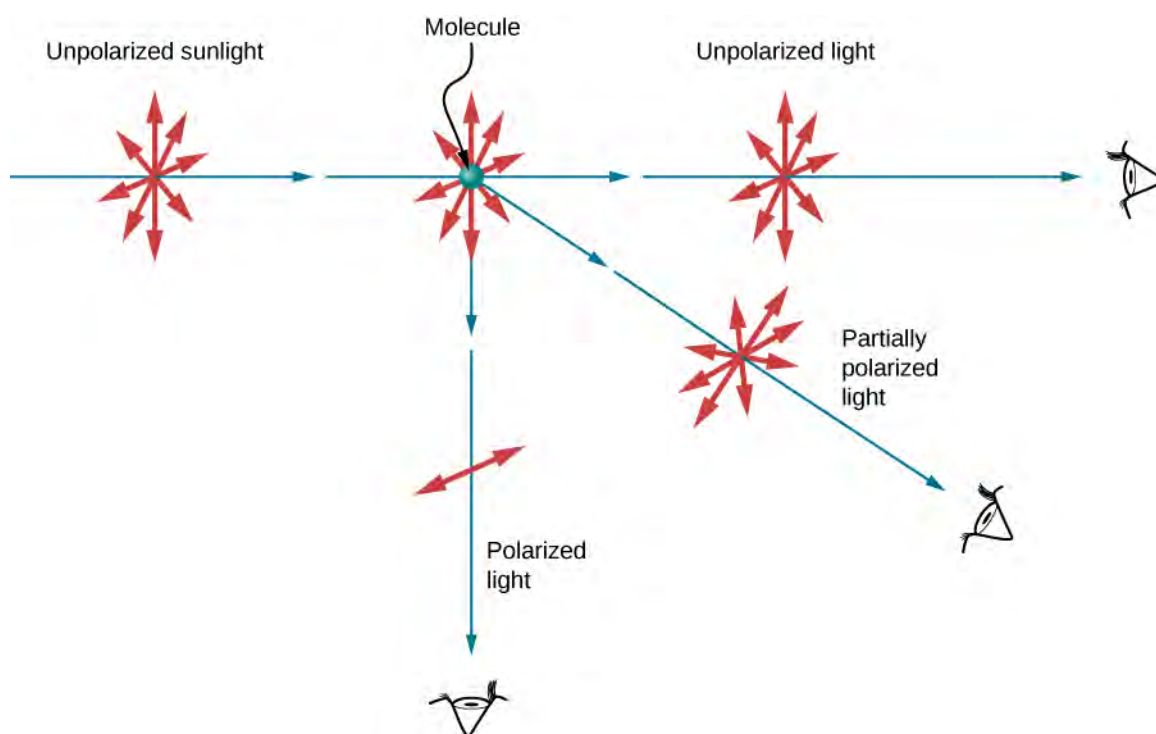


Figure 1.41 Polarization by scattering. Unpolarized light scattering from air molecules shakes their electrons perpendicular to the direction of the original ray. The scattered light therefore has a polarization perpendicular to the original direction and none parallel to the original direction.

Photographs of the sky can be darkened by polarizing filters, a trick used by many photographers to make clouds brighter by contrast. Scattering from other particles, such as smoke or dust, can also polarize light. Detecting polarization in scattered EM waves can be a useful analytical tool in determining the scattering source.

A range of optical effects are used in sunglasses. Besides being polarizing, sunglasses may have colored pigments embedded in them, whereas others use either a nonreflective or reflective coating. A recent development is photochromic lenses, which darken in the sunlight and become clear indoors. Photochromic lenses are embedded with organic microcrystalline molecules that change their properties when exposed to UV in sunlight, but become clear in artificial lighting with no UV.

Liquid Crystals and Other Polarization Effects in Materials

Although you are undoubtedly aware of liquid crystal displays (LCDs) found in watches, calculators, computer screens, cellphones, flat screen televisions, and many other places, you may not be aware that they are based on polarization. Liquid crystals are so named because their molecules can be aligned even though they are in a liquid. Liquid crystals have the property that they can rotate the polarization of light passing through them by 90° . Furthermore, this property can be turned off by the application of a voltage, as illustrated in [Figure 1.42](#). It is possible to manipulate this characteristic quickly and in small, well-defined regions to create the contrast patterns we see in so many LCD devices.

In flat screen LCD televisions, a large light is generated at the back of the TV. The light travels to the front screen through millions of tiny units called pixels (picture elements). One of these is shown in [Figure 1.42\(a\)](#) and (b). Each unit has three cells, with red, blue, or green filters, each controlled independently. When the voltage across a liquid crystal is switched off, the liquid crystal passes the light through the particular filter. We can vary the picture contrast by varying the strength of the voltage applied to the liquid crystal.

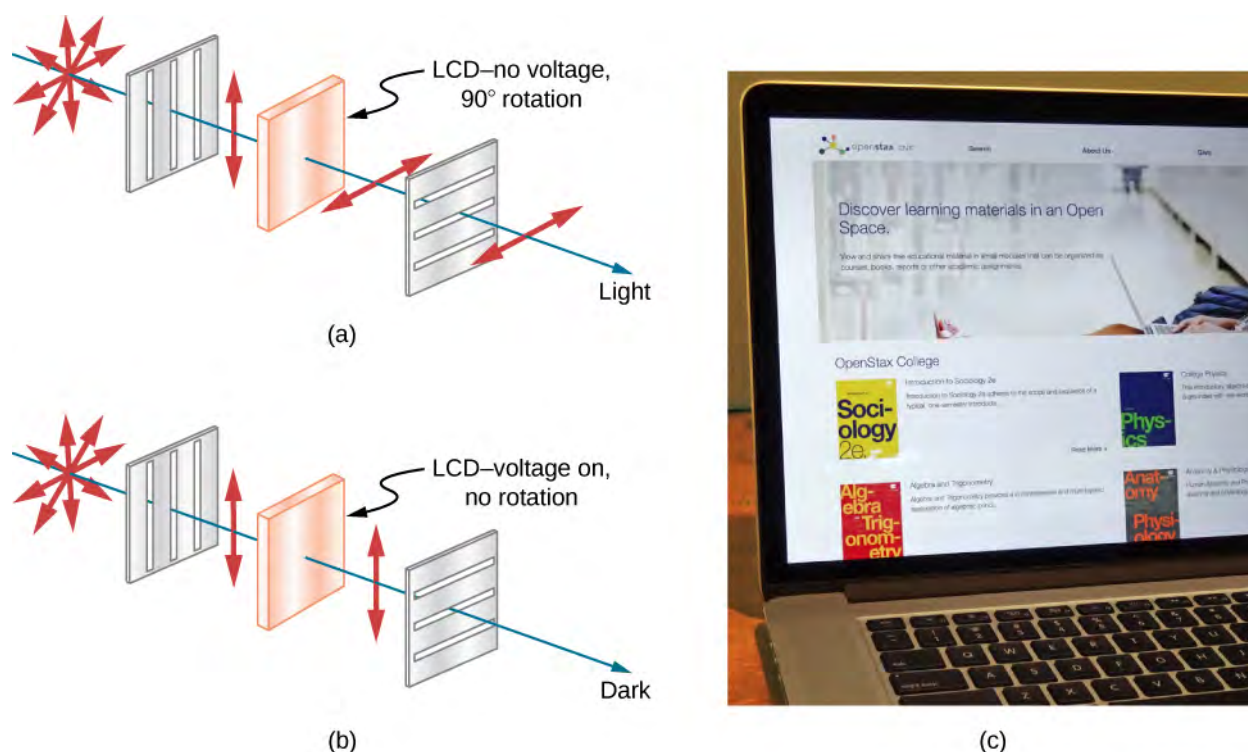


Figure 1.42 (a) Polarized light is rotated 90° by a liquid crystal and then passed by a polarizing filter that has its axis perpendicular to the direction of the original polarization. (b) When a voltage is applied to the liquid crystal, the polarized light is not rotated and is blocked by the filter, making the region dark in comparison with its surroundings. (c) LCDs can be made color specific, small, and fast enough to use in laptop computers and TVs. (credit c: modification of work by Jane Whitney)

Many crystals and solutions rotate the plane of polarization of light passing through them. Such substances are said to be **optically active**. Examples include sugar water, insulin, and collagen ([Figure 1.43](#)). In addition to depending on the type of substance, the amount and direction of rotation depend on several other factors. Among these is the concentration of the substance, the distance the light travels through it, and the wavelength of light. Optical activity is due to the asymmetrical shape of molecules in the substance, such as being helical. Measurements of the rotation of polarized light passing through substances can thus be used to measure concentrations, a standard technique for sugars. It can also give information on the shapes of molecules, such as proteins, and factors that affect their shapes, such as temperature and pH.

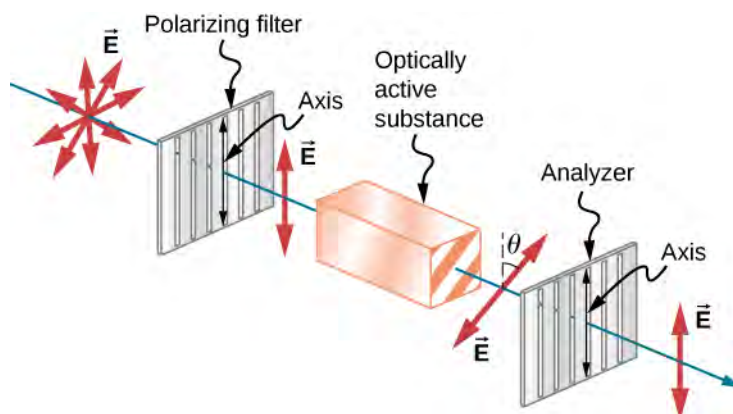


Figure 1.43 Optical activity is the ability of some substances to rotate the plane of polarization of light passing through them. The rotation is detected with a polarizing filter or analyzer.

Glass and plastic become optically active when stressed: the greater the stress, the greater the effect. Optical stress analysis on complicated shapes can be performed by making plastic models of them and observing them through crossed filters, as seen in [Figure 1.44](#). It is apparent that the effect depends on wavelength as

well as stress. The wavelength dependence is sometimes also used for artistic purposes.

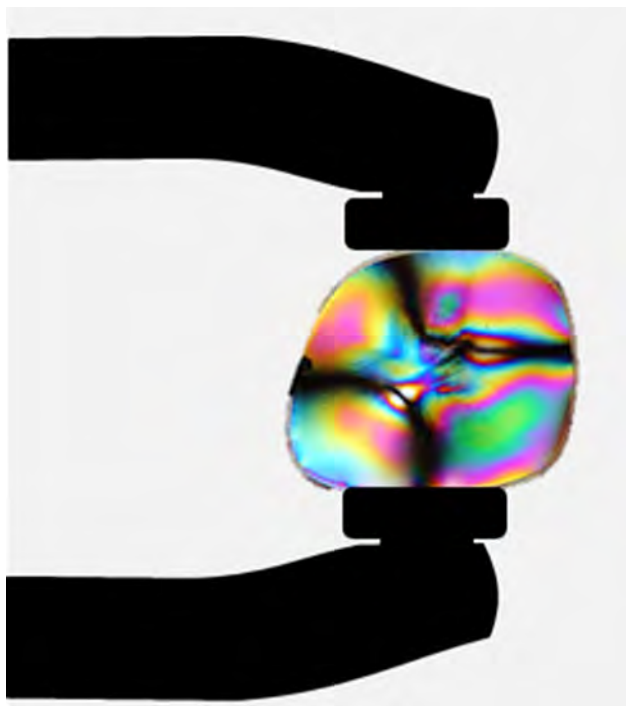


Figure 1.44 Optical stress analysis of a plastic lens placed between crossed polarizers. (credit: "Infopro"/Wikimedia Commons)

Another interesting phenomenon associated with polarized light is the ability of some crystals to split an unpolarized beam of light into two polarized beams. This occurs because the crystal has one value for the index of refraction of polarized light but a different value for the index of refraction of light polarized in the perpendicular direction. Such crystals are said to be **birefringent**, and, when aligned properly, two perpendicularly polarized beams will emerge from the crystal ([Figure 1.45](#)). Birefringent crystals can be used to produce polarized beams from unpolarized light. Some birefringent materials preferentially absorb one of the polarizations. These materials are called dichroic and can produce polarization by this preferential absorption. This is fundamentally how polarizing filters and other polarizers work.

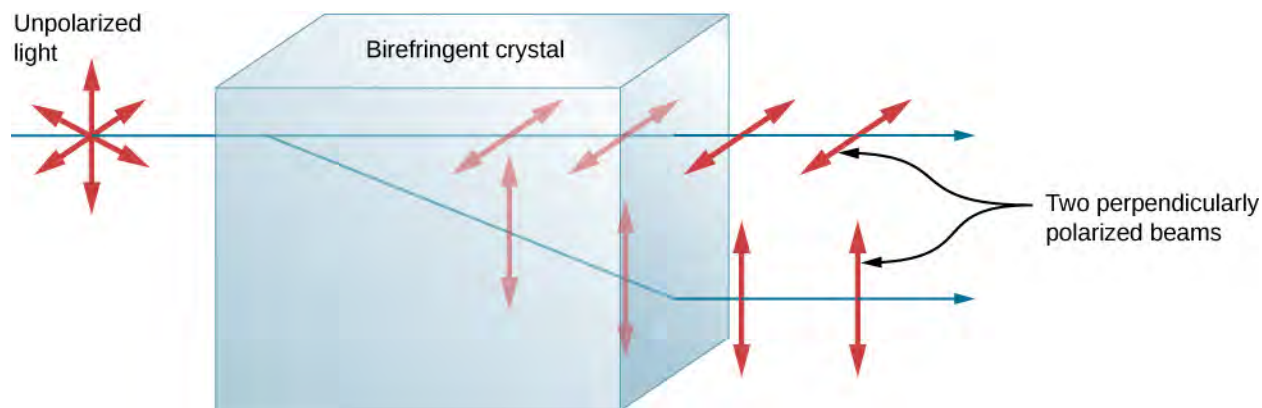


Figure 1.45 Birefringent materials, such as the common mineral calcite, split unpolarized beams of light into two with two different values of index of refraction.

Summary

1.1 The Propagation of Light

- The speed of light in a vacuum is $c = 2.99792458 \times 10^8 \text{ m/s} \approx 3.00 \times 10^8 \text{ m/s}$.
- The index of refraction of a material is $n = c/v$, where v is the speed of light in a material and c is the speed of light in a vacuum.
- The ray model of light describes the path of light as straight lines. The part of optics dealing with the ray aspect of light is called geometric optics.
- Light can travel in three ways from a source to another location: (1) directly from the source through empty space; (2) through various media; and (3) after being reflected from a mirror.

1.2 The Law of Reflection

- When a light ray strikes a smooth surface, the angle of reflection equals the angle of incidence.
- A mirror has a smooth surface and reflects light at specific angles.
- Light is diffused when it reflects from a rough surface.

1.3 Refraction

- The change of a light ray's direction when it passes through variations in matter is called refraction.
- The law of refraction, also called Snell's law, relates the indices of refraction for two media at an interface to the change in angle of a light ray passing through that interface.

1.4 Total Internal Reflection

- The incident angle that produces an angle of refraction of 90° is called the critical angle.
- Total internal reflection is a phenomenon that occurs at the boundary between two media, such that if the incident angle in the first medium is greater than the critical angle, then all the light is reflected back into that medium.
- Fiber optics involves the transmission of light down fibers of plastic or glass, applying the principle of total internal reflection.
- Cladding prevents light from being transmitted between fibers in a bundle.
- Diamonds sparkle due to total internal reflection coupled with a large index of refraction.

1.5 Dispersion

- The spreading of white light into its full

spectrum of wavelengths is called dispersion.

- Rainbows are produced by a combination of refraction and reflection, and involve the dispersion of sunlight into a continuous distribution of colors.
- Dispersion produces beautiful rainbows but also causes problems in certain optical systems.

1.6 Huygens's Principle

- According to Huygens's principle, every point on a wave front is a source of wavelets that spread out in the forward direction at the same speed as the wave itself. The new wave front is tangent to all of the wavelets.
- A mirror reflects an incoming wave at an angle equal to the incident angle, verifying the law of reflection.
- The law of refraction can be explained by applying Huygens's principle to a wave front passing from one medium to another.
- The bending of a wave around the edges of an opening or an obstacle is called diffraction.

1.7 Polarization

- Polarization is the attribute that wave oscillations have a definite direction relative to the direction of propagation of the wave. The direction of polarization is defined to be the direction parallel to the electric field of the EM wave.
- Unpolarized light is composed of many rays having random polarization directions.
- Unpolarized light can be polarized by passing it through a polarizing filter or other polarizing material. The process of polarizing light decreases its intensity by a factor of 2.
- The intensity, I , of polarized light after passing through a polarizing filter is $I = I_0 \cos^2 \theta$, where I_0 is the incident intensity and θ is the angle between the direction of polarization and the axis of the filter.
- Polarization is also produced by reflection.
- Brewster's law states that reflected light is completely polarized at the angle of reflection θ_b , known as Brewster's angle.
- Polarization can also be produced by scattering.
- Several types of optically active substances rotate the direction of polarization of light passing through them.

Conceptual Questions

1.1 The Propagation of Light

- Under what conditions can light be modeled like a ray? Like a wave?
- Why is the index of refraction always greater than or equal to 1?
- Does the fact that the light flash from lightning reaches you before its sound prove that the speed of light is extremely large or simply that it is greater than the speed of sound? Discuss how you could use this effect to get an estimate of the speed of light.
- Speculate as to what physical process might be responsible for light traveling more slowly in a medium than in a vacuum.

1.2 The Law of Reflection

- Using the law of reflection, explain how powder takes the shine off of a person's nose. What is the name of the optical effect?

1.3 Refraction

- Diffusion by reflection from a rough surface is described in this chapter. Light can also be diffused by refraction. Describe how this occurs in a specific situation, such as light interacting with crushed ice.
- Will light change direction toward or away from the perpendicular when it goes from air to water? Water to glass? Glass to air?
- Explain why an object in water always appears to be at a depth shallower than it actually is?
- Explain why a person's legs appear very short when wading in a pool. Justify your explanation with a ray diagram showing the path of rays from the feet to the eye of an observer who is out of the water.
- Explain why an oar that is partially submerged in water appears bent.

1.4 Total Internal Reflection

- A ring with a colorless gemstone is dropped into water. The gemstone becomes invisible when submerged. Can it be a diamond? Explain.
- The most common type of mirage is an illusion that light from faraway objects is reflected by a pool of water that is not really there. Mirages are generally observed in deserts, when there is a hot layer of air near the ground. Given that the refractive index of air is lower for air at higher

temperatures, explain how mirages can be formed.

- How can you use total internal reflection to estimate the index of refraction of a medium?

1.5 Dispersion

- Is it possible that total internal reflection plays a role in rainbows? Explain in terms of indices of refraction and angles, perhaps referring to that shown below. Some of us have seen the formation of a double rainbow; is it physically possible to observe a triple rainbow?

(credit: "Chad"/Flickr)



- A high-quality diamond may be quite clear and colorless, transmitting all visible wavelengths with little absorption. Explain how it can sparkle with flashes of brilliant color when illuminated by white light.

1.6 Huygens's Principle

- How do wave effects depend on the size of the object with which the wave interacts? For example, why does sound bend around the corner of a building while light does not?
- Does Huygens's principle apply to all types of waves?
- If diffraction is observed for some phenomenon, it is evidence that the phenomenon is a wave. Does the reverse hold true? That is, if diffraction is not observed, does that mean the phenomenon is not a wave?

1.7 Polarization

- Can a sound wave in air be polarized? Explain.
- No light passes through two perfect polarizing filters with perpendicular axes. However, if a third polarizing filter is placed between the original two, some light can pass. Why is this?

Under what circumstances does most of the light pass?

21. Explain what happens to the energy carried by light that it is dimmed by passing it through two crossed polarizing filters.
22. When particles scattering light are much smaller than its wavelength, the amount of scattering is proportional to $\frac{1}{\lambda}$. Does this mean there is more scattering for small λ than large λ ? How does this relate to the fact that the sky is blue?
23. Using the information given in the preceding question, explain why sunsets are red.

Problems

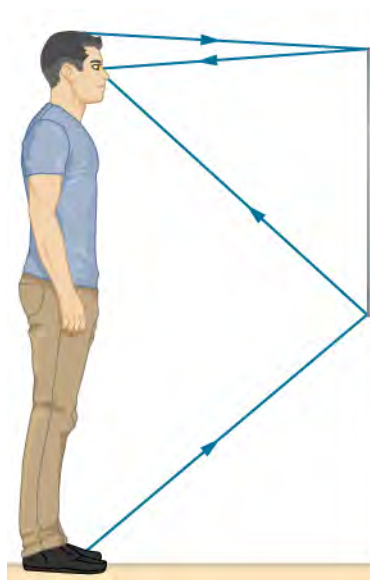
1.1 The Propagation of Light

26. What is the speed of light in water? In glycerine?
27. What is the speed of light in air? In crown glass?
28. Calculate the index of refraction for a medium in which the speed of light is 2.012×10^8 m/s, and identify the most likely substance based on [Table 1.1](#).
29. In what substance in [Table 1.1](#) is the speed of light 2.290×10^8 m/s?
30. There was a major collision of an asteroid with the Moon in medieval times. It was described by monks at Canterbury Cathedral in England as a red glow on and around the Moon. How long after the asteroid hit the Moon, which is 3.84×10^5 km away, would the light first arrive on Earth?
31. Components of some computers communicate with each other through optical fibers having an index of refraction $n = 1.55$. What time in nanoseconds is required for a signal to travel 0.200 m through such a fiber?
32. Compare the time it takes for light to travel 1000 m on the surface of Earth and in outer space.
33. How far does light travel underwater during a time interval of 1.50×10^{-6} s?

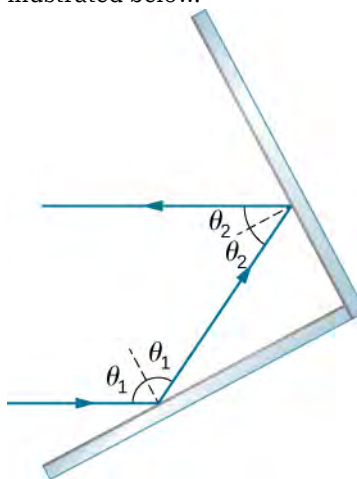
1.2 The Law of Reflection

34. Suppose a man stands in front of a mirror as shown below. His eyes are 1.65 m above the floor and the top of his head is 0.13 m higher. Find the height above the floor of the top and bottom of the smallest mirror in which he can see both the top of his head and his feet. How is this distance related to the man's height?

24. When light is reflected at Brewster's angle from a smooth surface, it is 100% polarized parallel to the surface. Part of the light will be refracted into the surface. Describe how you would do an experiment to determine the polarization of the refracted light. What direction would you expect the polarization to have and would you expect it to be 100%?
25. If you lie on a beach looking at the water with your head tipped slightly sideways, your polarized sunglasses do not work very well. Why not?



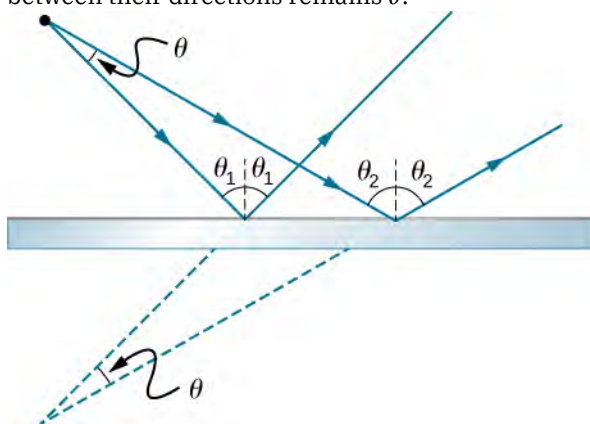
35. Show that when light reflects from two mirrors that meet each other at a right angle, the outgoing ray is parallel to the incoming ray, as illustrated below.



36. On the Moon's surface, lunar astronauts placed a corner reflector, off which a laser beam is

periodically reflected. The distance to the Moon is calculated from the round-trip time. What percent correction is needed to account for the delay in time due to the slowing of light in Earth's atmosphere? Assume the distance to the Moon is precisely 3.84×10^8 m and Earth's atmosphere (which varies in density with altitude) is equivalent to a layer 30.0 km thick with a constant index of refraction $n = 1.000293$.

37. A flat mirror is neither converging nor diverging. To prove this, consider two rays originating from the same point and diverging at an angle θ (see below). Show that after striking a plane mirror, the angle between their directions remains θ .



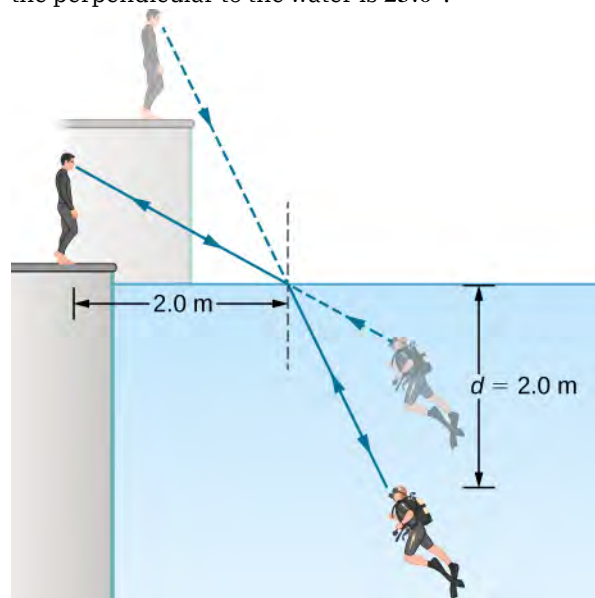
1.3 Refraction

Unless otherwise specified, for problems 1 through 10, the indices of refraction of glass and water should be taken to be 1.50 and 1.333, respectively.

38. A light beam in air has an angle of incidence of 35° at the surface of a glass plate. What are the angles of reflection and refraction?
39. A light beam in air is incident on the surface of a pond, making an angle of 20° with respect to the surface. What are the angles of reflection and refraction?
40. When a light ray crosses from water into glass, it emerges at an angle of 30° with respect to the normal of the interface. What is its angle of incidence?
41. A pencil flashlight submerged in water sends a light beam toward the surface at an angle of incidence of 30° . What is the angle of refraction in air?
42. Light rays from the Sun make a 30° angle to the vertical when seen from below the surface of a body of water. At what angle above the horizon is the Sun?
43. The path of a light beam in air goes from an

angle of incidence of 35° to an angle of refraction of 22° when it enters a rectangular block of plastic. What is the index of refraction of the plastic?

44. A scuba diver training in a pool looks at his instructor as shown below. What angle does the ray from the instructor's face make with the perpendicular to the water at the point where the ray enters? The angle between the ray in the water and the perpendicular to the water is 25.0° .



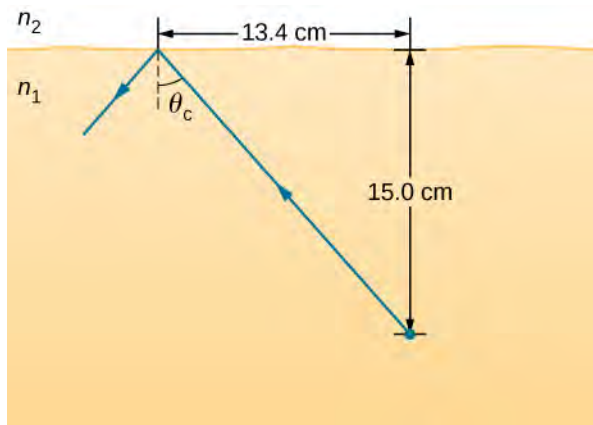
45. (a) Using information in the preceding problem, find the height of the instructor's head above the water, noting that you will first have to calculate the angle of incidence. (b) Find the apparent depth of the diver's head below water as seen by the instructor.

1.4 Total Internal Reflection

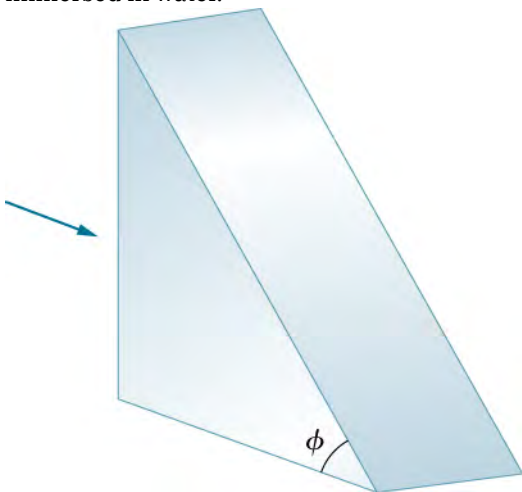
46. Verify that the critical angle for light going from water to air is 48.6° , as discussed at the end of [Example 1.4](#), regarding the critical angle for light traveling in a polystyrene (a type of plastic) pipe surrounded by air.
47. (a) At the end of [Example 1.4](#), it was stated that the critical angle for light going from diamond to air is 24.4° . Verify this. (b) What is the critical angle for light going from zircon to air?
48. An optical fiber uses flint glass clad with crown glass. What is the critical angle?
49. At what minimum angle will you get total internal reflection of light traveling in water and reflected from ice?
50. Suppose you are using total internal reflection to make an efficient corner reflector. If there is air outside and the incident angle is 45.0° , what

must be the minimum index of refraction of the material from which the reflector is made?

51. You can determine the index of refraction of a substance by determining its critical angle. (a) What is the index of refraction of a substance that has a critical angle of 68.4° when submerged in water? What is the substance, based on [Table 1.1](#)? (b) What would the critical angle be for this substance in air?
52. A ray of light, emitted beneath the surface of an unknown liquid with air above it, undergoes total internal reflection as shown below. What is the index of refraction for the liquid and its likely identification?



53. Light rays fall normally on the vertical surface of the glass prism ($n = 1.50$) shown below. (a) What is the largest value for ϕ such that the ray is totally reflected at the slanted face? (b) Repeat the calculation of part (a) if the prism is immersed in water.

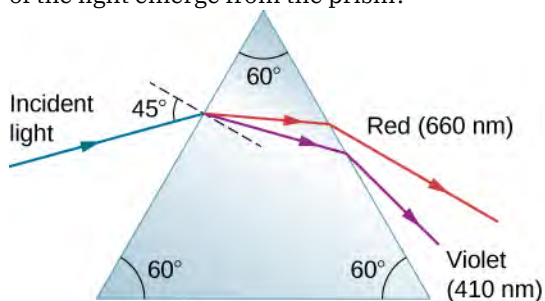


1.5 Dispersion

54. (a) What is the ratio of the speed of red light to violet light in diamond, based on [Table 1.2](#)? (b) What is this ratio in polystyrene? (c) Which is

more dispersive?

55. A beam of white light goes from air into water at an incident angle of 75.0° . At what angles are the red (660 nm) and violet (410 nm) parts of the light refracted?
56. By how much do the critical angles for red (660 nm) and violet (410 nm) light differ in a diamond surrounded by air?
57. (a) A narrow beam of light containing yellow (580 nm) and green (550 nm) wavelengths goes from polystyrene to air, striking the surface at a 30.0° incident angle. What is the angle between the colors when they emerge? (b) How far would they have to travel to be separated by 1.00 mm?
58. A parallel beam of light containing orange (610 nm) and violet (410 nm) wavelengths goes from fused quartz to water, striking the surface between them at a 60.0° incident angle. What is the angle between the two colors in water?
59. A ray of 610-nm light goes from air into fused quartz at an incident angle of 55.0° . At what incident angle must 470 nm light enter flint glass to have the same angle of refraction?
60. A narrow beam of light containing red (660 nm) and blue (470 nm) wavelengths travels from air through a 1.00-cm-thick flat piece of crown glass and back to air again. The beam strikes at a 30.0° incident angle. (a) At what angles do the two colors emerge? (b) By what distance are the red and blue separated when they emerge?
61. A narrow beam of white light enters a prism made of crown glass at a 45.0° incident angle, as shown below. At what angles, θ_R and θ_V , do the red (660 nm) and violet (410 nm) components of the light emerge from the prism?



1.7 Polarization

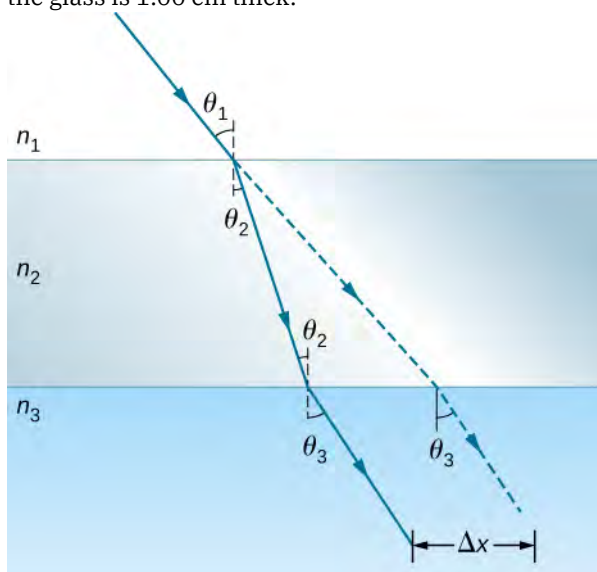
62. What angle is needed between the direction of polarized light and the axis of a polarizing filter to cut its intensity in half?
63. The angle between the axes of two polarizing filters is 45.0° . By how much does the second filter reduce the intensity of the light coming through the first?

64. Two polarizing sheets P_1 and P_2 are placed together with their transmission axes oriented at an angle θ to each other. What is θ when only 25% of the maximum transmitted light intensity passes through them?
65. Suppose that in the preceding problem the light incident on P_1 is unpolarized. At the determined value of θ , what fraction of the incident light passes through the combination?
66. If you have completely polarized light of intensity 150 W/m^2 , what will its intensity be after passing through a polarizing filter with its axis at an 89.0° angle to the light's polarization direction?
67. What angle would the axis of a polarizing filter need to make with the direction of polarized light of intensity 1.00 kW/m^2 to reduce the intensity to 10.0 W/m^2 ?
68. At the end of [Example 1.7](#), it was stated that the intensity of polarized light is reduced to 90.0% of its original value by passing through a polarizing filter with its axis at an angle of 18.4° to the direction of polarization. Verify this statement.
69. Show that if you have three polarizing filters, with the second at an angle of 45.0° to the first and the third at an angle of 90.0° to the first, the intensity of light passed by the first will be reduced to 25.0% of its value. (This is in contrast to having only the first and third, which reduces the intensity to zero, so that placing the second between them increases the intensity of the transmitted light.)
70. Three polarizing sheets are placed together such that the transmission axis of the second sheet is oriented at 25.0° to the axis of the first, whereas the transmission axis of the third sheet is oriented at 40.0° (in the same sense) to the axis of the first. What fraction of the intensity of an incident unpolarized beam is transmitted by the combination?
71. In order to rotate the polarization axis of a beam of linearly polarized light by 90.0° , a student places sheets P_1 and P_2 with their transmission axes at 45.0° and 90.0° , respectively, to the beam's axis of polarization. (a) What fraction of the incident light passes through P_1 and (b) through the combination? (c) Repeat your calculations for part (b) for transmission-axis angles of 30.0° and 90.0° , respectively.
72. It is found that when light traveling in water falls on a plastic block, Brewster's angle is 50.0° . What is the refractive index of the plastic?
73. At what angle will light reflected from diamond be completely polarized?
74. What is Brewster's angle for light traveling in water that is reflected from crown glass?
75. A scuba diver sees light reflected from the water's surface. At what angle relative to the water's surface will this light be completely polarized?

Additional Problems

76. From his measurements, Roemer estimated that it took 22 min for light to travel a distance equal to the diameter of Earth's orbit around the Sun. (a) Use this estimate along with the known diameter of Earth's orbit to obtain a rough value of the speed of light. (b) Light actually takes 16.5 min to travel this distance. Use this time to calculate the speed of light.
77. Cornu performed Fizeau's measurement of the speed of light using a wheel of diameter 4.00 cm that contained 180 teeth. The distance from the wheel to the mirror was 22.9 km. Assuming he measured the speed of light accurately, what was the angular velocity of the wheel?
78. Suppose you have an unknown clear substance immersed in water, and you wish to identify it by finding its index of refraction. You arrange to have a beam of light enter it at an angle of 45.0° , and you observe the angle of refraction to be 40.3° . What is the index of refraction of the substance and its likely identity?

79. Shown below is a ray of light going from air through crown glass into water, such as going into a fish tank. Calculate the amount the ray is displaced by the glass (Δx), given that the incident angle is 40.0° and the glass is 1.00 cm thick.



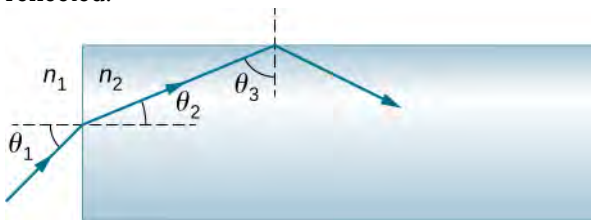
80. Considering the previous problem, show that θ_3 is the same as it would be if the second medium were not present.
81. At what angle is light inside crown glass completely polarized when reflected from water, as in a fish tank?
82. Light reflected at 55.6° from a window is completely polarized. What is the window's index of refraction and the likely substance of which it is made?
83. (a) Light reflected at 62.5° from a gemstone in a ring is completely polarized. Can the gem be a diamond? (b) At what angle would the light be completely polarized if the gem was in water?
84. If θ_b is Brewster's angle for light reflected from the top of an interface between two substances, and θ'_b is Brewster's angle for light reflected from below, prove that $\theta_b + \theta'_b = 90.0^\circ$.
85. **Unreasonable results** Suppose light travels from water to another substance, with an angle of incidence of 10.0° and an angle of refraction of 14.9° . (a) What is the index of refraction of the other substance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?
86. **Unreasonable results** Light traveling from water to a gemstone strikes the surface at an angle of 80.0° and has an angle of refraction of 15.2° . (a) What is the speed of light in the gemstone? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?
87. If a polarizing filter reduces the intensity of polarized light to 50.0% of its original value, by how much are the electric and magnetic fields reduced?
88. Suppose you put on two pairs of polarizing sunglasses with their axes at an angle of 15.0° . How much longer will it take the light to deposit a given amount of energy in your eye compared with a single pair of sunglasses? Assume the lenses are clear except for their polarizing characteristics.
89. (a) On a day when the intensity of sunlight is 1.00 kW/m^2 , a circular lens 0.200 m in diameter focuses light onto water in a black beaker. Two polarizing sheets of plastic are placed in front of the lens with their axes at an angle of 20.0° . Assuming the sunlight is unpolarized and the polarizers are 100% efficient, what is the initial rate of heating of the water in $^\circ\text{C/s}$, assuming it is 80.0% absorbed? The aluminum beaker has a mass of 30.0 grams and contains 250 grams of water. (b) Do the polarizing filters get hot? Explain.

Challenge Problems

90. Light shows staged with lasers use moving mirrors to swing beams and create colorful effects. Show that a light ray reflected from a mirror changes direction by 2θ when the mirror is rotated by an angle θ .

- 91.** Consider sunlight entering Earth's atmosphere at sunrise and sunset—that is, at a 90.0° incident angle. Taking the boundary between nearly empty space and the atmosphere to be sudden, calculate the angle of refraction for sunlight. This lengthens the time the Sun appears to be above the horizon, both at sunrise and sunset. Now construct a problem in which you determine the angle of refraction for different models of the atmosphere, such as various layers of varying density. Your instructor may wish to guide you on the level of complexity to consider and on how the index of refraction varies with air density.

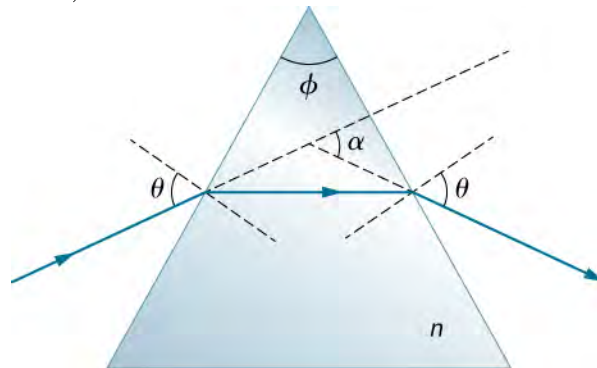
- 92.** A light ray entering an optical fiber surrounded by air is first refracted and then reflected as shown below. Show that if the fiber is made from crown glass, any incident ray will be totally internally reflected.



- 93.** A light ray falls on the left face of a prism (see below) at the angle of incidence θ for which the emerging beam has an angle of refraction θ at the right face. Show that the index of refraction n of the glass prism is given by

$$n = \frac{\sin \frac{1}{2}(\alpha + \phi)}{\sin \frac{1}{2}\phi}$$

where ϕ is the vertex angle of the prism and α is the angle through which the beam has been deviated. If $\alpha = 37.0^\circ$ and the base angles of the prism are each 50.0° , what is n ?



- 94.** If the apex angle ϕ in the previous problem is 20.0° and $n = 1.50$, what is the value of α ?
- 95.** The light incident on polarizing sheet P_1 is linearly polarized at an angle of 30.0° with respect to the transmission axis of P_1 . Sheet P_2 is placed so that its axis is parallel to the polarization axis of the incident light, that is, also at 30.0° with respect to P_1 . (a) What fraction of the incident light passes through P_1 ? (b) What fraction of the incident light is passed by the combination? (c) By rotating P_2 , a maximum in transmitted intensity is obtained. What is the ratio of this maximum intensity to the intensity of transmitted light when P_2 is at 30.0° with respect to P_1 ?
- 96.** Prove that if I is the intensity of light transmitted by two polarizing filters with axes at an angle θ and I' is the intensity when the axes are at an angle $90.0^\circ - \theta$, then $I + I' = I_0$, the original intensity. (*Hint:* Use the trigonometric identities $\cos 90.0^\circ - \theta = \sin \theta$ and $\cos^2 \theta + \sin^2 \theta = 1$.)

CHAPTER 3

Interference

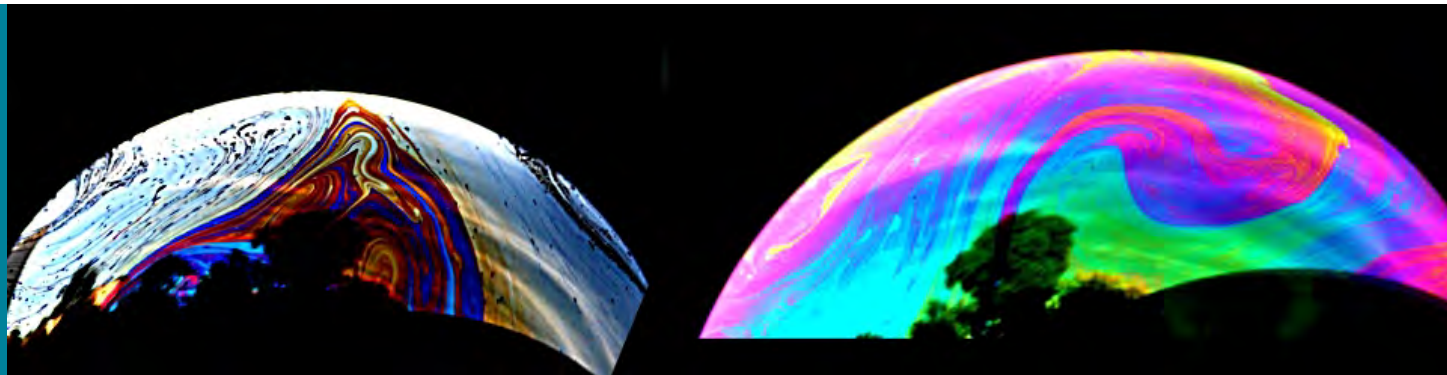


Figure 3.1 Soap bubbles are blown from clear fluid into very thin films. The colors we see are not due to any pigmentation but are the result of light interference, which enhances specific wavelengths for a given thickness of the film.

Chapter Outline

[3.1 Young's Double-Slit Interference](#)

[3.2 Mathematics of Interference](#)

[3.3 Multiple-Slit Interference](#)

[3.4 Interference in Thin Films](#)

[3.5 The Michelson Interferometer](#)

INTRODUCTION The most certain indication of a wave is interference. This wave characteristic is most prominent when the wave interacts with an object that is not large compared with the wavelength. Interference is observed for water waves, sound waves, light waves, and, in fact, all types of waves.

If you have ever looked at the reds, blues, and greens in a sunlit soap bubble and wondered how straw-colored soapy water could produce them, you have hit upon one of the many phenomena that can only be explained by the wave character of light (see [Figure 3.1](#)). The same is true for the colors seen in an oil slick or in the light reflected from a DVD disc. These and other interesting phenomena cannot be explained fully by geometric optics. In these cases, light interacts with objects and exhibits wave characteristics. The branch of optics that considers the behavior of light when it exhibits wave characteristics is called wave optics (sometimes called physical optics). It is the topic of this chapter.

3.1 Young's Double-Slit Interference

Learning Objectives

By the end of this section, you will be able to:

- Explain the phenomenon of interference
- Define constructive and destructive interference for a double slit

The Dutch physicist Christiaan Huygens (1629–1695) thought that light was a wave, but Isaac Newton did not.

Newton thought that there were other explanations for color, and for the interference and diffraction effects that were observable at the time. Owing to Newton's tremendous reputation, his view generally prevailed; the fact that Huygens's principle worked was not considered direct evidence proving that light is a wave. The acceptance of the wave character of light came many years later in 1801, when the English physicist and physician Thomas Young (1773–1829) demonstrated optical interference with his now-classic double-slit experiment.

If there were not one but two sources of waves, the waves could be made to interfere, as in the case of waves on water ([Figure 3.2](#)). If light is an electromagnetic wave, it must therefore exhibit interference effects under appropriate circumstances. In Young's experiment, sunlight was passed through a pinhole on a board. The emerging beam fell on two pinholes on a second board. The light emanating from the two pinholes then fell on a screen where a pattern of bright and dark spots was observed. This pattern, called fringes, can only be explained through interference, a wave phenomenon.

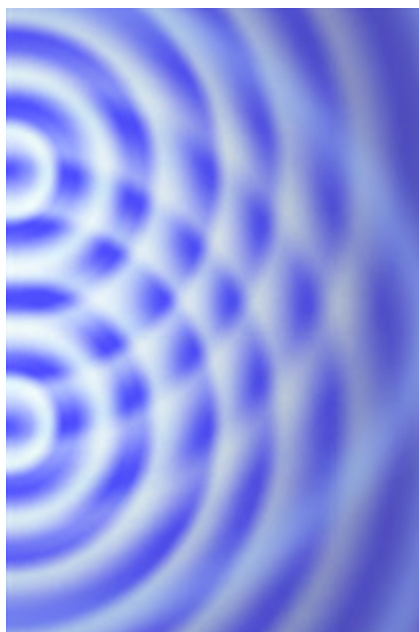


Figure 3.2 Photograph of an interference pattern produced by circular water waves in a ripple tank. Two thin plungers are vibrated up and down in phase at the surface of the water. Circular water waves are produced by and emanate from each plunger.

We can analyze double-slit interference with the help of [Figure 3.3](#), which depicts an apparatus analogous to Young's. Light from a monochromatic source falls on a slit S_0 . The light emanating from S_0 is incident on two other slits S_1 and S_2 that are equidistant from S_0 . A pattern of *interference fringes* on the screen is then produced by the light emanating from S_1 and S_2 . All slits are assumed to be so narrow that they can be considered secondary point sources for Huygens' wavelets ([The Nature of Light](#)). Slits S_1 and S_2 are a distance d apart ($d \leq 1$ mm), and the distance between the screen and the slits is D (≈ 1 m), which is much greater than d .

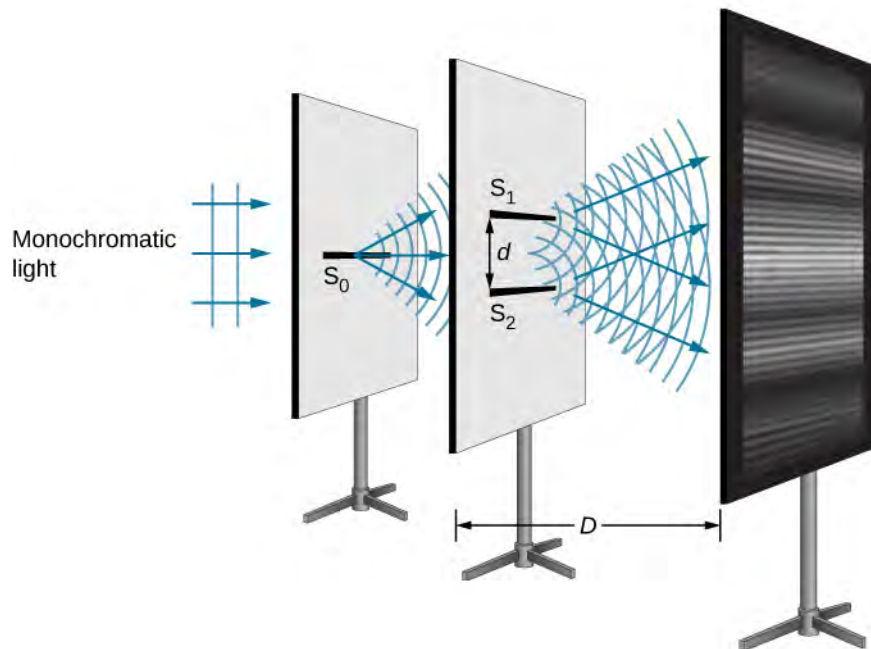


Figure 3.3 The double-slit interference experiment using monochromatic light and narrow slits. Fringes produced by interfering Huygens wavelets from slits S_1 and S_2 are observed on the screen.

Since S_0 is assumed to be a point source of monochromatic light, the secondary Huygens wavelets leaving S_1 and S_2 always maintain a constant phase difference (zero in this case because S_1 and S_2 are equidistant from S_0) and have the same frequency. The sources S_1 and S_2 are then said to be **coherent**. By **coherent waves**, we mean the waves are in phase or have a definite phase relationship. The term **incoherent** means the waves have random phase relationships, which would be the case if S_1 and S_2 were illuminated by two independent light sources, rather than a single source S_0 . Two independent light sources (which may be two separate areas within the same lamp or the Sun) would generally not emit their light in unison, that is, not coherently. Also, because S_1 and S_2 are the same distance from S_0 , the amplitudes of the two Huygens wavelets are equal.

Young used sunlight, where each wavelength forms its own pattern, making the effect more difficult to see. In the following discussion, we illustrate the double-slit experiment with **monochromatic** light (single λ) to clarify the effect. [Figure 3.4](#) shows the pure constructive and destructive interference of two waves having the same wavelength and amplitude.

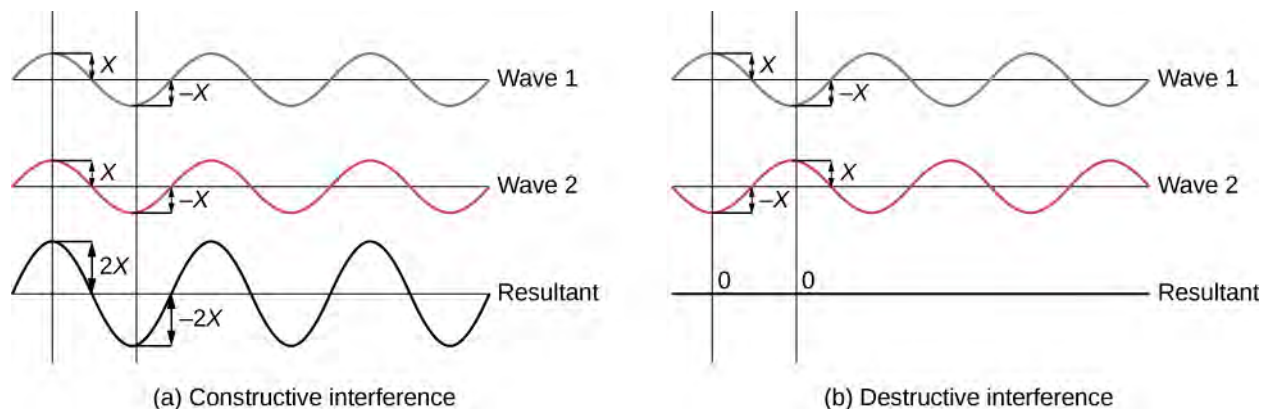


Figure 3.4 The amplitudes of waves add. (a) Pure constructive interference is obtained when identical waves are in phase. (b) Pure destructive interference occurs when identical waves are exactly out of phase, or shifted by half a wavelength.

When light passes through narrow slits, the slits act as sources of coherent waves and light spreads out as

semicircular waves, as shown in [Figure 3.5\(a\)](#). Pure *constructive interference* occurs where the waves are crest to crest or trough to trough. Pure *destructive interference* occurs where they are crest to trough. The light must fall on a screen and be scattered into our eyes for us to see the pattern. An analogous pattern for water waves is shown in [Figure 3.2](#). Note that regions of constructive and destructive interference move out from the slits at well-defined angles to the original beam. These angles depend on wavelength and the distance between the slits, as we shall see below.

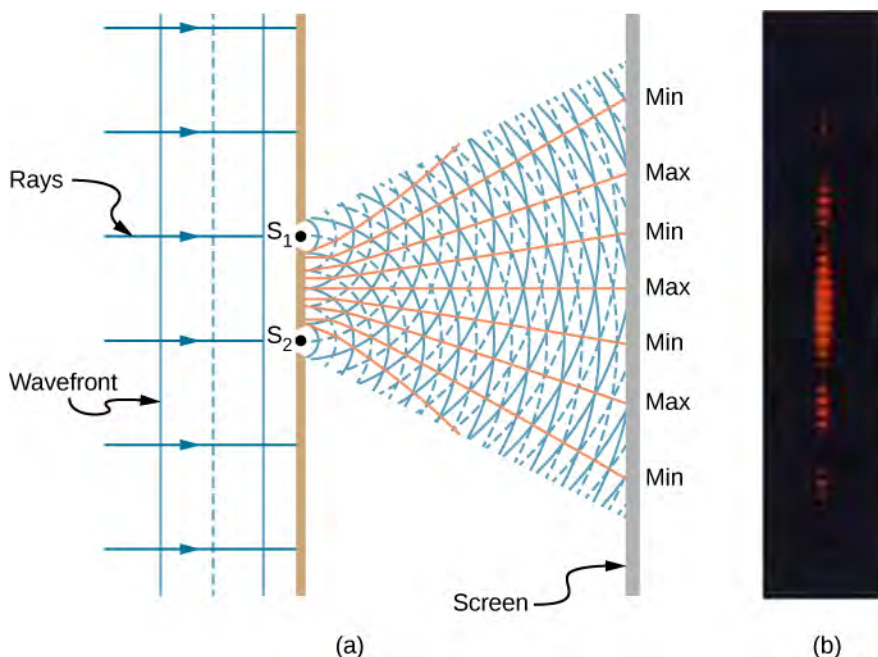


Figure 3.5 Double slits produce two coherent sources of waves that interfere. (a) Light spreads out (diffracts) from each slit, because the slits are narrow. These waves overlap and interfere constructively (bright lines) and destructively (dark regions). We can only see this if the light falls onto a screen and is scattered into our eyes. (b) When light that has passed through double slits falls on a screen, we see a pattern such as this.

To understand the double-slit interference pattern, consider how two waves travel from the slits to the screen ([Figure 3.6](#)). Each slit is a different distance from a given point on the screen. Thus, different numbers of wavelengths fit into each path. Waves start out from the slits in phase (crest to crest), but they may end up out of phase (crest to trough) at the screen if the paths differ in length by half a wavelength, interfering destructively. If the paths differ by a whole wavelength, then the waves arrive in phase (crest to crest) at the screen, interfering constructively. More generally, if the path length difference Δl between the two waves is any half-integral number of wavelengths $[(1/2)\lambda, (3/2)\lambda, (5/2)\lambda, \text{etc.}]$, then destructive interference occurs. Similarly, if the path length difference is any integral number of wavelengths $(\lambda, 2\lambda, 3\lambda, \text{etc.})$, then constructive interference occurs. These conditions can be expressed as equations:

$$\Delta l = m\lambda, \quad \text{for } m = 0, \pm 1, \pm 2, \pm 3 \dots \text{ (constructive interference)} \quad 3.1$$

$$\Delta l = (m + \frac{1}{2})\lambda, \quad \text{for } m = 0, \pm 1, \pm 2, \pm 3 \dots \text{ (destructive interference)} \quad 3.2$$

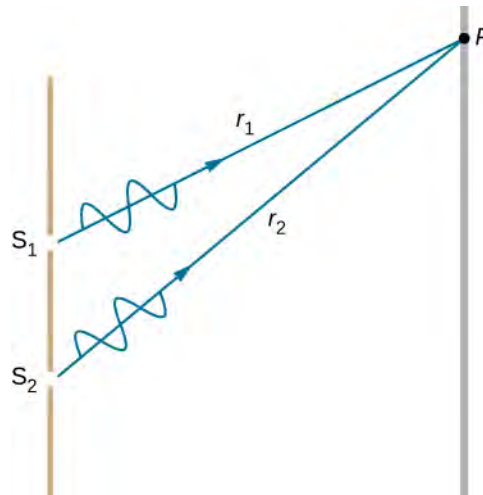


Figure 3.6 Waves follow different paths from the slits to a common point P on a screen. Destructive interference occurs where one path is a half wavelength longer than the other—the waves start in phase but arrive out of phase. Constructive interference occurs where one path is a whole wavelength longer than the other—the waves start out and arrive in phase.

3.2 Mathematics of Interference

Learning Objectives

By the end of this section, you will be able to:

- Determine the angles for bright and dark fringes for double slit interference
- Calculate the positions of bright fringes on a screen

[Figure 3.7\(a\)](#) shows how to determine the path length difference Δl for waves traveling from two slits to a common point on a screen. If the screen is a large distance away compared with the distance between the slits, then the angle θ between the path and a line from the slits to the screen [part (b)] is nearly the same for each path. In other words, r_1 and r_2 are essentially parallel. The lengths of r_1 and r_2 differ by Δl , as indicated by the two dashed lines in the figure. Simple trigonometry shows

$$\Delta l = d \sin \theta \quad 3.3$$

where d is the distance between the slits. Combining this result with [Equation 3.1](#), we obtain constructive interference for a double slit when the path length difference is an integral multiple of the wavelength, or

$$d \sin \theta = m\lambda, \text{ for } m = 0, \pm 1, \pm 2, \pm 3, \dots \text{ (constructive interference).} \quad 3.4$$

Similarly, to obtain destructive interference for a double slit, the path length difference must be a half-integral multiple of the wavelength, or

$$d \sin \theta = (m + \frac{1}{2})\lambda, \text{ for } m = 0, \pm 1, \pm 2, \pm 3, \dots \text{ (destructive interference)} \quad 3.5$$

where λ is the wavelength of the light, d is the distance between slits, and θ is the angle from the original direction of the beam as discussed above. We call m the **order** of the interference. For example, $m = 4$ is fourth-order interference.

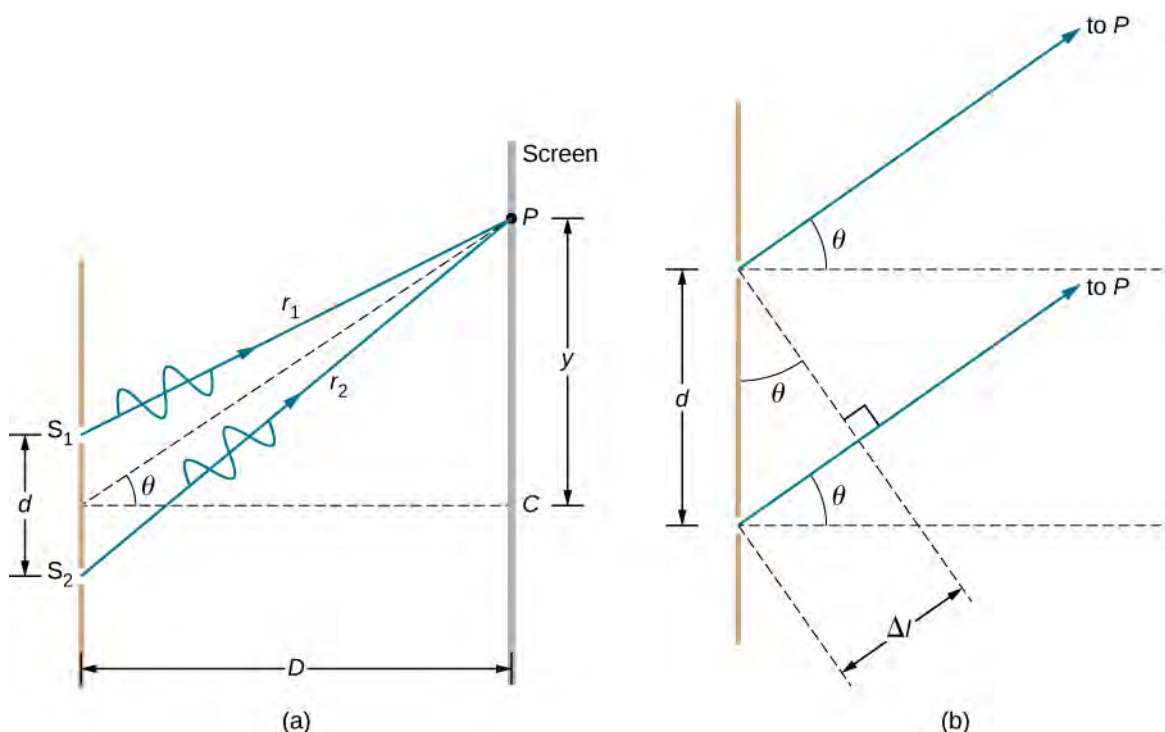


Figure 3.7 (a) To reach P , the light waves from S_1 and S_2 must travel different distances. (b) The path difference between the two rays is Δl .

The equations for double-slit interference imply that a series of bright and dark lines are formed. For vertical slits, the light spreads out horizontally on either side of the incident beam into a pattern called **interference fringes** (Figure 3.8). The closer the slits are, the more the bright fringes spread apart. We can see this by examining the equation

$d \sin \theta = m\lambda$, for $m = 0, \pm 1, \pm 2, \pm 3, \dots$. For fixed λ and m , the smaller d is, the larger θ must be, since $\sin \theta = m\lambda/d$. This is consistent with our contention that wave effects are most noticeable when the object the wave encounters (here, slits a distance d apart) is small. Small d gives large θ , hence, a large effect.

Referring back to part (a) of the figure, θ is typically small enough that $\sin \theta \approx \tan \theta \approx y_m/D$, where y_m is the distance from the central maximum to the m th bright fringe and D is the distance between the slit and the screen. Equation 3.4 may then be written as

$$d \frac{y_m}{D} = m\lambda$$

or

$$y_m = \frac{m\lambda D}{d}.$$

3.6

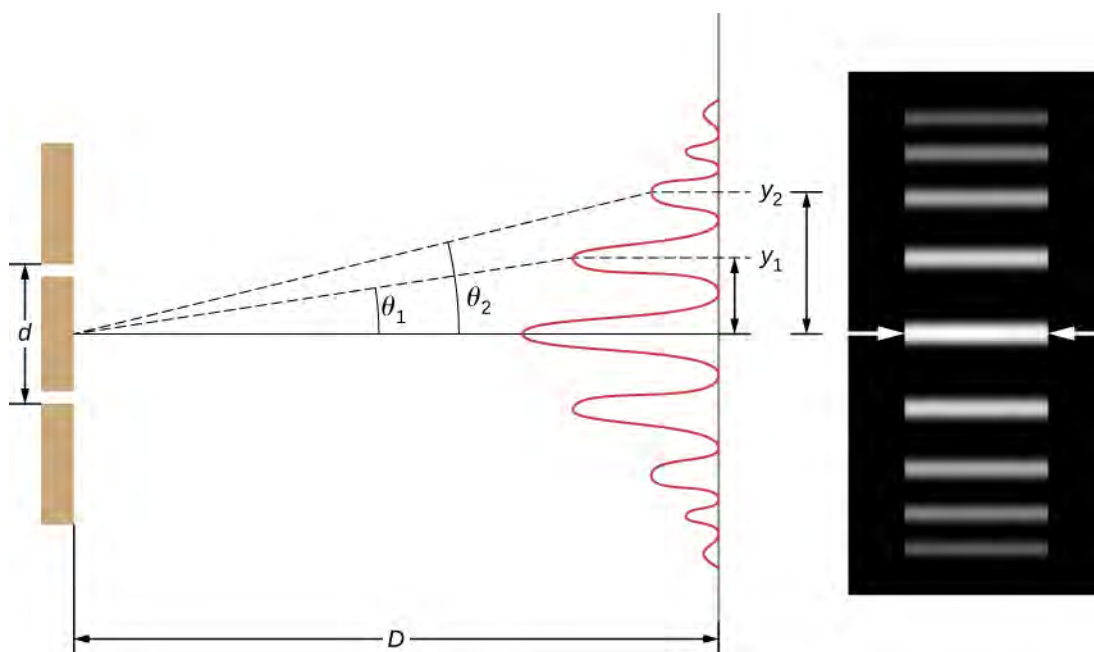


Figure 3.8 The interference pattern for a double slit has an intensity that falls off with angle. The image shows multiple bright and dark lines, or fringes, formed by light passing through a double slit.

EXAMPLE 3.1

Finding a Wavelength from an Interference Pattern

Suppose you pass light from a He-Ne laser through two slits separated by 0.0100 mm and find that the third bright line on a screen is formed at an angle of 10.95° relative to the incident beam. What is the wavelength of the light?

Strategy

The phenomenon is two-slit interference as illustrated in [Figure 3.8](#) and the third bright line is due to third-order constructive interference, which means that $m = 3$. We are given $d = 0.0100$ mm and $\theta = 10.95^\circ$. The wavelength can thus be found using the equation $d \sin \theta = m\lambda$ for constructive interference.

Solution

Solving $d \sin \theta = m\lambda$ for the wavelength λ gives

$$\lambda = \frac{d \sin \theta}{m}.$$

Substituting known values yields

$$\lambda = \frac{(0.0100 \text{ mm})(\sin 10.95^\circ)}{3} = 6.33 \times 10^{-4} \text{ mm} = 633 \text{ nm}.$$

Significance

To three digits, this is the wavelength of light emitted by the common He-Ne laser. Not by coincidence, this red color is similar to that emitted by neon lights. More important, however, is the fact that interference patterns can be used to measure wavelength. Young did this for visible wavelengths. This analytical technique is still widely used to measure electromagnetic spectra. For a given order, the angle for constructive interference increases with λ , so that spectra (measurements of intensity versus wavelength) can be obtained.

EXAMPLE 3.2

Calculating the Highest Order Possible

Interference patterns do not have an infinite number of lines, since there is a limit to how big m can be. What is the highest-order constructive interference possible with the system described in the preceding example?

Strategy

The equation $d \sin \theta = m\lambda$ (for $m = 0, \pm 1, \pm 2, \pm 3 \dots$) describes constructive interference from two slits. For fixed values of d and λ , the larger m is, the larger $\sin \theta$ is. However, the maximum value that $\sin \theta$ can have is 1, for an angle of 90° . (Larger angles imply that light goes backward and does not reach the screen at all.) Let us find what value of m corresponds to this maximum diffraction angle.

Solution

Solving the equation $d \sin \theta = m\lambda$ for m gives

$$m = \frac{d \sin \theta}{\lambda}.$$

Taking $\sin \theta = 1$ and substituting the values of d and λ from the preceding example gives

$$m = \frac{(0.0100 \text{ mm})(1)}{633 \text{ nm}} \approx 15.8.$$

Therefore, the largest integer m can be is 15, or $m = 15$.

Significance

The number of fringes depends on the wavelength and slit separation. The number of fringes is very large for large slit separations. However, recall (see [The Propagation of Light](#) and the introduction for this chapter) that wave interference is only prominent when the wave interacts with objects that are not large compared to the wavelength. Therefore, if the slit separation and the sizes of the slits become much greater than the wavelength, the intensity pattern of light on the screen changes, so there are simply two bright lines cast by the slits, as expected, when light behaves like rays. We also note that the fringes get fainter farther away from the center. Consequently, not all 15 fringes may be observable.

CHECK YOUR UNDERSTANDING 3.1

In the system used in the preceding examples, at what angles are the first and the second bright fringes formed?

3.3 Multiple-Slit Interference

Learning Objectives

By the end of this section, you will be able to:

- Describe the locations and intensities of secondary maxima for multiple-slit interference

Analyzing the interference of light passing through two slits lays out the theoretical framework of interference and gives us a historical insight into Thomas Young's experiments. However, much of the modern-day application of slit interference uses not just two slits but many, approaching infinity for practical purposes. The key optical element is called a diffraction grating, an important tool in optical analysis, which we discuss in detail in [Diffraction](#). Here, we start the analysis of multiple-slit interference by taking the results from our analysis of the double slit ($N = 2$) and extending it to configurations with three, four, and much larger numbers of slits.

[Figure 3.9](#) shows the simplest case of multiple-slit interference, with three slits, or $N = 3$. The spacing between slits is d , and the path length difference between adjacent slits is $d \sin \theta$, same as the case for the double slit. What is new is that the path length difference for the first and the third slits is $2d \sin \theta$. The

condition for constructive interference is the same as for the double slit, that is

$$d \sin \theta = m\lambda.$$

When this condition is met, $2d \sin \theta$ is automatically a multiple of λ , so all three rays combine constructively, and the bright fringes that occur here are called **principal maxima**. But what happens when the path length difference between adjacent slits is only $\lambda/2$? We can think of the first and second rays as interfering destructively, but the third ray remains unaltered. Instead of obtaining a dark fringe, or a minimum, as we did for the double slit, we see a **secondary maximum** with intensity lower than the principal maxima.

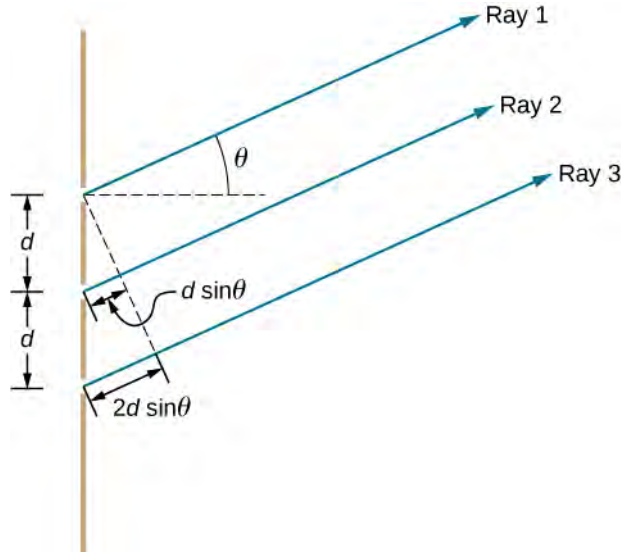


Figure 3.9 Interference with three slits. Different pairs of emerging rays can combine constructively or destructively at the same time, leading to secondary maxima.

In general, for N slits, these secondary maxima occur whenever an unpaired ray is present that does not go away due to destructive interference. This occurs at $(N - 2)$ evenly spaced positions between the principal maxima. The amplitude of the electromagnetic wave is correspondingly diminished to $1/N$ of the wave at the principal maxima, and the light intensity, being proportional to the square of the wave amplitude, is diminished to $1/N^2$ of the intensity compared to the principal maxima. As [Figure 3.10](#) shows, a dark fringe is located between every maximum (principal or secondary). As N grows larger and the number of bright and dark fringes increase, the widths of the maxima become narrower due to the closely located neighboring dark fringes. Because the total amount of light energy remains unaltered, narrower maxima require that each maximum reaches a correspondingly higher intensity.

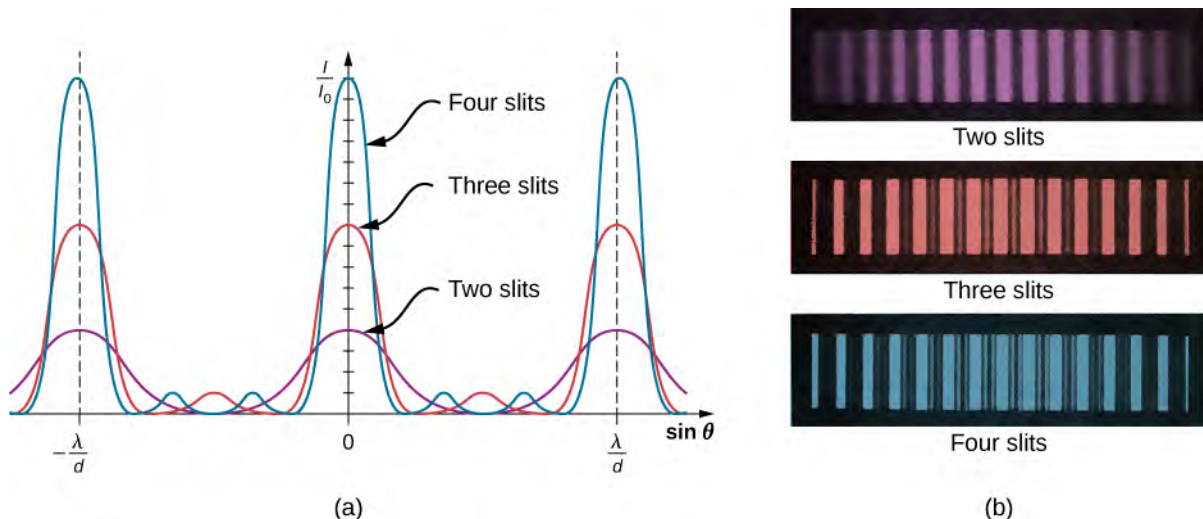


Figure 3.10 Interference fringe patterns for two, three and four slits. As the number of slits increases, more secondary maxima appear,

but the principal maxima become brighter and narrower. (a) Graph and (b) photographs of fringe patterns.

3.4 Interference in Thin Films

Learning Objectives

By the end of this section, you will be able to:

- Describe the phase changes that occur upon reflection
- Describe fringes established by reflected rays of a common source
- Explain the appearance of colors in thin films

The bright colors seen in an oil slick floating on water or in a sunlit soap bubble are caused by interference. The brightest colors are those that interfere constructively. This interference is between light reflected from different surfaces of a thin film; thus, the effect is known as **thin-film interference**.

As we noted before, interference effects are most prominent when light interacts with something having a size similar to its wavelength. A thin film is one having a thickness t smaller than a few times the wavelength of light, λ . Since color is associated indirectly with λ and because all interference depends in some way on the ratio of λ to the size of the object involved, we should expect to see different colors for different thicknesses of a film, as in [Figure 3.11](#).

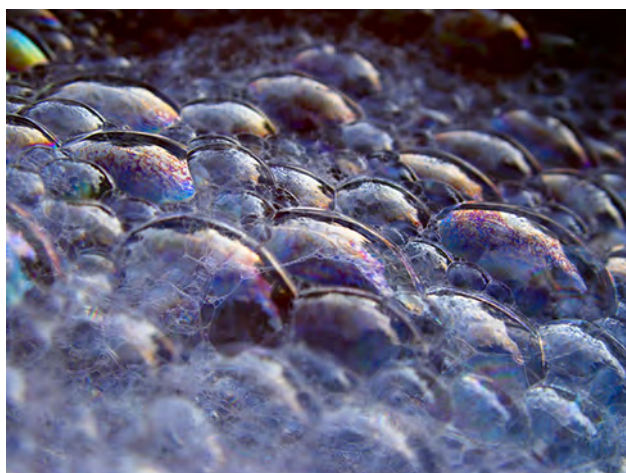


Figure 3.11 These soap bubbles exhibit brilliant colors when exposed to sunlight. (credit: Scott Robinson)

What causes thin-film interference? [Figure 3.12](#) shows how light reflected from the top and bottom surfaces of a film can interfere. Incident light is only partially reflected from the top surface of the film (ray 1). The remainder enters the film and is itself partially reflected from the bottom surface. Part of the light reflected from the bottom surface can emerge from the top of the film (ray 2) and interfere with light reflected from the top (ray 1). The ray that enters the film travels a greater distance, so it may be in or out of phase with the ray reflected from the top. However, consider for a moment, again, the bubbles in [Figure 3.11](#). The bubbles are darkest where they are thinnest. Furthermore, if you observe a soap bubble carefully, you will note it gets dark at the point where it breaks. For very thin films, the difference in path lengths of rays 1 and 2 in [Figure 3.12](#) is negligible, so why should they interfere destructively and not constructively? The answer is that a phase change can occur upon reflection, as discussed next.

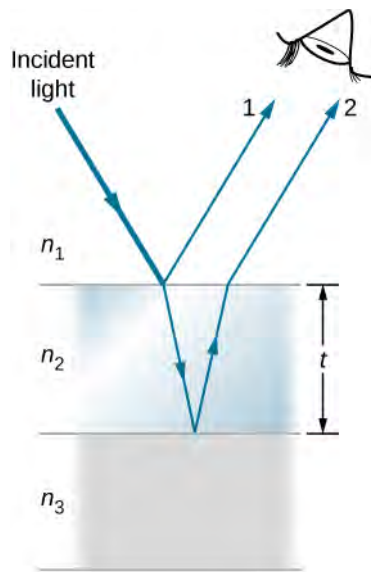


Figure 3.12 Light striking a thin film is partially reflected (ray 1) and partially refracted at the top surface. The refracted ray is partially reflected at the bottom surface and emerges as ray 2. These rays interfere in a way that depends on the thickness of the film and the indices of refraction of the various media.

Changes in Phase due to Reflection

We saw earlier ([Waves](#)) that reflection of mechanical waves can involve a 180° phase change. For example, a traveling wave on a string is inverted (i.e., a 180° phase change) upon reflection at a boundary to which a heavier string is tied. However, if the second string is lighter (or more precisely, of a lower linear density), no inversion occurs. Light waves produce the same effect, but the deciding parameter for light is the index of refraction. Light waves undergo a 180° or π radians phase change upon reflection at an interface beyond which is a medium of higher index of refraction. No phase change takes place when reflecting from a medium of lower refractive index ([Figure 3.13](#)). Because of the periodic nature of waves, this phase change or inversion is equivalent to $\pm\lambda/2$ in distance travelled, or path length. Both the path length and refractive indices are important factors in thin-film interference.

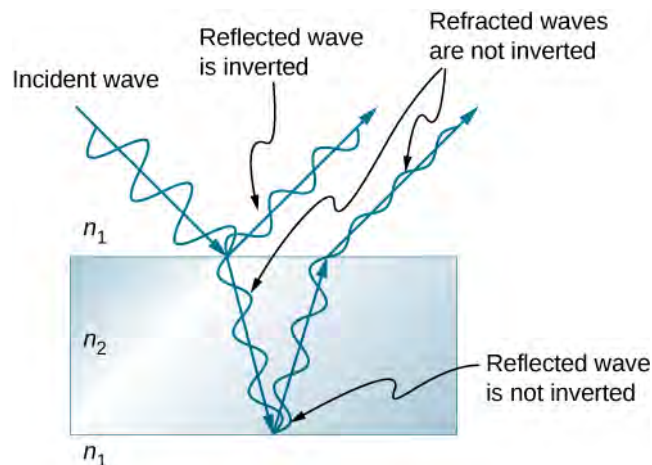


Figure 3.13 Reflection at an interface for light traveling from a medium with index of refraction n_1 to a medium with index of refraction n_2 , $n_1 < n_2$, causes the phase of the wave to change by π radians.

If the film in [Figure 3.12](#) is a **soap bubble** (essentially water with air on both sides), then a phase shift of $\lambda/2$ occurs for ray 1 but not for ray 2. Thus, when the film is very thin and the path length difference between the two rays is negligible, they are exactly out of phase, and destructive interference occurs at all wavelengths. Thus, the soap bubble is dark here. The thickness of the film relative to the wavelength of light is the other

crucial factor in thin-film interference. Ray 2 in [Figure 3.12](#) travels a greater distance than ray 1. For light incident perpendicular to the surface, ray 2 travels a distance approximately $2t$ farther than ray 1. When this distance is an integral or half-integral multiple of the wavelength in the medium ($\lambda_n = \lambda/n$, where λ is the wavelength in vacuum and n is the index of refraction), constructive or destructive interference occurs, depending also on whether there is a phase change in either ray.



EXAMPLE 3.3

Calculating the Thickness of a Nonreflective Lens Coating

Sophisticated cameras use a series of several lenses. Light can reflect from the surfaces of these various lenses and degrade image clarity. To limit these reflections, lenses are coated with a thin layer of magnesium fluoride, which causes destructive thin-film interference. What is the thinnest this film can be, if its index of refraction is 1.38 and it is designed to limit the reflection of 550-nm light, normally the most intense visible wavelength? Assume the index of refraction of the glass is 1.52.

Strategy

Refer to [Figure 3.12](#) and use $n_1 = 1.00$ for air, $n_2 = 1.38$, and $n_3 = 1.52$. Both ray 1 and ray 2 have a $\lambda/2$ shift upon reflection. Thus, to obtain destructive interference, ray 2 needs to travel a half wavelength farther than ray 1. For rays incident perpendicularly, the path length difference is $2t$.

Solution

To obtain destructive interference here,

$$2t = \frac{\lambda_{n2}}{2}$$

where λ_{n2} is the wavelength in the film and is given by $\lambda_{n2} = \lambda/n_2$. Thus,

$$2t = \frac{\lambda/n_2}{2}.$$

Solving for t and entering known values yields

$$t = \frac{\lambda/n_2}{4} = \frac{(500 \text{ nm})/1.38}{4} = 90.6 \text{ nm}.$$

Significance

Films such as the one in this example are most effective in producing destructive interference when the thinnest layer is used, since light over a broader range of incident angles is reduced in intensity. These films are called nonreflective coatings; this is only an approximately correct description, though, since other wavelengths are only partially cancelled. Nonreflective coatings are also used in car windows and sunglasses.

Combining Path Length Difference with Phase Change

Thin-film interference is most constructive or most destructive when the path length difference for the two rays is an integral or half-integral wavelength. That is, for rays incident perpendicularly,

$$2t = \lambda_n, 2\lambda_n, 3\lambda_n, \dots \text{ or } 2t = \lambda_n/2, 3\lambda_n/2, 5\lambda_n/2, \dots$$

To know whether interference is constructive or destructive, you must also determine if there is a phase change upon reflection. Thin-film interference thus depends on film thickness, the wavelength of light, and the refractive indices. For white light incident on a film that varies in thickness, you can observe rainbow colors of constructive interference for various wavelengths as the thickness varies.



EXAMPLE 3.4

Soap Bubbles

(a) What are the three smallest thicknesses of a soap bubble that produce constructive interference for red light with a wavelength of 650 nm? The index of refraction of soap is taken to be the same as that of water. (b) What three smallest thicknesses give destructive interference?

Strategy

Use [Figure 3.12](#) to visualize the bubble, which acts as a thin film between two layers of air. Thus $n_1 = n_3 = 1.00$ for air, and $n_2 = 1.333$ for soap (equivalent to water). There is a $\lambda/2$ shift for ray 1 reflected from the top surface of the bubble and no shift for ray 2 reflected from the bottom surface. To get constructive interference, then, the path length difference ($2t$) must be a half-integral multiple of the wavelength—the first three being $\lambda_n/2$, $3\lambda_n/2$, and $5\lambda_n/2$. To get destructive interference, the path length difference must be an integral multiple of the wavelength—the first three being 0, λ_n , and $2\lambda_n$.

Solution

a. Constructive interference occurs here when

$$2t_c = \frac{\lambda_n}{2}, \frac{3\lambda_n}{2}, \frac{5\lambda_n}{2}, \dots$$

Thus, the smallest constructive thickness t_c is

$$t_c = \frac{\lambda_n}{4} = \frac{\lambda/n}{4} = \frac{(650 \text{ nm})/1.333}{4} = 122 \text{ nm}.$$

The next thickness that gives constructive interference is $t'_c = 3\lambda_n/4$, so that

$$t'_c = 366 \text{ nm}.$$

Finally, the third thickness producing constructive interference is $t''_c = 5\lambda_n/4$, so that

$$t''_c = 610 \text{ nm}.$$

b. For destructive interference, the path length difference here is an integral multiple of the wavelength. The first occurs for zero thickness, since there is a phase change at the top surface, that is,

$$t_d = 0,$$

the very thin (or negligibly thin) case discussed above. The first non-zero thickness producing destructive interference is

$$2t'_d = \lambda_n.$$

Substituting known values gives

$$t'_d = \frac{\lambda}{2} = \frac{\lambda/n}{2} = \frac{(650 \text{ nm})/1.333}{2} = 244 \text{ nm}.$$

Finally, the third destructive thickness is $2t''_d = 2\lambda_n$, so that

$$t''_d = \lambda_n = \frac{\lambda}{n} = \frac{650 \text{ nm}}{1.333} = 488 \text{ nm}.$$

Significance

If the bubble were illuminated with pure red light, we would see bright and dark bands at very uniform increases in thickness. First would be a dark band at 0 thickness, then bright at 122 nm thickness, then dark at 244 nm, bright at 366 nm, dark at 488 nm, and bright at 610 nm. If the bubble varied smoothly in thickness, like a smooth wedge, then the bands would be evenly spaced.



CHECK YOUR UNDERSTANDING 3.2

Going further with [Example 3.4](#), what are the next two thicknesses of soap bubble that would lead to (a) constructive interference, and (b) destructive interference?

Another example of thin-film interference can be seen when microscope slides are separated (see [Figure 3.14](#)). The slides are very flat, so that the wedge of air between them increases in thickness very uniformly. A phase change occurs at the second surface but not the first, so a dark band forms where the slides touch. The rainbow colors of constructive interference repeat, going from violet to red again and again as the distance between the slides increases. As the layer of air increases, the bands become more difficult to see, because slight changes in incident angle have greater effects on path length differences. If monochromatic light instead of white light is used, then bright and dark bands are obtained rather than repeating rainbow colors.

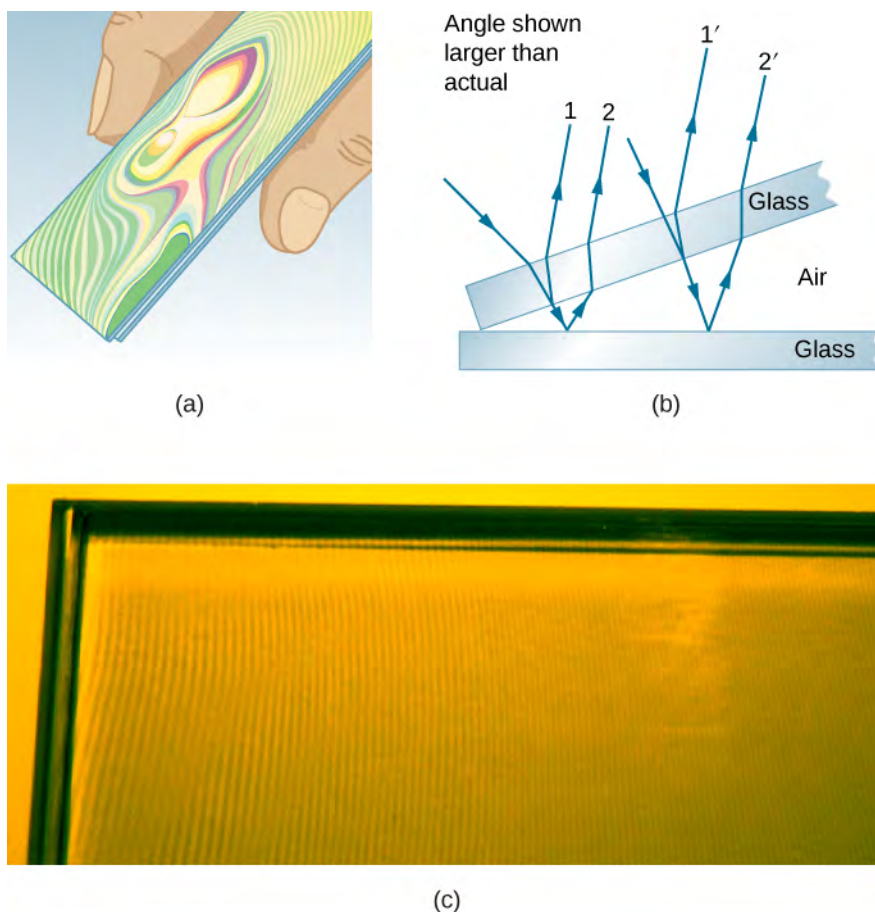


Figure 3.14 (a) The rainbow-color bands are produced by thin-film interference in the air between the two glass slides. (b) Schematic of the paths taken by rays in the wedge of air between the slides. (c) If the air wedge is illuminated with monochromatic light, bright and dark bands are obtained rather than repeating rainbow colors.

An important application of thin-film interference is found in the manufacturing of optical instruments. A lens or mirror can be compared with a master as it is being ground, allowing it to be shaped to an accuracy of less than a wavelength over its entire surface. [Figure 3.15](#) illustrates the phenomenon called **Newton's rings**, which occurs when the plane surfaces of two lenses are placed together. (The circular bands are called Newton's rings because Isaac Newton described them and their use in detail. Newton did not discover them; Robert Hooke did, and Newton did not believe they were due to the wave character of light.) Each successive ring of a given color indicates an increase of only half a wavelength in the distance between the lens and the blank, so that great precision can be obtained. Once the lens is perfect, no rings appear.

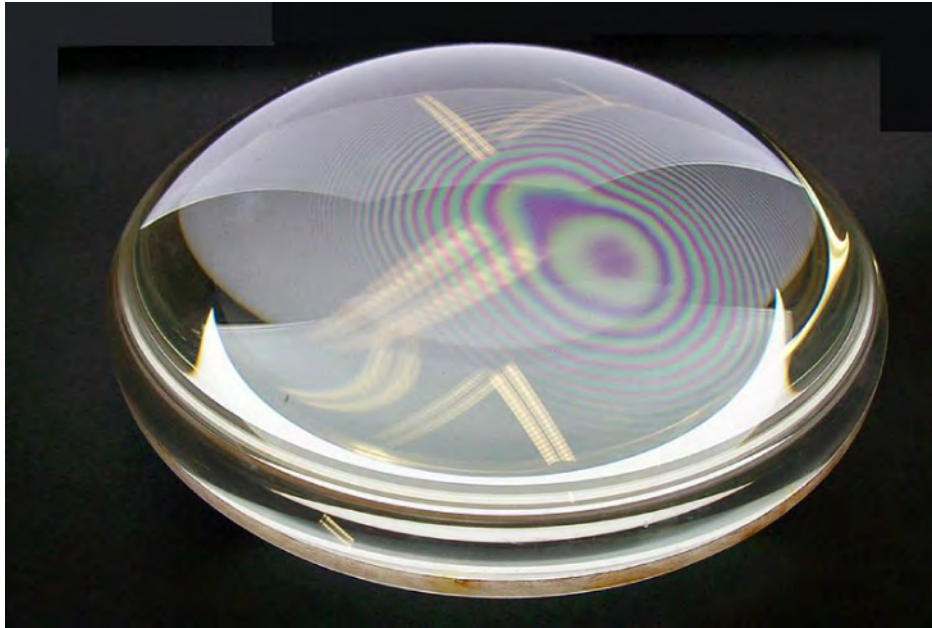


Figure 3.15 “Newton’s rings” interference fringes are produced when two plano-convex lenses are placed together with their plane surfaces in contact. The rings are created by interference between the light reflected off the two surfaces as a result of a slight gap between them, indicating that these surfaces are not precisely plane but are slightly convex. (credit: Ulf Seifert)

Thin-film interference has many other applications, both in nature and in manufacturing. The wings of certain moths and butterflies have nearly iridescent colors due to thin-film interference. In addition to pigmentation, the wing’s color is affected greatly by constructive interference of certain wavelengths reflected from its film-coated surface. Some car manufacturers offer special paint jobs that use thin-film interference to produce colors that change with angle. This expensive option is based on variation of thin-film path length differences with angle. Security features on credit cards, banknotes, driving licenses, and similar items prone to forgery use thin-film interference, diffraction gratings, or holograms. As early as 1998, Australia led the way with dollar bills printed on polymer with a diffraction grating security feature, making the currency difficult to forge. Other countries, such as Canada, New Zealand, and Taiwan, are using similar technologies, while US currency includes a thin-film interference effect.

3.5 The Michelson Interferometer

Learning Objectives

By the end of this section, you will be able to:

- Explain changes in fringes observed with a Michelson interferometer caused by mirror movements
- Explain changes in fringes observed with a Michelson interferometer caused by changes in medium

The Michelson **interferometer** (invented by the American physicist Albert A. Michelson, 1852–1931) is a precision instrument that produces interference fringes by splitting a light beam into two parts and then recombining them after they have traveled different optical paths. [Figure 3.16](#) depicts the interferometer and the path of a light beam from a single point on the extended source S , which is a ground-glass plate that diffuses the light from a monochromatic lamp of wavelength λ_0 . The beam strikes the half-silvered mirror M , where half of it is reflected to the side and half passes through the mirror. The reflected light travels to the movable plane mirror M_1 , where it is reflected back through M to the observer. The transmitted half of the original beam is reflected back by the stationary mirror M_2 and then toward the observer by M .

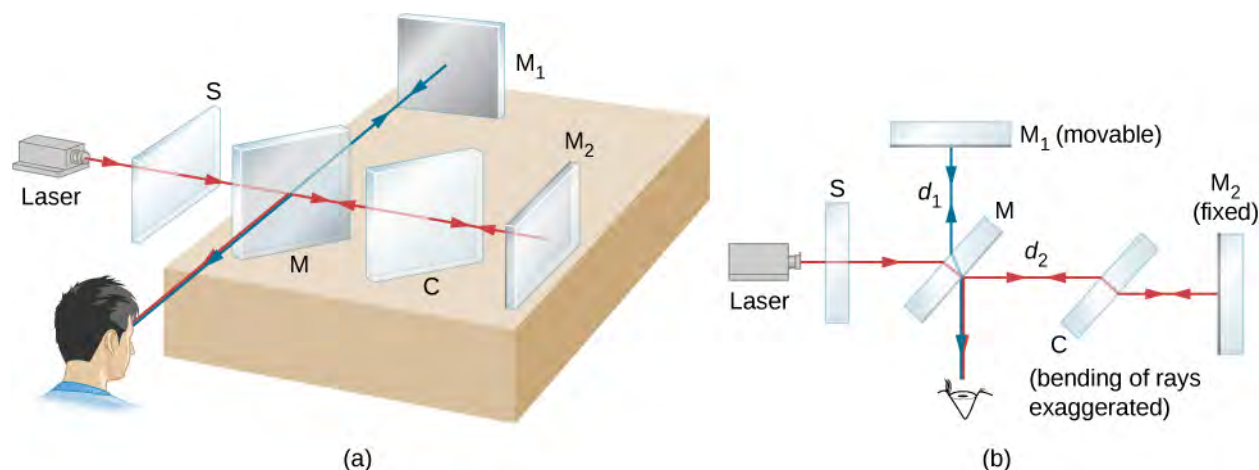


Figure 3.16 (a) The Michelson interferometer. The extended light source is a ground-glass plate that diffuses the light from a laser. (b) A planar view of the interferometer.

Because both beams originate from the same point on the source, they are coherent and therefore interfere. Notice from the figure that one beam passes through M three times and the other only once. To ensure that both beams traverse the same thickness of glass, a compensator plate C of transparent glass is placed in the arm containing M_2 . This plate is a duplicate of M (without the silvering) and is usually cut from the same piece of glass used to produce M. With the compensator in place, any phase difference between the two beams is due solely to the difference in the distances they travel.

The path difference of the two beams when they recombine is $2d_1 - 2d_2$, where d_1 is the distance between M and M_1 , and d_2 is the distance between M and M_2 . Suppose this path difference is an integer number of wavelengths $m\lambda_0$. Then, constructive interference occurs and a bright image of the point on the source is seen at the observer. Now the light from any other point on the source whose two beams have this same path difference also undergoes constructive interference and produces a bright image. The collection of these point images is a bright fringe corresponding to a path difference of $m\lambda_0$ (Figure 3.17). When M_1 is moved a distance $\Delta d = \lambda_0/2$, this path difference changes by λ_0 , and each fringe moves to the position previously occupied by an adjacent fringe. Consequently, by counting the number of fringes m passing a given point as M_1 is moved, an observer can measure minute displacements that are accurate to a fraction of a wavelength, as shown by the relation

$$\Delta d = m \frac{\lambda_0}{2}. \quad 3.7$$

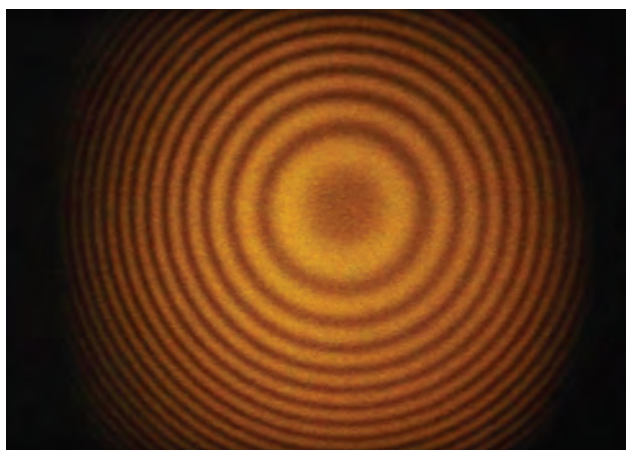


Figure 3.17 Fringes produced with a Michelson interferometer. (credit: "SILLAGESvideos"/YouTube)



EXAMPLE 3.5

Precise Distance Measurements by Michelson Interferometer

A red laser light of wavelength 630 nm is used in a Michelson interferometer. While keeping the mirror M_1 fixed, mirror M_2 is moved. The fringes are found to move past a fixed cross-hair in the viewer. Find the distance the mirror M_2 is moved for a single fringe to move past the reference line.

Strategy

Refer to [Figure 3.16](#) for the geometry. We use the result of the Michelson interferometer interference condition to find the distance moved, Δd .

Solution

For a 630-nm red laser light, and for each fringe crossing ($m = 1$), the distance traveled by M_2 if you keep M_1 fixed is

$$\Delta d = m \frac{\lambda_0}{2} = 1 \times \frac{630 \text{ nm}}{2} = 315 \text{ nm} = 0.315 \mu\text{m}.$$

Significance

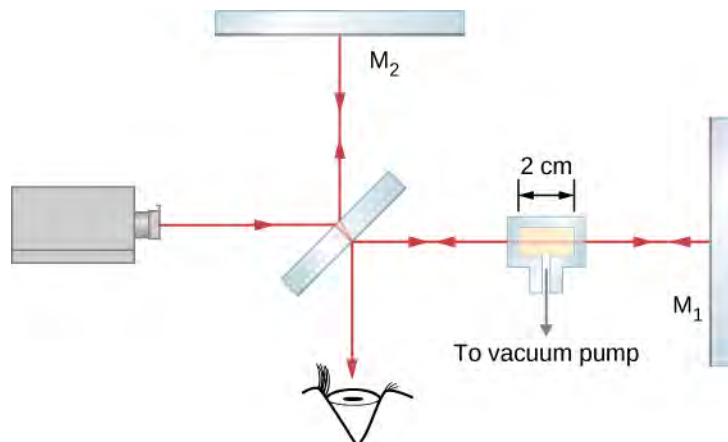
An important application of this measurement is the definition of the standard meter. As mentioned in [Units and Measurement](#), the length of the standard meter was once defined as the mirror displacement in a Michelson interferometer corresponding to 1,650,763.73 wavelengths of the particular fringe of krypton-86 in a gas discharge tube.



EXAMPLE 3.6

Measuring the Refractive Index of a Gas

In one arm of a Michelson interferometer, a glass chamber is placed with attachments for evacuating the inside and putting gases in it. The space inside the container is 2 cm wide. Initially, the container is empty. As gas is slowly let into the chamber, you observe that dark fringes move past a reference line in the field of observation. By the time the chamber is filled to the desired pressure, you have counted 122 fringes move past the reference line. The wavelength of the light used is 632.8 nm. What is the refractive index of this gas?



Strategy

The $m = 122$ fringes observed compose the difference between the number of wavelengths that fit within the empty chamber (vacuum) and the number of wavelengths that fit within the same chamber when it is gas-filled. The wavelength in the filled chamber is shorter by a factor of n , the index of refraction.

Solution

The ray travels a distance $t = 2$ cm to the right through the glass chamber and another distance t to the left upon reflection. The total travel is $L = 2t$. When empty, the number of wavelengths that fit in this chamber is

$$N_0 = \frac{L}{\lambda_0} = \frac{2t}{\lambda_0}$$

where $\lambda_0 = 632.8$ nm is the wavelength in vacuum of the light used. In any other medium, the wavelength is $\lambda = \lambda_0/n$ and the number of wavelengths that fit in the gas-filled chamber is

$$N = \frac{L}{\lambda} = \frac{2t}{\lambda_0/n}.$$

The number of fringes observed in the transition is

$$\begin{aligned} m &= N - N_0, \\ &= \frac{2t}{\lambda_0/n} - \frac{2t}{\lambda_0}, \\ &= \frac{2t}{\lambda_0}(n - 1). \end{aligned}$$

Solving for $(n - 1)$ gives

$$n - 1 = m \left(\frac{\lambda_0}{2t} \right) = 122 \left(\frac{632.8 \times 10^{-9} \text{ m}}{2(2 \times 10^{-2} \text{ m})} \right) = 0.0019$$

and $n = 1.0019$.

Significance

The indices of refraction for gases are so close to that of vacuum, that we normally consider them equal to 1. The difference between 1 and 1.0019 is so small that measuring it requires a correspondingly sensitive technique such as interferometry. We cannot, for example, hope to measure this value using techniques based simply on Snell's law.

✓ CHECK YOUR UNDERSTANDING 3.3

Although m , the number of fringes observed, is an integer, which is often regarded as having zero uncertainty, in practical terms, it is all too easy to lose track when counting fringes. In [Example 3.6](#), if you estimate that you might have missed as many as five fringes when you reported $m = 122$ fringes, (a) is the value for the index of refraction worked out in [Example 3.6](#) too large or too small? (b) By how much?

**PROBLEM-SOLVING STRATEGY****Wave Optics**

Step 1. *Examine the situation to determine that interference is involved.* Identify whether slits, thin films, or interferometers are considered in the problem.

Step 2. *If slits are involved,* note that diffraction gratings and double slits produce very similar interference patterns, but that gratings have narrower (sharper) maxima. Single-slit patterns are characterized by a large central maximum and smaller maxima to the sides.

Step 3. *If thin-film interference or an interferometer is involved, take note of the path length difference between the two rays that interfere.* Be certain to use the wavelength in the medium involved, since it differs from the wavelength in vacuum. Note also that there is an additional $\lambda/2$ phase shift when light reflects from a medium with a greater index of refraction.

Step 4. *Identify exactly what needs to be determined in the problem (identify the unknowns).* A written list is

useful. Draw a diagram of the situation. Labeling the diagram is useful.

Step 5. *Make a list of what is given or can be inferred from the problem as stated (identify the knowns).*

Step 6. *Solve the appropriate equation for the quantity to be determined (the unknown) and enter the knowns.* Slits, gratings, and the Rayleigh limit involve equations.

Step 7. *For thin-film interference, you have constructive interference for a total shift that is an integral number of wavelengths. You have destructive interference for a total shift of a half-integral number of wavelengths.* Always keep in mind that crest to crest is constructive whereas crest to trough is destructive.

Step 8. *Check to see if the answer is reasonable: Does it make sense?* Angles in interference patterns cannot be greater than 90° , for example.

CHAPTER REVIEW

Key Terms

coherent waves waves are in phase or have a definite phase relationship

fringes bright and dark patterns of interference

incoherent waves have random phase relationships

interferometer instrument that uses interference of waves to make measurements

monochromatic light composed of one wavelength only

Newton's rings circular interference pattern created by interference between the light

reflected off two surfaces as a result of a slight gap between them

order integer m used in the equations for constructive and destructive interference for a double slit

principal maximum brightest interference fringes seen with multiple slits

secondary maximum bright interference fringes of intensity lower than the principal maxima

thin-film interference interference between light reflected from different surfaces of a thin film

Key Equations

Constructive interference

$$\Delta l = m\lambda, \text{ for } m = 0, \pm 1, \pm 2, \pm 3 \dots$$

Destructive interference

$$\Delta l = (m + \frac{1}{2})\lambda, \text{ for } m = 0, \pm 1, \pm 2, \pm 3 \dots$$

Path length difference for waves from two slits to a common point on a screen

$$\Delta l = d \sin \theta$$

Constructive interference

$$d \sin \theta = m\lambda, \text{ for } m = 0, \pm 1, \pm 2, \pm 3, \dots$$

Destructive interference

$$d \sin \theta = (m + \frac{1}{2})\lambda, \text{ for } m = 0, \pm 1, \pm 2, \pm 3, \dots$$

Distance from central maximum to the m th bright fringe

$$y_m = \frac{m\lambda D}{d}$$

Displacement measured by a Michelson interferometer

$$\Delta d = m \frac{\lambda_0}{2}$$

Summary

3.1 Young's Double-Slit Interference

- Young's double-slit experiment gave definitive proof of the wave character of light.
- An interference pattern is obtained by the superposition of light from two slits.

3.2 Mathematics of Interference

- In double-slit diffraction, constructive interference occurs when $d \sin \theta = m\lambda$ (for $m = 0, \pm 1, \pm 2, \pm 3 \dots$), where d is the distance between the slits, θ is the angle relative to the incident direction, and m is the order of the interference.
- Destructive interference occurs when $d \sin \theta = (m + \frac{1}{2})\lambda$ for $m = 0, \pm 1, \pm 2, \pm 3, \dots$

3.3 Multiple-Slit Interference

- Interference from multiple slits ($N > 2$) produces principal as well as secondary maxima.
- As the number of slits is increased, the intensity of the principal maxima increases and the width decreases.

3.4 Interference in Thin Films

- When light reflects from a medium having an index of refraction greater than that of the medium in which it is traveling, a 180° phase change (or a $\lambda/2$ shift) occurs.
- Thin-film interference occurs between the light reflected from the top and bottom surfaces of a film. In addition to the path length difference,

there can be a phase change.

3.5 The Michelson Interferometer

- When the mirror in one arm of the

interferometer moves a distance of $\lambda/2$ each fringe in the interference pattern moves to the position previously occupied by the adjacent fringe.

Conceptual Questions

3.1 Young's Double-Slit Interference

1. Young's double-slit experiment breaks a single light beam into two sources. Would the same pattern be obtained for two independent sources of light, such as the headlights of a distant car? Explain.
2. Is it possible to create an experimental setup in which there is only destructive interference? Explain.
3. Why won't two small sodium lamps, held close together, produce an interference pattern on a distant screen? What if the sodium lamps were replaced by two laser pointers held close together?

3.2 Mathematics of Interference

4. Suppose you use the same double slit to perform Young's double-slit experiment in air and then repeat the experiment in water. Do the angles to the same parts of the interference pattern get larger or smaller? Does the color of the light change? Explain.
5. Why is monochromatic light used in the double slit experiment? What would happen if white light were used?

3.4 Interference in Thin Films

6. What effect does increasing the wedge angle have on the spacing of interference fringes? If the wedge angle is too large, fringes are not observed. Why?
7. How is the difference in paths taken by two originally in-phase light waves related to whether they interfere constructively or destructively? How can this be affected by reflection? By refraction?
8. Is there a phase change in the light reflected from either surface of a contact lens floating on a

person's tear layer? The index of refraction of the lens is about 1.5, and its top surface is dry.

9. In placing a sample on a microscope slide, a glass cover is placed over a water drop on the glass slide. Light incident from above can reflect from the top and bottom of the glass cover and from the glass slide below the water drop. At which surfaces will there be a phase change in the reflected light?
10. Answer the above question if the fluid between the two pieces of crown glass is carbon disulfide.
11. While contemplating the food value of a slice of ham, you notice a rainbow of color reflected from its moist surface. Explain its origin.
12. An inventor notices that a soap bubble is dark at its thinnest and realizes that destructive interference is taking place for all wavelengths. How could she use this knowledge to make a nonreflective coating for lenses that is effective at all wavelengths? That is, what limits would there be on the index of refraction and thickness of the coating? How might this be impractical?
13. A nonreflective coating like the one described in [Example 3.3](#) works ideally for a single wavelength and for perpendicular incidence. What happens for other wavelengths and other incident directions? Be specific.
14. Why is it much more difficult to see interference fringes for light reflected from a thick piece of glass than from a thin film? Would it be easier if monochromatic light were used?

3.5 The Michelson Interferometer

15. Describe how a Michelson interferometer can be used to measure the index of refraction of a gas (including air).

Problems

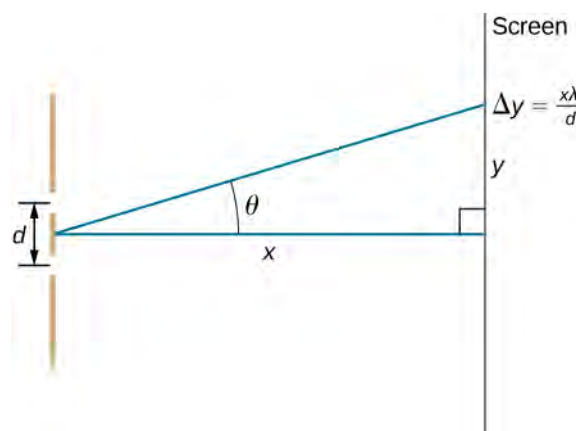
3.2 Mathematics of Interference

16. At what angle is the first-order maximum for 450-nm wavelength blue light falling on double slits separated by 0.0500 mm?

17. Calculate the angle for the third-order maximum of 580-nm wavelength yellow light falling on double slits separated by 0.100 mm.
18. What is the separation between two slits for

which 610-nm orange light has its first maximum at an angle of 30.0° ?

19. Find the distance between two slits that produces the first minimum for 410-nm violet light at an angle of 45.0° .
20. Calculate the wavelength of light that has its third minimum at an angle of 30.0° when falling on double slits separated by $3.00 \mu\text{m}$. Explicitly show how you follow the steps from the [Problem-Solving Strategy: Wave Optics](#), located at the end of the chapter.
21. What is the wavelength of light falling on double slits separated by $2.00 \mu\text{m}$ if the third-order maximum is at an angle of 60.0° ?
22. At what angle is the second-order maximum for the situation in the preceding problem?
23. What is the highest-order maximum for 400-nm light falling on double slits separated by $25.0 \mu\text{m}$?
24. Find the largest wavelength of light falling on double slits separated by $1.20 \mu\text{m}$ for which there is a first-order maximum. Is this in the visible part of the spectrum?
25. What is the smallest separation between two slits that will produce a second-order maximum for 720-nm red light?
26. (a) What is the smallest separation between two slits that will produce a second-order maximum for any visible light? (b) For all visible light?
27. (a) If the first-order maximum for monochromatic light falling on a double slit is at an angle of 10.0° , at what angle is the second-order maximum? (b) What is the angle of the first minimum? (c) What is the highest-order maximum possible here?
28. Shown below is a double slit located a distance x from a screen, with the distance from the center of the screen given by y . When the distance d between the slits is relatively large, numerous bright spots appear, called fringes. Show that, for small angles (where $\sin \theta \approx \theta$, with θ in radians), the distance between fringes is given by $\Delta y = x\lambda/d$



29. Using the result of the preceding problem, (a) calculate the distance between fringes for 633-nm light falling on double slits separated by 0.0800 mm , located 3.00 m from a screen. (b) What would be the distance between fringes if the entire apparatus were submerged in water, whose index of refraction is 1.33?
30. Using the result of the problem two problems prior, find the wavelength of light that produces fringes 7.50 mm apart on a screen 2.00 m from double slits separated by 0.120 mm .
31. In a double-slit experiment, the fifth maximum is 2.8 cm from the central maximum on a screen that is 1.5 m away from the slits. If the slits are 0.15 mm apart, what is the wavelength of the light being used?
32. The source in Young's experiment emits at two wavelengths. On the viewing screen, the fourth maximum for one wavelength is located at the same spot as the fifth maximum for the other wavelength. What is the ratio of the two wavelengths?
33. If 500-nm and 650-nm light illuminates two slits that are separated by 0.50 mm , how far apart are the second-order maxima for these two wavelengths on a screen 2.0 m away?
34. Red light of wavelength of 700 nm falls on a double slit separated by 400 nm . (a) At what angle is the first-order maximum in the diffraction pattern? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

3.3 Multiple-Slit Interference

35. Ten narrow slits are equally spaced 0.25 mm apart and illuminated with yellow light of wavelength 580 nm . (a) What are the angular positions of the third and fourth principal maxima? (b) What is the separation of these maxima on a screen 2.0 m from the slits?

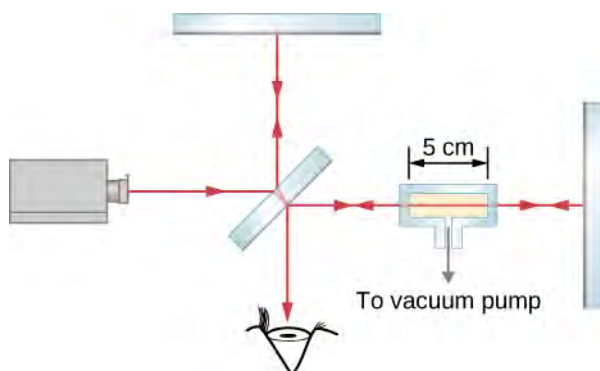
36. The width of bright fringes can be calculated as the separation between the two adjacent dark fringes on either side. Find the angular widths of the third- and fourth-order bright fringes from the preceding problem.
37. For a three-slit interference pattern, find the ratio of the peak intensities of a secondary maximum to a principal maximum.
38. What is the angular width of the central fringe of the interference pattern of (a) 20 slits separated by $d = 2.0 \times 10^{-3} \text{ mm}$? (b) 50 slits with the same separation? Assume that $\lambda = 600 \text{ nm}$.

3.4 Interference in Thin Films

39. A soap bubble is 100 nm thick and illuminated by white light incident perpendicular to its surface. What wavelength and color of visible light is most constructively reflected, assuming the same index of refraction as water?
40. An oil slick on water is 120 nm thick and illuminated by white light incident perpendicular to its surface. What color does the oil appear (what is the most constructively reflected wavelength), given its index of refraction is 1.40?
41. Calculate the minimum thickness of an oil slick on water that appears blue when illuminated by white light perpendicular to its surface. Take the blue wavelength to be 470 nm and the index of refraction of oil to be 1.40.
42. Find the minimum thickness of a soap bubble that appears red when illuminated by white light perpendicular to its surface. Take the wavelength to be 680 nm, and assume the same index of refraction as water.
43. A film of soapy water ($n = 1.33$) on top of a plastic cutting board has a thickness of 233 nm. What color is most strongly reflected if it is illuminated perpendicular to its surface?
44. What are the three smallest non-zero thicknesses of soapy water ($n = 1.33$) on Plexiglas if it appears green (constructively reflecting 520-nm light) when illuminated perpendicularly by white light?
45. Suppose you have a lens system that is to be used primarily for 700-nm red light. What is the second thinnest coating of fluorite (magnesium fluoride) that would be nonreflective for this wavelength?
46. (a) As a soap bubble thins it becomes dark, because the path length difference becomes small compared with the wavelength of light and there is a phase shift at the top surface. If it becomes dark when the path length difference is less than one-fourth the wavelength, what is the thickest the bubble can be and appear dark at all visible wavelengths? Assume the same index of refraction as water. (b) Discuss the fragility of the film considering the thickness found.
47. To save money on making military aircraft invisible to radar, an inventor decides to coat them with a nonreflective material having an index of refraction of 1.20, which is between that of air and the surface of the plane. This, he reasons, should be much cheaper than designing Stealth bombers. (a) What thickness should the coating be to inhibit the reflection of 4.00-cm wavelength radar? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

3.5 The Michelson Interferometer

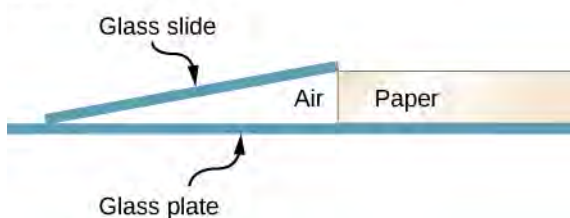
48. A Michelson interferometer has two equal arms. A mercury light of wavelength 546 nm is used for the interferometer and stable fringes are found. One of the arms is moved by $1.5 \mu\text{m}$. How many fringes will cross the observing field?
49. What is the distance moved by the traveling mirror of a Michelson interferometer that corresponds to 1500 fringes passing by a point of the observation screen? Assume that the interferometer is illuminated with a 606 nm spectral line of krypton-86.
50. When the traveling mirror of a Michelson interferometer is moved $2.40 \times 10^{-5} \text{ m}$, 90 fringes pass by a point on the observation screen. What is the wavelength of the light used?
51. In a Michelson interferometer, light of wavelength 632.8 nm from a He-Ne laser is used. When one of the mirrors is moved by a distance D , 8 fringes move past the field of view. What is the value of the distance D ?
52. A chamber 5.0 cm long with flat, parallel windows at the ends is placed in one arm of a Michelson interferometer (see below). The light used has a wavelength of 500 nm in a vacuum. While all the air is being pumped out of the chamber, 29 fringes pass by a point on the observation screen. What is the refractive index of the air?



Additional Problems

53. For 600-nm wavelength light and a slit separation of 0.12 mm, what are the angular positions of the first and third maxima in the double slit interference pattern?
54. If the light source in the preceding problem is changed, the angular position of the third maximum is found to be 0.57° . What is the wavelength of light being used now?
55. Red light ($\lambda = 710. \text{ nm}$) illuminates double slits separated by a distance $d = 0.150 \text{ mm}$. The screen and the slits are 3.00 m apart. (a) Find the distance on the screen between the central maximum and the third maximum. (b) What is the distance between the second and the fourth maxima?
56. Two sources as in phase and emit waves with $\lambda = 0.42 \text{ m}$. Determine whether constructive or destructive interference occurs at points whose distances from the two sources are (a) 0.84 and 0.42 m, (b) 0.21 and 0.42 m, (c) 1.26 and 0.42 m, (d) 1.87 and 1.45 m, (e) 0.63 and 0.84 m and (f) 1.47 and 1.26 m.
57. Two slits $4.0 \times 10^{-6} \text{ m}$ apart are illuminated by light of wavelength 600 nm. What is the highest order fringe in the interference pattern?
58. Suppose that the highest order fringe that can be observed is the eighth in a double-slit experiment where 550-nm wavelength light is used. What is the minimum separation of the slits?
59. The interference pattern of a He-Ne laser light ($\lambda = 632.9 \text{ nm}$) passing through two slits 0.031 mm apart is projected on a screen 10.0 m away. Determine the distance between the adjacent bright fringes.
60. Young's double-slit experiment is performed immersed in water ($n = 1.333$). The light source is a He-Ne laser, $\lambda = 632.9 \text{ nm}$ in vacuum. (a) What is the wavelength of this light in water? (b) What is the angle for the third order maximum for two slits separated by 0.100 mm.
61. A double-slit experiment is to be set up so that the bright fringes appear 1.27 cm apart on a screen 2.13 m away from the two slits. The light source was wavelength 500 nm. What should be the separation between the two slits?
62. An effect analogous to two-slit interference can occur with sound waves, instead of light. In an open field, two speakers placed 1.30 m apart are powered by a single-function generator producing sine waves at 1200-Hz frequency. A student walks along a line 12.5 m away and parallel to the line between the speakers. She hears an alternating pattern of loud and quiet, due to constructive and destructive interference. What is (a) the wavelength of this sound and (b) the distance between the central maximum and the first maximum (loud) position along this line?
63. A hydrogen gas discharge lamp emits visible light at four wavelengths, $\lambda = 410, 434, 486,$ and 656 nm . (a) If light from this lamp falls on a N slits separated by 0.025 mm, how far from the central maximum are the third maxima when viewed on a screen 2.0 m from the slits? (b) By what distance are the second and third maxima separated for $\lambda = 486 \text{ nm}$?
64. Monochromatic light of frequency $5.5 \times 10^{14} \text{ Hz}$ falls on 10 slits separated by 0.020 mm. What is the separation between the first and third maxima on a screen that is 2.0 m from the slits?

65. Eight slits equally separated by 0.149 mm is uniformly illuminated by a monochromatic light at $\lambda = 523$ nm. What is the width of the central principal maximum on a screen 2.35 m away?
66. Eight slits equally separated by 0.149 mm is uniformly illuminated by a monochromatic light at $\lambda = 523$ nm. What is the intensity of a secondary maxima compared to that of the principal maxima?
67. A transparent film of thickness 250 nm and index of refraction of 1.40 is surrounded by air. What wavelength in a beam of white light at near-normal incidence to the film undergoes destructive interference when reflected?
68. An intensity minimum is found for 450 nm light transmitted through a transparent film ($n = 1.20$) in air. (a) What is minimum thickness of the film? (b) If this wavelength is the longest for which the intensity minimum occurs, what are the next three lower values of λ for which this happens?
69. A thin film with $n = 1.32$ is surrounded by air. What is the minimum thickness of this film such that the reflection of normally incident light with $\lambda = 500$ nm is minimized?
70. Repeat your calculation of the previous problem with the thin film placed on a flat glass ($n = 1.50$) surface.
71. After a minor oil spill, a thin film of oil ($n = 1.40$) of thickness 450 nm floats on the water surface in a bay. (a) What predominant color is seen by a bird flying overhead? (b) What predominant color is seen by a seal swimming underwater?
72. A microscope slide 10 cm long is separated from a glass plate at one end by a sheet of paper. As shown below, the other end of the slide is in contact with the plate. The slide is illuminated from above by light from a sodium lamp ($\lambda = 589$ nm), and 14 fringes per centimeter are seen along the slide. What is the thickness of the piece of paper?
(Not to scale)



73. Suppose that the setup of the preceding problem is immersed in an unknown liquid. If 18 fringes per centimeter are now seen along the slide, what is the index of refraction of the liquid?
74. A thin wedge filled with air is produced when two flat glass plates are placed on top of one another and a slip of paper is inserted between them at one edge. Interference fringes are observed when monochromatic light falling vertically on the plates are seen in reflection. Is the first fringe near the edge where the plates are in contact a bright fringe or a dark fringe? Explain.
75. Two identical pieces of rectangular plate glass are used to measure the thickness of a hair. The glass plates are in direct contact at one edge and a single hair is placed between them near the opposite edge. When illuminated with a sodium lamp ($\lambda = 589$ nm), the hair is seen between the 180th and 181st dark fringes. What are the lower and upper limits on the hair's diameter?
76. Two microscope slides made of glass are illuminated by monochromatic ($\lambda = 589$ nm) light incident perpendicularly. The top slide touches the bottom slide at one end and rests on a thin copper wire at the other end, forming a wedge of air. The diameter of the copper wire is $29.45 \mu\text{m}$. How many bright fringes are seen across these slides?
77. A good quality camera "lens" is actually a system of lenses, rather than a single lens, but a side effect is that a reflection from the surface of one lens can bounce around many times within the system, creating artifacts in the photograph. To counteract this problem, one of the lenses in such a system is coated with a thin layer of material ($n = 1.28$) on one side. The index of refraction of the lens glass is 1.68. What is the smallest thickness of the coating that reduces the reflection at 640 nm by destructive interference? (In other words, the coating's effect is to be optimized for $\lambda = 640$ nm.)
78. Constructive interference is observed from directly above an oil slick for wavelengths (in air) 440 nm and 616 nm. The index of refraction of this oil is $n = 1.54$. What is the film's minimum possible thickness?
79. A soap bubble is blown outdoors. What colors (indicate by wavelengths) of the reflected sunlight are seen enhanced? The soap bubble has index of refraction 1.36 and thickness 380 nm.

80. A Michelson interferometer with a He-Ne laser light source ($\lambda = 632.8 \text{ nm}$) projects its interference pattern on a screen. If the movable mirror is caused to move by $8.54 \mu\text{m}$, how many fringes will be observed shifting through a reference point on a screen?
81. An experimenter detects 251 fringes when the movable mirror in a Michelson interferometer is displaced. The light source used is a sodium lamp, wavelength 589 nm . By what distance did the movable mirror move?
82. A Michelson interferometer is used to measure the wavelength of light put through it. When the movable mirror is moved by exactly 0.100 mm , the number of fringes observed moving through is 316. What is the wavelength of the light?
83. A 5.08-cm -long rectangular glass chamber is inserted into one arm of a Michelson interferometer using a 633-nm light source. This chamber is initially filled with air ($n = 1.000293$) at standard atmospheric pressure but the air is gradually pumped out using a vacuum pump until a near perfect vacuum is achieved. How many fringes are observed moving by during the transition?
84. Into one arm of a Michelson interferometer, a plastic sheet of thickness $75 \mu\text{m}$ is inserted, which causes a shift in the interference pattern by 86 fringes. The light source has wavelength of 610 nm in air. What is the index of refraction of this plastic?
85. The thickness of an aluminum foil is measured using a Michelson interferometer that has its movable mirror mounted on a micrometer. There is a difference of 27 fringes in the observed interference pattern when the micrometer clamps down on the foil compared to when the micrometer is empty. The light source is a He-Ne laser with wavelength 632.8 nm . Calculate the thickness of the foil.
86. The movable mirror of a Michelson interferometer is attached to one end of a thin metal rod of length 23.3 mm . The other end of the rod is anchored so it does not move. As the temperature of the rod changes from 15°C to 25°C , a change of 14 fringes is observed. The light source is a He Ne laser, $\lambda = 632.8 \text{ nm}$. What is the change in length of the metal bar, and what is its thermal expansion coefficient?
87. In a thermally stabilized lab, a Michelson interferometer is used to monitor the temperature to ensure it stays constant. The movable mirror is mounted on the end of a 1.00-m -long aluminum rod, held fixed at the other end. The light source is a He Ne laser, $\lambda = 632.8 \text{ nm}$. The resolution of this apparatus corresponds to the temperature difference when a change of just one fringe is observed. What is this temperature difference?
88. A 65-fringe shift results in a Michelson interferometer when a $42.0\text{-}\mu\text{m}$ film made of an unknown material is placed in one arm. The light source has wavelength 632.9 nm . Identify the material using the indices of refraction found in [Table 1.1](#).

Challenge Problems

89. Determine what happens to the double-slit interference pattern if one of the slits is covered with a thin, transparent film whose thickness is $\lambda/[2(n - 1)]$, where λ is the wavelength of the incident light and n is the index of refraction of the film.
90. Fifty-one narrow slits are equally spaced and separated by 0.10 mm . The slits are illuminated by blue light of wavelength 400 nm . What is angular position of the twenty-fifth secondary maximum? What is its peak intensity in comparison with that of the primary maximum?
91. A film of oil on water will appear dark when it is very thin, because the path length difference becomes small compared with the wavelength of light and there is a phase shift at the top surface. If it becomes dark when the path length difference is less than one-fourth the wavelength, what is the thickest the oil can be and appear dark at all visible wavelengths? Oil has an index of refraction of 1.40 .

92. [Figure 3.14](#) shows two glass slides illuminated by monochromatic light incident perpendicularly. The top slide touches the bottom slide at one end and rests on a 0.100-mm-diameter hair at the other end, forming a wedge of air. (a) How far apart are the dark bands, if the slides are 7.50 cm long and 589-nm light is used? (b) Is there any difference if the slides are made from crown or flint glass? Explain.
93. [Figure 3.14](#) shows two 7.50-cm-long glass slides illuminated by pure 589-nm wavelength light incident perpendicularly. The top slide touches the bottom slide at one end and rests on some debris at the other end, forming a wedge of air. How thick is the debris, if the dark bands are 1.00 mm apart?
94. A soap bubble is 100 nm thick and illuminated by white light incident at a 45° angle to its surface. What wavelength and color of visible light is most constructively reflected, assuming the same index of refraction as water?
95. An oil slick on water is 120 nm thick and illuminated by white light incident at a 45° angle to its surface. What color does the oil appear (what is the most constructively reflected wavelength), given its index of refraction is 1.40?

CHAPTER 4

Diffraction

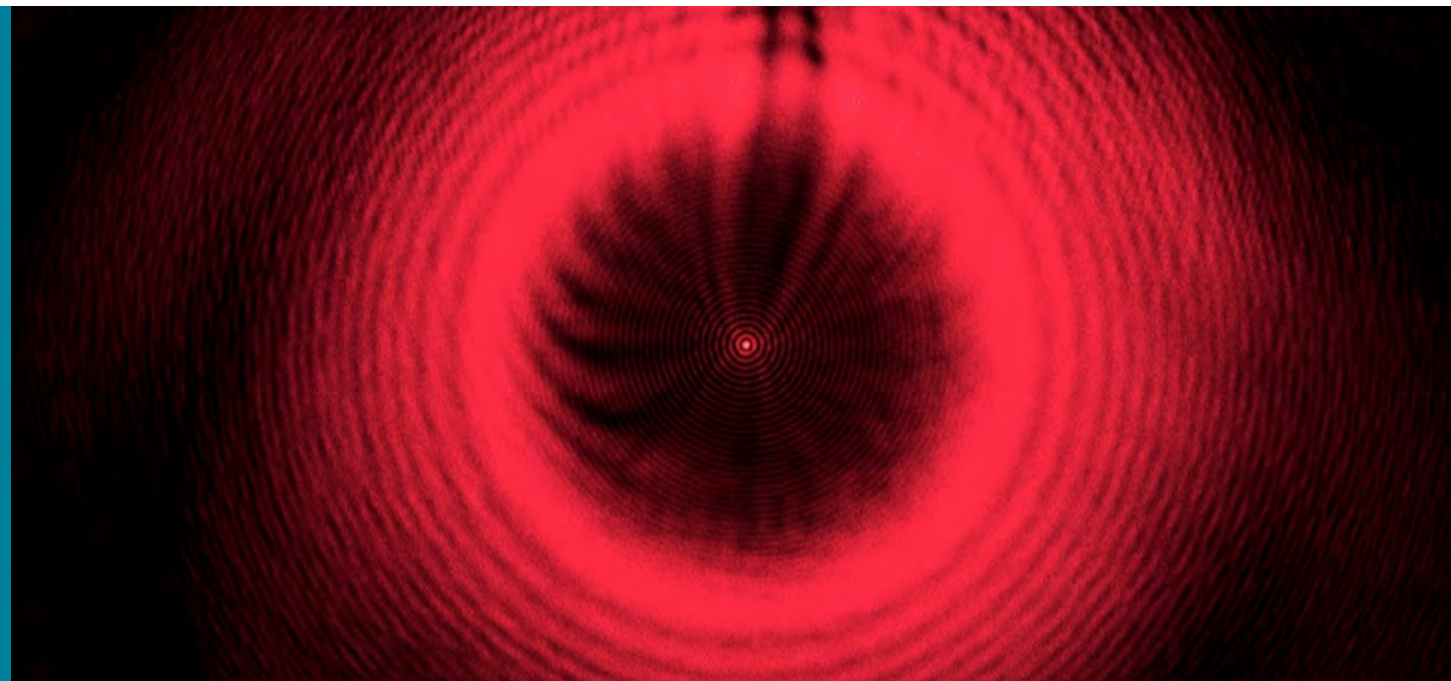


Figure 4.1 A steel ball bearing illuminated by a laser does not cast a sharp, circular shadow. Instead, a series of diffraction fringes and a central bright spot are observed. Known as Poisson's spot, the effect was first predicted by Augustin-Jean Fresnel (1788–1827) as a consequence of diffraction of light waves. Based on principles of ray optics, Siméon-Denis Poisson (1781–1840) argued against Fresnel's prediction. (credit: modification of work by Harvard Natural Science Lecture Demonstrations)

Chapter Outline

[4.1 Single-Slit Diffraction](#)

[4.2 Intensity in Single-Slit Diffraction](#)

[4.3 Double-Slit Diffraction](#)

[4.4 Diffraction Gratings](#)

[4.5 Circular Apertures and Resolution](#)

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INTRODUCTION Imagine passing a monochromatic light beam through a narrow opening—a slit just a little wider than the wavelength of the light. Instead of a simple shadow of the slit on the screen, you will see that an interference pattern appears, even though there is only one slit.

In the chapter on interference, we saw that you need two sources of waves for interference to occur. How can

there be an interference pattern when we have only one slit? In [The Nature of Light](#), we learned that, due to Huygens's principle, we can imagine a wave front as equivalent to infinitely many point sources of waves. Thus, a wave from a slit can behave not as one wave but as an infinite number of point sources. These waves can interfere with each other, resulting in an interference pattern without the presence of a second slit. This phenomenon is called *diffraction*.

Another way to view this is to recognize that a slit has a small but finite width. In the preceding chapter, we implicitly regarded slits as objects with positions but no size. The widths of the slits were considered negligible. When the slits have finite widths, each point along the opening can be considered a point source of light—a foundation of Huygens's principle. Because real-world optical instruments must have finite apertures (otherwise, no light can enter), diffraction plays a major role in the way we interpret the output of these optical instruments. For example, diffraction places limits on our ability to resolve images or objects. This is a problem that we will study later in this chapter.

4.1 Single-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Explain the phenomenon of diffraction and the conditions under which it is observed
- Describe diffraction through a single slit

After passing through a narrow aperture (opening), a wave propagating in a specific direction tends to spread out. For example, sound waves that enter a room through an open door can be heard even if the listener is in a part of the room where the geometry of ray propagation dictates that there should only be silence. Similarly, ocean waves passing through an opening in a breakwater can spread throughout the bay inside. ([Figure 4.2](#)). The spreading and bending of sound and ocean waves are two examples of **diffraction**, which is the bending of a wave around the edges of an opening or an obstacle—a phenomenon exhibited by all types of waves.

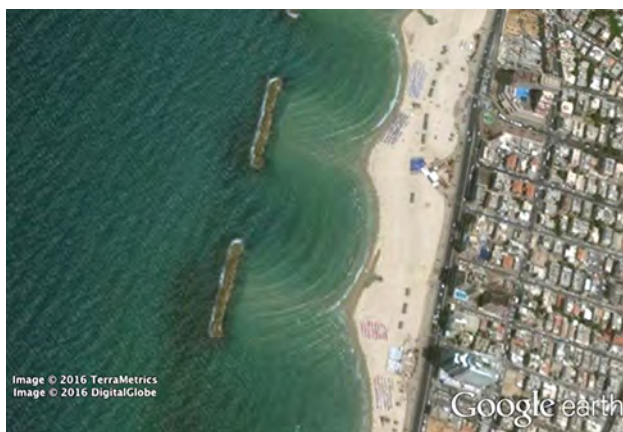


Figure 4.2 Because of the diffraction of waves, ocean waves entering through an opening in a breakwater can spread throughout the bay. (credit: modification of map data from Google Earth)

The diffraction of sound waves is apparent to us because wavelengths in the audible region are approximately the same size as the objects they encounter, a condition that must be satisfied if diffraction effects are to be observed easily. Since the wavelengths of visible light range from approximately 390 to 770 nm, most objects do not diffract light significantly. However, situations do occur in which apertures are small enough that the diffraction of light is observable. For example, if you place your middle and index fingers close together and look through the opening at a light bulb, you can see a rather clear diffraction pattern, consisting of light and dark lines running parallel to your fingers.

Diffraction through a Single Slit

Light passing through a single slit forms a diffraction pattern somewhat different from those formed by double slits or diffraction gratings, which we discussed in the chapter on interference. [Figure 4.3](#) shows a single-slit diffraction pattern. Note that the central maximum is larger than maxima on either side and that the intensity

decreases rapidly on either side. In contrast, a diffraction grating ([Diffraction Gratings](#)) produces evenly spaced lines that dim slowly on either side of the center.

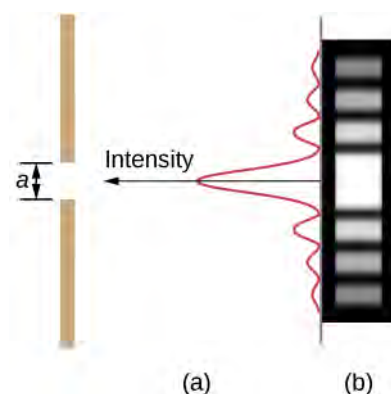


Figure 4.3 Single-slit diffraction pattern. (a) Monochromatic light passing through a single slit has a central maximum and many smaller and dimmer maxima on either side. The central maximum is six times higher than shown. (b) The diagram shows the bright central maximum, and the dimmer and thinner maxima on either side.

The analysis of single-slit diffraction is illustrated in [Figure 4.4](#). Here, the light arrives at the slit, illuminating it uniformly and is in phase across its width. We then consider light propagating onwards from different parts of the *same* slit. According to Huygens's principle, every part of the wave front in the slit emits wavelets, as we discussed in [The Nature of Light](#). These are like rays that start out in phase and head in all directions. (Each ray is perpendicular to the wave front of a wavelet.) Assuming the screen is very far away compared with the size of the slit, rays heading toward a common destination are nearly parallel. When they travel straight ahead, as in part (a) of the figure, they remain in phase, and we observe a central maximum. However, when rays travel at an angle θ relative to the original direction of the beam, each ray travels a different distance to a common location, and they can arrive in or out of phase. In part (b), the ray from the bottom travels a distance of one wavelength λ farther than the ray from the top. Thus, a ray from the center travels a distance $\lambda/2$ less than the one at the bottom edge of the slit, arrives out of phase, and interferes destructively. A ray from slightly above the center and one from slightly above the bottom also cancel one another. In fact, each ray from the slit interferes destructively with another ray. In other words, a pair-wise cancellation of all rays results in a dark minimum in intensity at this angle. By symmetry, another minimum occurs at the same angle to the right of the incident direction (toward the bottom of the figure) of the light.

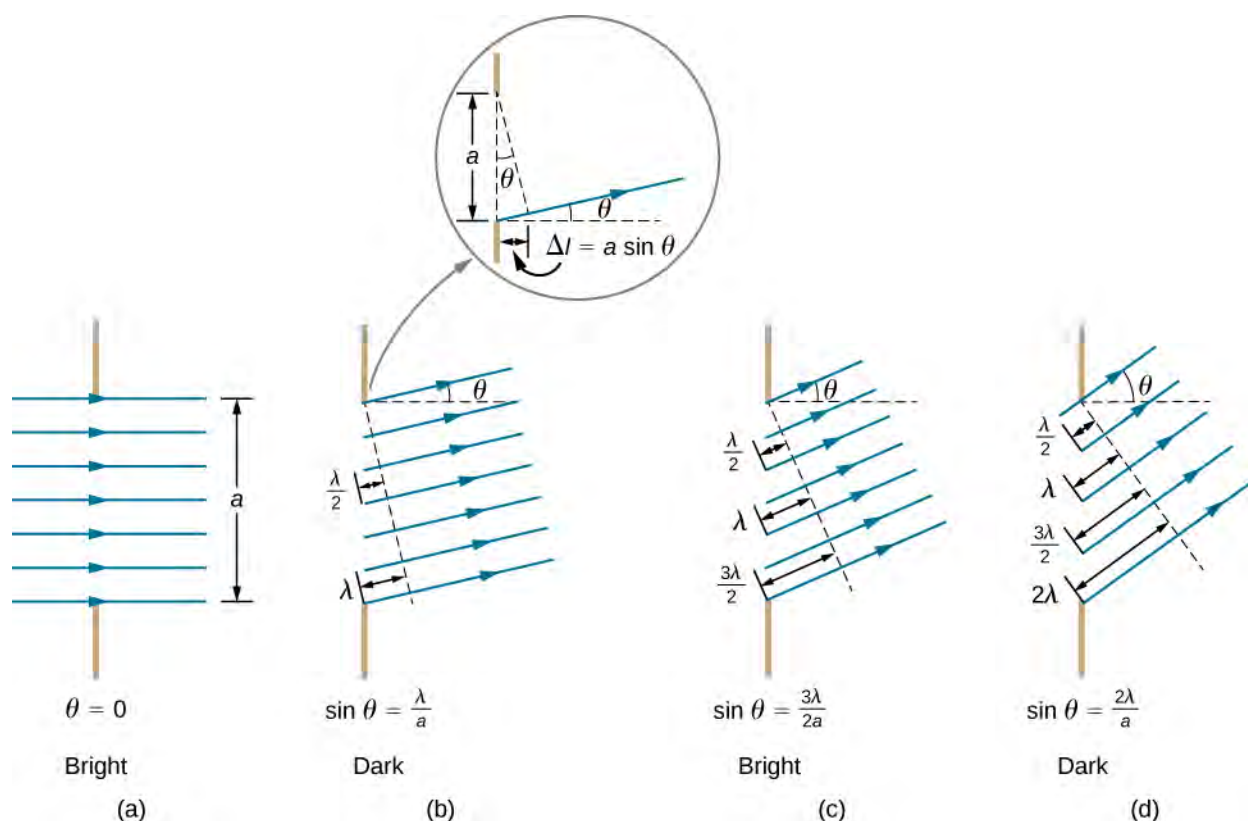


Figure 4.4 Light passing through a single slit is diffracted in all directions and may interfere constructively or destructively, depending on the angle. The difference in path length for rays from either side of the slit is seen to be $a \sin \theta$.

At the larger angle shown in part (c), the path lengths differ by $3\lambda/2$ for rays from the top and bottom of the slit. One ray travels a distance λ different from the ray from the bottom and arrives in phase, interfering constructively. Two rays, each from slightly above those two, also add constructively. Most rays from the slit have another ray to interfere with constructively, and a maximum in intensity occurs at this angle. However, not all rays interfere constructively for this situation, so the maximum is not as intense as the central maximum. Finally, in part (d), the angle shown is large enough to produce a second minimum. As seen in the figure, the difference in path length for rays from either side of the slit is $a \sin \theta$, and we see that a destructive minimum is obtained when this distance is an integral multiple of the wavelength.

Thus, to obtain **destructive interference for a single slit**,

$$a \sin \theta = m\lambda, \text{ for } m = \pm 1, \pm 2, \pm 3, \dots (\text{destructive}), \quad 4.1$$

where a is the slit width, λ is the light's wavelength, θ is the angle relative to the original direction of the light, and m is the order of the minimum. [Figure 4.5](#) shows a graph of intensity for single-slit interference, and it is apparent that the maxima on either side of the central maximum are much less intense and not as wide. This effect is explored in [Double-Slit Diffraction](#).

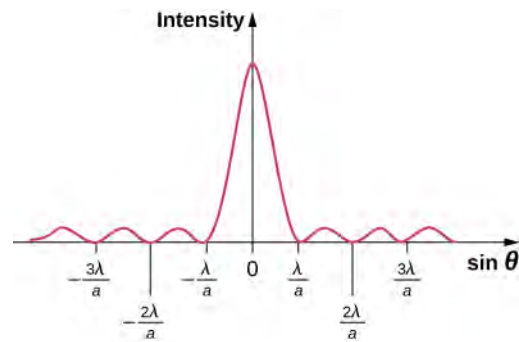


Figure 4.5 A graph of single-slit diffraction intensity showing the central maximum to be wider and much more intense than those to the sides. In fact, the central maximum is six times higher than shown here.

EXAMPLE 4.1

Calculating Single-Slit Diffraction

Visible light of wavelength 550 nm falls on a single slit and produces its second diffraction minimum at an angle of 45.0° relative to the incident direction of the light, as in [Figure 4.6](#). (a) What is the width of the slit? (b) At what angle is the first minimum produced?

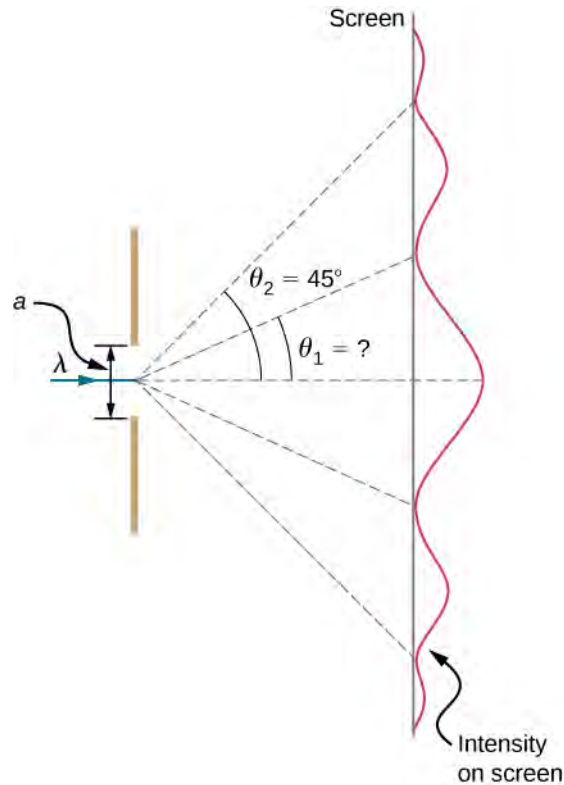


Figure 4.6 In this example, we analyze a graph of the single-slit diffraction pattern.

Strategy

From the given information, and assuming the screen is far away from the slit, we can use the equation $a \sin \theta = m\lambda$ first to find a , and again to find the angle for the first minimum θ_1 .

Solution

- We are given that $\lambda = 550 \text{ nm}$, $m = 2$, and $\theta_2 = 45.0^\circ$. Solving the equation $a \sin \theta = m\lambda$ for a and substituting known values gives

$$a = \frac{m\lambda}{\sin \theta_2} = \frac{2(550 \text{ nm})}{\sin 45.0^\circ} = \frac{1100 \times 10^{-9} \text{ m}}{0.707} = 1.56 \times 10^{-6} \text{ m}.$$

b. Solving the equation $a \sin \theta = m\lambda$ for $\sin \theta_1$ and substituting the known values gives

$$\sin \theta_1 = \frac{m\lambda}{a} = \frac{1(550 \times 10^{-9} \text{ m})}{1.56 \times 10^{-6} \text{ m}}.$$

Thus the angle θ_1 is

$$\theta_1 = \sin^{-1} 0.354 = 20.7^\circ.$$

Significance

We see that the slit is narrow (it is only a few times greater than the wavelength of light). This is consistent with the fact that light must interact with an object comparable in size to its wavelength in order to exhibit significant wave effects such as this single-slit diffraction pattern. We also see that the central maximum extends 20.7° on either side of the original beam, for a width of about 41° . The angle between the first and second minima is only about $24^\circ (45.0^\circ - 20.7^\circ)$. Thus, the second maximum is only about half as wide as the central maximum.

CHECK YOUR UNDERSTANDING 4.1

Suppose the slit width in [Example 4.1](#) is increased to $1.8 \times 10^{-6} \text{ m}$. What are the new angular positions for the first, second, and third minima? Would a fourth minimum exist?

4.2 Intensity in Single-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Calculate the intensity relative to the central maximum of the single-slit diffraction peaks
- Calculate the intensity relative to the central maximum of an arbitrary point on the screen

To calculate the intensity of the diffraction pattern, we follow the phasor method used for calculations with ac circuits in [Alternating-Current Circuits](#). If we consider that there are N Huygens sources across the slit shown in [Figure 4.4](#), with each source separated by a distance a/N from its adjacent neighbors, the path difference between waves from adjacent sources reaching the arbitrary point P on the screen is $(a/N) \sin \theta$. This distance is equivalent to a phase difference of $(2\pi a/\lambda N) \sin \theta$. The phasor diagram for the waves arriving at the point whose angular position is θ is shown in [Figure 4.7](#). The amplitude of the phasor for each Huygens wavelet is ΔE_0 , the amplitude of the resultant phasor is E , and the phase difference between the wavelets from the first and the last sources is

$$\phi = \left(\frac{2\pi}{\lambda} \right) a \sin \theta.$$

With $N \rightarrow \infty$, the phasor diagram approaches a circular arc of length $N\Delta E_0$ and radius r . Since the length of the arc is $N\Delta E_0$ for any ϕ , the radius r of the arc must decrease as ϕ increases (or equivalently, as the phasors form tighter spirals).

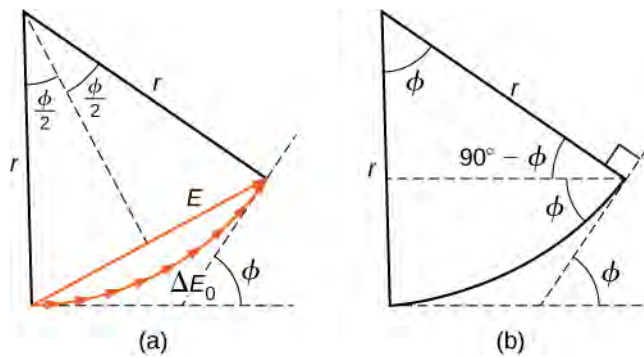


Figure 4.7 (a) Phasor diagram corresponding to the angular position θ in the single-slit diffraction pattern. The phase difference between the wavelets from the first and last sources is $\phi = (2\pi/\lambda)a \sin \theta$. (b) The geometry of the phasor diagram.

The phasor diagram for $\phi = 0$ (the center of the diffraction pattern) is shown in [Figure 4.8\(a\)](#) using $N = 30$. In this case, the phasors are laid end to end in a straight line of length $N\Delta E_0$, the radius r goes to infinity, and the resultant has its maximum value $E = N\Delta E_0$. The intensity of the light can be obtained using the relation $I = \frac{1}{2}c\epsilon_0 E^2$ from [Electromagnetic Waves](#). The intensity of the maximum is then

$$I_0 = \frac{1}{2}c\epsilon_0 (N\Delta E_0)^2 = \frac{1}{2\mu_0 c} (N\Delta E_0)^2,$$

where $\epsilon_0 = 1/\mu_0 c^2$. The phasor diagrams for the first two zeros of the diffraction pattern are shown in parts (b) and (d) of the figure. In both cases, the phasors add to zero, after rotating through $\phi = 2\pi$ rad for $m = 1$ and 4π rad for $m = 2$.

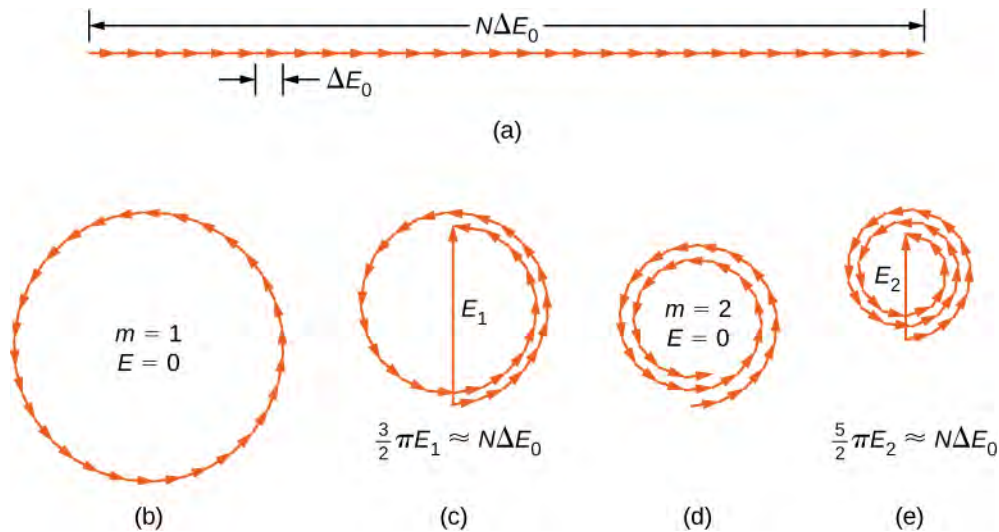


Figure 4.8 Phasor diagrams (with 30 phasors) for various points on the single-slit diffraction pattern. Multiple rotations around a given circle have been separated slightly so that the phasors can be seen. (a) Central maximum, (b) first minimum, (c) first maximum beyond central maximum, (d) second minimum, and (e) second maximum beyond central maximum.

The next two maxima beyond the central maxima are represented by the phasor diagrams of parts (c) and (e). In part (c), the phasors have rotated through $\phi = 3\pi$ rad and have formed a resultant phasor of magnitude E_1 . The length of the arc formed by the phasors is $N\Delta E_0$. Since this corresponds to 1.5 rotations around a circle of diameter E_1 , we have

$$\frac{3}{2}\pi E_1 \approx N\Delta E_0,$$

so

$$E_1 = \frac{2N\Delta E_0}{3\pi}$$

and

$$I_1 = \frac{1}{2\mu_0 c} E_1^2 = \frac{4(N\Delta E_0)^2}{(9\pi^2)(2\mu_0 c)} \approx 0.045 I_0,$$

where

$$I_0 = \frac{(N\Delta E_0)^2}{2\mu_0 c}.$$

In part (e), the phasors have rotated through $\phi = 5\pi$ rad, corresponding to 2.5 rotations around a circle of diameter E_2 and arc length $N\Delta E_0$. This results in $I_2 \approx 0.016 I_0$. The proof is left as an exercise for the student ([Exercise 4.119](#)).

These two maxima actually correspond to values of ϕ slightly less than 3π rad and 5π rad. Since the total length of the arc of the phasor diagram is always $N\Delta E_0$, the radius of the arc decreases as ϕ increases. As a result, E_1 and E_2 turn out to be slightly larger for arcs that have not quite curled through 3π rad and 5π rad, respectively. The exact values of ϕ for the maxima are investigated in [Exercise 4.120](#). In solving that problem, you will find that they are less than, but very close to, $\phi = 3\pi, 5\pi, 7\pi, \dots$ rad.

To calculate the intensity at an arbitrary point P on the screen, we return to the phasor diagram of [Figure 4.7](#). Since the arc subtends an angle ϕ at the center of the circle,

$$N\Delta E_0 = r\phi$$

and

$$\sin\left(\frac{\phi}{2}\right) = \frac{E}{2r}.$$

where E is the amplitude of the resultant field. Solving the second equation for E and then substituting r from the first equation, we find

$$E = 2r \sin \frac{\phi}{2} = 2 \frac{N\Delta E_0}{\phi} \sin \frac{\phi}{2}.$$

Now defining

$$\beta = \frac{\phi}{2} = \frac{\pi a \sin \theta}{\lambda} \quad 4.2$$

we obtain

$$E = N\Delta E_0 \frac{\sin \beta}{\beta} \quad 4.3$$

This equation relates the amplitude of the resultant field at any point in the diffraction pattern to the amplitude $N\Delta E_0$ at the central maximum. The intensity is proportional to the square of the amplitude, so

$$I = I_0 \left(\frac{\sin \beta}{\beta} \right)^2 \quad 4.4$$

where $I_0 = (N\Delta E_0)^2/2\mu_0 c$ is the intensity at the center of the pattern.

For the central maximum, $\phi = 0$, β is also zero and we see from l'Hôpital's rule that $\lim_{\beta \rightarrow 0} (\sin \beta / \beta) = 1$, so that $\lim_{\phi \rightarrow 0} I = I_0$. For the next maximum, $\phi = 3\pi$ rad, we have $\beta = 3\pi/2$ rad and when substituted into [Equation 4.4](#), it yields

$$I_1 = I_0 \left(\frac{\sin 3\pi/2}{3\pi/2} \right)^2 \approx 0.045 I_0,$$

in agreement with what we found earlier in this section using the diameters and circumferences of phasor diagrams. Substituting $\phi = 5\pi$ rad into [Equation 4.4](#) yields a similar result for I_2 .

A plot of [Equation 4.4](#) is shown in [Figure 4.9](#) and directly below it is a photograph of an actual diffraction pattern. Notice that the central peak is much brighter than the others, and that the zeros of the pattern are located at those points where $\sin \beta = 0$, which occurs when $\beta = m\pi$ rad. This corresponds to

$$\frac{\pi a \sin \theta}{\lambda} = m\pi,$$

or

$$a \sin \theta = m\lambda,$$

which is [Equation 4.1](#).

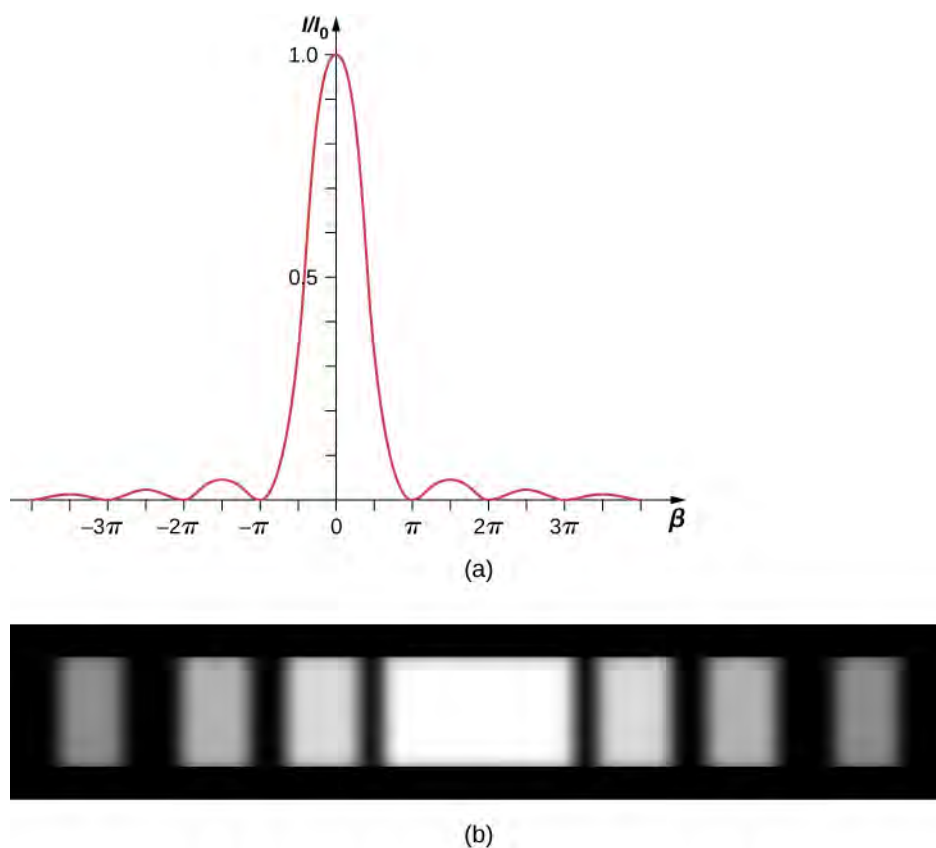


Figure 4.9 (a) The calculated intensity distribution of a single-slit diffraction pattern. (b) The actual diffraction pattern.

EXAMPLE 4.2

Intensity in Single-Slit Diffraction

Light of wavelength 550 nm passes through a slit of width $2.00 \mu\text{m}$ and produces a diffraction pattern similar to that shown in [Figure 4.9](#). (a) Find the locations of the first two minima in terms of the angle from the central maximum and (b) determine the intensity relative to the central maximum at a point halfway between these two minima.

Strategy

The minima are given by [Equation 4.1](#), $a \sin \theta = m\lambda$. The first two minima are for $m = 1$ and $m = 2$. [Equation](#)

4.4 and Equation 4.2 can be used to determine the intensity once the angle has been worked out.

Solution

- a. Solving Equation 4.1 for θ_m gives us $\theta_m = \sin^{-1}(m\lambda/a)$, so that

$$\theta_1 = \sin^{-1}\left(\frac{(+1)(550 \times 10^{-9} \text{ m})}{2.00 \times 10^{-6} \text{ m}}\right) = +16.0^\circ$$

and

$$\theta_2 = \sin^{-1}\left(\frac{(+2)(550 \times 10^{-9} \text{ m})}{2.00 \times 10^{-6} \text{ m}}\right) = +33.4^\circ.$$

- b. The halfway point between θ_1 and θ_2 is
 $\theta = (\theta_1 + \theta_2)/2 = (16.0^\circ + 33.4^\circ)/2 = 24.7^\circ.$

Equation 4.2 gives

$$\beta = \frac{\pi a \sin \theta}{\lambda} = \frac{\pi (2.00 \times 10^{-6} \text{ m}) \sin (24.7^\circ)}{(550 \times 10^{-9} \text{ m})} = 1.52\pi \text{ or } 4.77 \text{ rad}.$$

From Equation 4.4, we can calculate

$$\frac{I}{I_o} = \left(\frac{\sin \beta}{\beta}\right)^2 = \left(\frac{\sin (4.77)}{4.77}\right)^2 = \left(\frac{-0.9985}{4.77}\right)^2 = 0.044.$$

Significance

This position, halfway between two minima, is very close to the location of the maximum, expected near $\beta = 3\pi/2$, or 1.5π .

✓ CHECK YOUR UNDERSTANDING 4.2

For the experiment in Example 4.2, at what angle from the center is the third maximum and what is its intensity relative to the central maximum?

If the slit width a is varied, the intensity distribution changes, as illustrated in Figure 4.10. The central peak is distributed over the region from $\sin \theta = -\lambda/a$ to $\sin \theta = +\lambda/a$. For small θ , this corresponds to an angular width $\Delta\theta \approx 2\lambda/a$. Hence, an increase in the slit width results in a decrease in the **width of the central peak**. For a slit with $a \gg \lambda$, the central peak is very sharp, whereas if $a \approx \lambda$, it becomes quite broad.

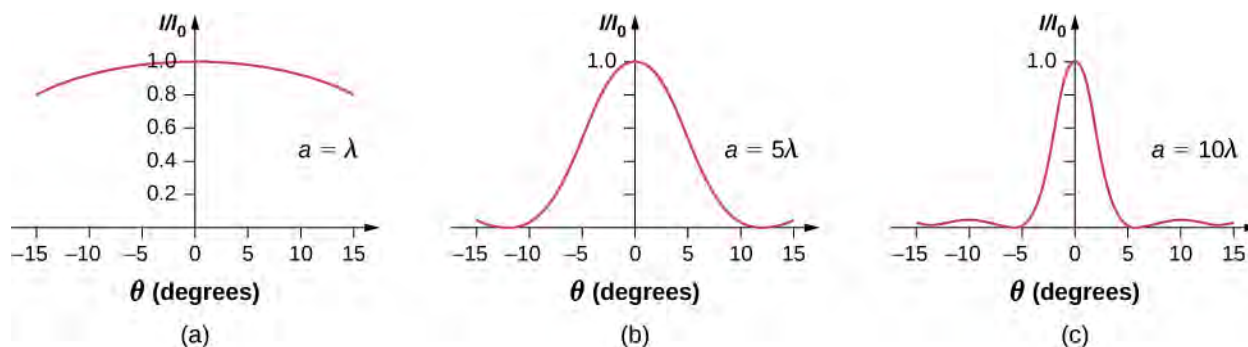


Figure 4.10 Single-slit diffraction patterns for various slit widths. As the slit width a increases from $a = \lambda$ to 5λ and then to 10λ , the width of the central peak decreases as the angles for the first minima decrease as predicted by Equation 4.1.

INTERACTIVE

A diffraction experiment in optics can require a lot of preparation but [this simulation \(https://openstax.org/l/21diffrexpoptsi\)](https://openstax.org/l/21diffrexpoptsi) by Andrew Duffy offers not only a quick set up but also the ability to change the slit width instantly. Run the simulation and select “Single slit.” You can adjust the slit width and see the effect on the diffraction pattern on a screen and as a graph.

4.3 Double-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Describe the combined effect of interference and diffraction with two slits, each with finite width
- Determine the relative intensities of interference fringes within a diffraction pattern
- Identify missing orders, if any

When we studied interference in Young’s double-slit experiment, we ignored the diffraction effect in each slit. We assumed that the slits were so narrow that on the screen you saw only the interference of light from just two point sources. If the slit is smaller than the wavelength, then [Figure 4.10\(a\)](#) shows that there is just a spreading of light and no peaks or troughs on the screen. Therefore, it was reasonable to leave out the diffraction effect in that chapter. However, if you make the slit wider, [Figure 4.10\(b\)](#) and (c) show that you cannot ignore diffraction. In this section, we study the complications to the double-slit experiment that arise when you also need to take into account the diffraction effect of each slit.

To calculate the diffraction pattern for two (or any number of) slits, we need to generalize the method we just used for a single slit. That is, across each slit, we place a uniform distribution of point sources that radiate Huygens wavelets, and then we sum the wavelets from all the slits. This gives the intensity at any point on the screen. Although the details of that calculation can be complicated, the final result is quite simple:

Two-Slit Diffraction Pattern

The diffraction pattern of two slits of width a that are separated by a distance d is the interference pattern of two point sources separated by d multiplied by the diffraction pattern of a slit of width a .

In other words, the *locations* of the interference fringes are given by the equation $d \sin \theta = m\lambda$, the same as when we considered the slits to be point sources, but the *intensities* of the fringes are now reduced by diffraction effects, according to [Equation 4.4](#). [Note that in the chapter on interference, we wrote $d \sin \theta = m\lambda$ and used the integer m to refer to interference fringes. [Equation 4.1](#) also uses m , but this time to refer to diffraction minima. If both equations are used simultaneously, it is good practice to use a different variable (such as n) for one of these integers in order to keep them distinct.]

Interference and diffraction effects operate simultaneously and generally produce minima at different angles. This gives rise to a complicated pattern on the screen, in which some of the maxima of interference from the two slits are missing if the maximum of the interference is in the same direction as the minimum of the diffraction. We refer to such a missing peak as a **missing order**. One example of a diffraction pattern on the screen is shown in [Figure 4.11](#). The solid line with multiple peaks of various heights is the intensity observed on the screen. It is a product of the interference pattern of waves from separate slits and the diffraction of waves from within one slit.

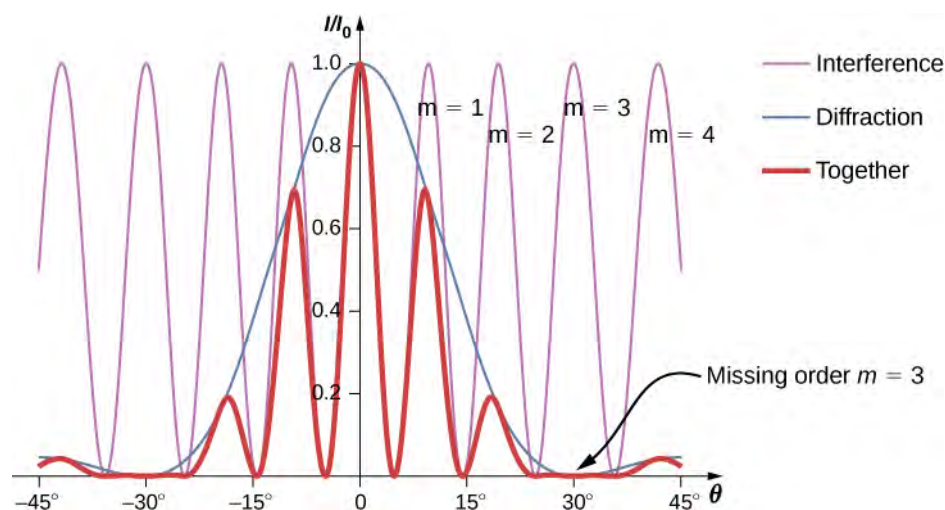


Figure 4.11 Diffraction from a double slit. The purple line with peaks of the same height are from the interference of the waves from two slits; the blue line with one big hump in the middle is the diffraction of waves from within one slit; and the thick red line is the product of the two, which is the pattern observed on the screen. The plot shows the expected result for a slit width $a = 2\lambda$ and slit separation $d = 6\lambda$. The maximum of $m = \pm 3$ order for the interference is missing because the minimum of the diffraction occurs in the same direction.

EXAMPLE 4.3

Intensity of the Fringes

Figure 4.11 shows that the intensity of the fringe for $m = 3$ is zero, but what about the other fringes? Calculate the intensity for the fringe at $m = 1$ relative to I_0 , the intensity of the central peak.

Strategy

Determine the angle for the double-slit interference fringe, using the equation from [Interference](#), then determine the relative intensity in that direction due to diffraction by using [Equation 4.4](#).

Solution

From the chapter on interference, we know that the bright interference fringes occur at $d \sin \theta = m\lambda$, or

$$\sin \theta = \frac{m\lambda}{d}.$$

From [Equation 4.4](#),

$$I = I_0 \left(\frac{\sin \beta}{\beta} \right)^2, \text{ where } \beta = \frac{\phi}{2} = \frac{\pi a \sin \theta}{\lambda}.$$

Substituting from above,

$$\beta = \frac{\pi a \sin \theta}{\lambda} = \frac{\pi a}{\lambda} \cdot \frac{m\lambda}{d} = \frac{m\pi a}{d}.$$

For $a = 2\lambda$, $d = 6\lambda$, and $m = 1$,

$$\beta = \frac{(1)\pi(2\lambda)}{(6\lambda)} = \frac{\pi}{3}.$$

Then, the intensity is

$$I = I_0 \left(\frac{\sin \beta}{\beta} \right)^2 = I_0 \left(\frac{\sin (\pi/3)}{\pi/3} \right)^2 = 0.684 I_0.$$

Significance

Note that this approach is relatively straightforward and gives a result that is almost exactly the same as the more complicated analysis using phasors to work out the intensity values of the double-slit interference (thin line in [Figure 4.11](#)). The phasor approach accounts for the downward slope in the diffraction intensity (blue line) so that the peak *near* $m = 1$ occurs at a value of θ ever so slightly smaller than we have shown here.

**EXAMPLE 4.4****Two-Slit Diffraction**

Suppose that in Young's experiment, slits of width 0.020 mm are separated by 0.20 mm. If the slits are illuminated by monochromatic light of wavelength 500 nm, how many bright fringes are observed in the central peak of the diffraction pattern?

Solution

From [Equation 4.1](#), the angular position of the first diffraction minimum is

$$\theta \approx \sin \theta = \frac{\lambda}{a} = \frac{5.0 \times 10^{-7} \text{ m}}{2.0 \times 10^{-5} \text{ m}} = 2.5 \times 10^{-2} \text{ rad.}$$

Using $d \sin \theta = m\lambda$ for $\theta = 2.5 \times 10^{-2}$ rad, we find

$$m = \frac{d \sin \theta}{\lambda} = \frac{(0.20 \text{ mm})(2.5 \times 10^{-2} \text{ rad})}{(5.0 \times 10^{-7} \text{ m})} = 10,$$

which is the maximum interference order that fits inside the central peak. We note that $m = \pm 10$ are missing orders as θ matches exactly. Accordingly, we observe bright fringes for

$$m = -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6, +7, +8, \text{ and } +9$$

for a total of 19 bright fringes.

**CHECK YOUR UNDERSTANDING 4.3**

For the experiment in [Example 4.4](#), show that $m = 20$ is also a missing order.

**INTERACTIVE**

Explore the effects of double-slit diffraction. In [this simulation \(https://openstax.org/l/21doubleslitdiff\)](https://openstax.org/l/21doubleslitdiff) written by Fu-Kwun Hwang, select $N = 2$ using the slider and see what happens when you control the slit width, slit separation and the wavelength. Can you make an order go “missing?”

4.4 Diffraction Gratings

Learning Objectives

By the end of this section, you will be able to:

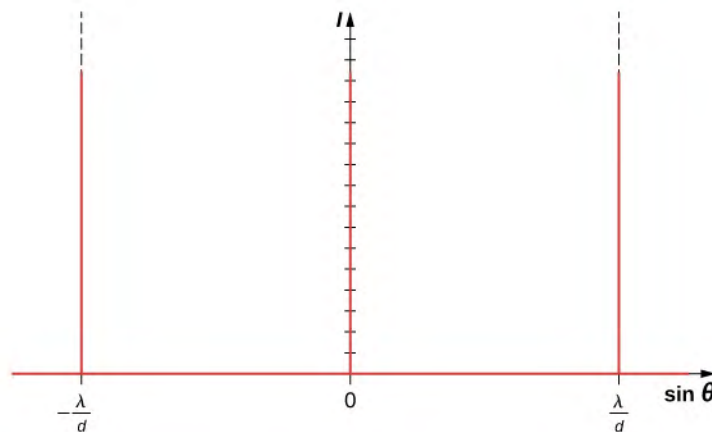
- Discuss the pattern obtained from diffraction gratings
- Explain diffraction grating effects

Analyzing the interference of light passing through two slits lays out the theoretical framework of interference and gives us a historical insight into Thomas Young's experiments. However, most modern-day applications of slit interference use not just two slits but many, approaching infinity for practical purposes. The key optical element is called a diffraction grating, an important tool in optical analysis.

Diffraction Gratings: An Infinite Number of Slits

The analysis of multi-slit interference in [Interference](#) allows us to consider what happens when the number of

slits N approaches infinity. Recall that $N-2$ secondary maxima appear between the principal maxima. We can see there will be an infinite number of secondary maxima that appear, and an infinite number of dark fringes between them. This makes the spacing between the fringes, and therefore the width of the maxima, infinitesimally small. Furthermore, because the intensity of the secondary maxima is proportional to $1/N^2$, it approaches zero so that the secondary maxima are no longer seen. What remains are only the principal maxima, now very bright and very narrow (Figure 4.12).



(a)



(b)

Figure 4.12 (a) Intensity of light transmitted through a large number of slits. When N approaches infinity, only the principal maxima remain as very bright and very narrow lines. (b) A laser beam passed through a diffraction grating. (credit b: modification of work by Sebastian Stapelberg)

In reality, the number of slits is not infinite, but it can be very large—large enough to produce the equivalent effect. A prime example is an optical element called a **diffraction grating**. A diffraction grating can be manufactured by carving glass with a sharp tool in a large number of precisely positioned parallel lines, with untouched regions acting like slits (Figure 4.13). This type of grating can be photographically mass produced rather cheaply. Because there can be over 1000 lines per millimeter across the grating, when a section as small as a few millimeters is illuminated by an incoming ray, the number of illuminated slits is effectively infinite, providing for very sharp principal maxima.

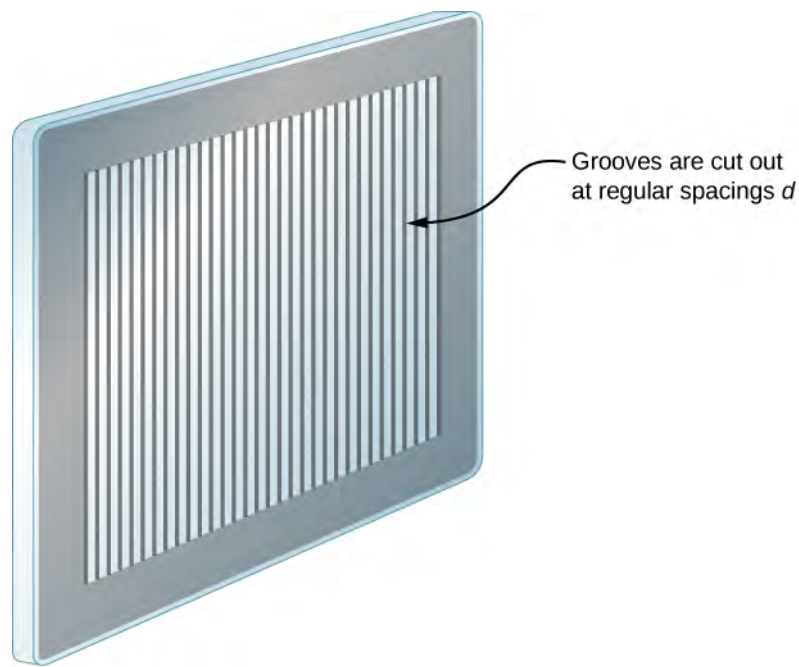


Figure 4.13 A diffraction grating can be manufactured by carving glass with a sharp tool in a large number of precisely positioned parallel lines.

Diffraction gratings work both for transmission of light, as in [Figure 4.14](#), and for reflection of light, as on butterfly wings and the Australian opal in [Figure 4.15](#). Natural diffraction gratings also occur in the feathers of certain birds such as the hummingbird. Tiny, finger-like structures in regular patterns act as reflection gratings, producing constructive interference that gives the feathers colors not solely due to their pigmentation. This is called iridescence.

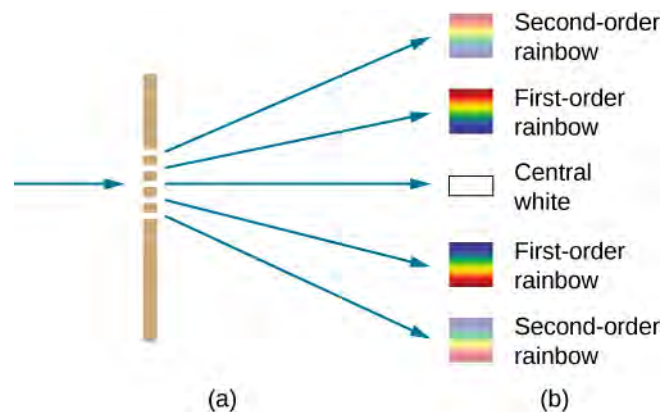


Figure 4.14 (a) Light passing through a diffraction grating is diffracted in a pattern similar to a double slit, with bright regions at various angles. (b) The pattern obtained for white light incident on a grating. The central maximum is white, and the higher-order maxima disperse white light into a rainbow of colors.

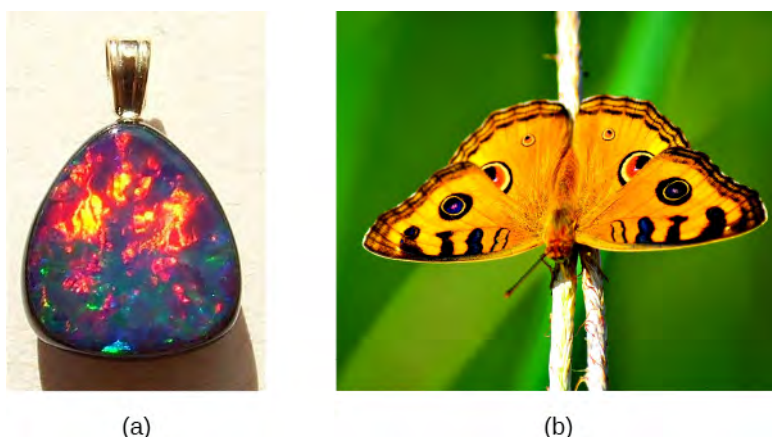


Figure 4.15 (a) This Australian opal and (b) butterfly wings have rows of reflectors that act like reflection gratings, reflecting different colors at different angles. (credit a: modification of work by "Opals-On-Black"/Flickr; credit b: modification of work by "whologwhy"/Flickr)

Applications of Diffraction Gratings

Where are diffraction gratings used in applications? Diffraction gratings are commonly used for spectroscopic dispersion and analysis of light. What makes them particularly useful is the fact that they form a sharper pattern than double slits do. That is, their bright fringes are narrower and brighter while their dark regions are darker. Diffraction gratings are key components of monochromators used, for example, in optical imaging of particular wavelengths from biological or medical samples. A diffraction grating can be chosen to specifically analyze a wavelength emitted by molecules in diseased cells in a biopsy sample or to help excite strategic molecules in the sample with a selected wavelength of light. Another vital use is in optical fiber technologies where fibers are designed to provide optimum performance at specific wavelengths. A range of diffraction gratings are available for selecting wavelengths for such use.



EXAMPLE 4.5

Calculating Typical Diffraction Grating Effects

Diffraction gratings with 10,000 lines per centimeter are readily available. Suppose you have one, and you send a beam of white light through it to a screen 2.00 m away. (a) Find the angles for the first-order diffraction of the shortest and longest wavelengths of visible light (380 and 760 nm, respectively). (b) What is the distance between the ends of the rainbow of visible light produced on the screen for first-order interference? (See [Figure 4.16](#).)

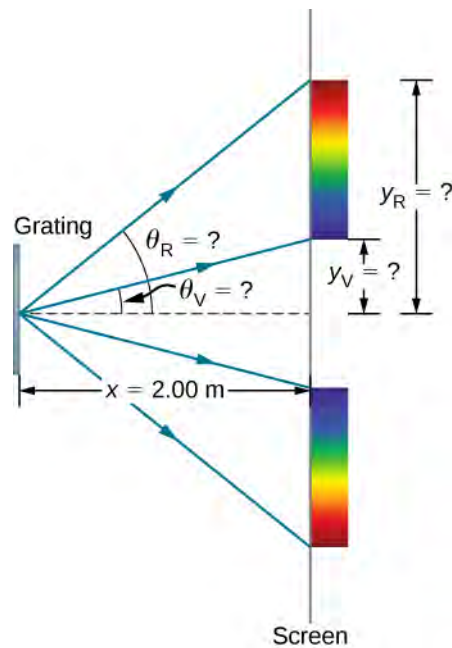


Figure 4.16 (a) The diffraction grating considered in this example produces a rainbow of colors on a screen a distance $x = 2.00$ m from the grating. The distances along the screen are measured perpendicular to the x -direction. In other words, the rainbow pattern extends out of the page.

(b) In a bird's-eye view, the rainbow pattern can be seen on a table where the equipment is placed.

Strategy

Once a value for the diffraction grating's slit spacing d has been determined, the angles for the sharp lines can be found using the equation

$$d \sin \theta = m\lambda \text{ for } m = 0, \pm 1, \pm 2, \dots$$

Since there are 10,000 lines per centimeter, each line is separated by $1/10,000$ of a centimeter. Once we know the angles, we can find the distances along the screen by using simple trigonometry.

Solution

- a. The distance between slits is $d = (1 \text{ cm})/10,000 = 1.00 \times 10^{-4} \text{ cm}$ or $1.00 \times 10^{-6} \text{ m}$. Let us call the two angles θ_V for violet (380 nm) and θ_R for red (760 nm). Solving the equation $d \sin \theta_V = m\lambda$ for $\sin \theta_V$,

$$\sin \theta_V = \frac{m\lambda_V}{d},$$

where $m = 1$ for the first-order and $\lambda_V = 380 \text{ nm} = 3.80 \times 10^{-7} \text{ m}$. Substituting these values gives

$$\sin \theta_V = \frac{3.80 \times 10^{-7} \text{ m}}{1.00 \times 10^{-6} \text{ m}} = 0.380.$$

Thus the angle θ_V is

$$\theta_V = \sin^{-1} 0.380 = 22.33^\circ.$$

Similarly,

$$\sin \theta_R = \frac{7.60 \times 10^{-7} \text{ m}}{1.00 \times 10^{-6} \text{ m}} = 0.760.$$

Thus the angle θ_R is

$$\theta_R = \sin^{-1} 0.760 = 49.46^\circ.$$

Notice that in both equations, we reported the results of these intermediate calculations to four significant figures to use with the calculation in part (b).

- b. The distances on the screen are labeled y_V and y_R in [Figure 4.16](#). Notice that $\tan \theta = y/x$. We can solve for y_V and y_R . That is,

$$y_V = x \tan \theta_V = (2.00 \text{ m})(\tan 22.33^\circ) = 0.815 \text{ m}$$

and

$$y_R = x \tan \theta_R = (2.00 \text{ m})(\tan 49.46^\circ) = 2.338 \text{ m}.$$

The distance between them is therefore

$$y_R - y_V = 1.523 \text{ m}.$$

Significance

The large distance between the red and violet ends of the rainbow produced from the white light indicates the potential this diffraction grating has as a spectroscopic tool. The more it can spread out the wavelengths (greater dispersion), the more detail can be seen in a spectrum. This depends on the quality of the diffraction grating—it must be very precisely made in addition to having closely spaced lines.

✓ CHECK YOUR UNDERSTANDING 4.4

If the line spacing of a diffraction grating d is not precisely known, we can use a light source with a well-determined wavelength to measure it. Suppose the first-order constructive fringe of the H_β emission line of hydrogen ($\lambda = 656.3 \text{ nm}$) is measured at 11.36° using a spectrometer with a diffraction grating. What is the line spacing of this grating?

🔗 INTERACTIVE

Take [the same simulation \(https://openstax.org/l/21doubleslitdiff\)](https://openstax.org/l/21doubleslitdiff) we used for double-slit diffraction and try increasing the number of slits from $N = 2$ to $N = 3, 4, 5, \dots$. The primary peaks become sharper, and the secondary peaks become less and less pronounced. By the time you reach the maximum number of $N = 20$, the system is behaving much like a diffraction grating.

4.5 Circular Apertures and Resolution

Learning Objectives

By the end of this section, you will be able to:

- Describe the diffraction limit on resolution
- Describe the diffraction limit on beam propagation

Light diffracts as it moves through space, bending around obstacles, interfering constructively and destructively. This can be used as a spectroscopic tool—a diffraction grating disperses light according to wavelength, for example, and is used to produce spectra—but diffraction also limits the detail we can obtain in images.

[Figure 4.17\(a\)](#) shows the effect of passing light through a small circular aperture. Instead of a bright spot with sharp edges, we obtain a spot with a fuzzy edge surrounded by circles of light. This pattern is caused by diffraction, similar to that produced by a single slit. Light from different parts of the circular aperture interferes constructively and destructively. The effect is most noticeable when the aperture is small, but the effect is there for large apertures as well.

CHAPTER REVIEW

Key Terms

Bragg planes families of planes within crystals that can give rise to X-ray diffraction

destructive interference for a single slit occurs when the width of the slit is comparable to the wavelength of light illuminating it

diffraction bending of a wave around the edges of an opening or an obstacle

diffraction grating large number of evenly spaced parallel slits

diffraction limit fundamental limit to resolution due to diffraction

hologram three-dimensional image recorded on film by lasers; the word hologram means *entire picture* (from the Greek word *holo*, as in holistic)

holography process of producing holograms with the use of lasers

missing order interference maximum that is not seen because it coincides with a diffraction

minimum

Rayleigh criterion two images are just-resolvable when the center of the diffraction pattern of one is directly over the first minimum of the diffraction pattern of the other

resolution ability, or limit thereof, to distinguish small details in images

two-slit diffraction pattern diffraction pattern of two slits of width D that are separated by a distance d is the interference pattern of two point sources separated by d multiplied by the diffraction pattern of a slit of width D

width of the central peak angle between the minimum for $m = 1$ and $m = -1$

X-ray diffraction technique that provides the detailed information about crystallographic structure of natural and manufactured materials

Key Equations

Destructive interference for a single slit $a \sin \theta = m\lambda$ for $m = \pm 1, \pm 2, \pm 3, \dots$

Half phase angle $\beta = \frac{\phi}{2} = \frac{\pi a \sin \theta}{\lambda}$

Field amplitude in the diffraction pattern $E = N \Delta E_0 \frac{\sin \beta}{\beta}$

Intensity in the diffraction pattern $I = I_0 \left(\frac{\sin \beta}{\beta} \right)^2$

Rayleigh criterion for circular apertures $\theta = 1.22 \frac{\lambda}{D}$

Bragg equation $m\lambda = 2d \sin \theta$, $m = 1, 2, 3, \dots$

Summary

4.1 Single-Slit Diffraction

- Diffraction can send a wave around the edges of an opening or other obstacle.
- A single slit produces an interference pattern characterized by a broad central maximum with narrower and dimmer maxima to the sides.

4.2 Intensity in Single-Slit Diffraction

- The intensity pattern for diffraction due to a single slit can be calculated using phasors as

$$I = I_0 \left(\frac{\sin \beta}{\beta} \right)^2,$$

where $\beta = \frac{\phi}{2} = \frac{\pi a \sin \theta}{\lambda}$, a is the slit width, λ is the wavelength, and θ is the angle from the central peak.

4.3 Double-Slit Diffraction

- With real slits with finite widths, the effects of interference and diffraction operate simultaneously to form a complicated intensity

pattern.

- Relative intensities of interference fringes within a diffraction pattern can be determined.
- Missing orders occur when an interference maximum and a diffraction minimum are located together.

4.4 Diffraction Gratings

- A diffraction grating consists of a large number of evenly spaced parallel slits that produce an interference pattern similar to but sharper than that of a double slit.
- Constructive interference occurs when $d \sin \theta = m\lambda$ for $m = 0, \pm 1, \pm 2, \dots$, where d is the distance between the slits, θ is the angle relative to the incident direction, and m is the order of the interference.

4.5 Circular Apertures and Resolution

- Diffraction limits resolution.

Conceptual Questions

4.1 Single-Slit Diffraction

1. As the width of the slit producing a single-slit diffraction pattern is reduced, how will the diffraction pattern produced change?
2. Compare interference and diffraction.
3. If you and a friend are on opposite sides of a hill, you can communicate with walkie-talkies but not with flashlights. Explain.
4. What happens to the diffraction pattern of a single slit when the entire optical apparatus is immersed in water?
5. In our study of diffraction by a single slit, we assume that the length of the slit is much larger than the width. What happens to the diffraction pattern if these two dimensions were comparable?
6. A rectangular slit is twice as wide as it is high. Is the central diffraction peak wider in the vertical direction or in the horizontal direction?

4.2 Intensity in Single-Slit Diffraction

7. In Equation 4.4, the parameter β looks like an angle but is not an angle that you can measure with a protractor in the physical world. Explain what β represents.

4.3 Double-Slit Diffraction

8. Shown below is the central part of the interference pattern for a pure wavelength of red light projected

- The Rayleigh criterion states that two images are just resolvable when the center of the diffraction pattern of one is directly over the first minimum of the diffraction pattern of the other.

4.6 X-Ray Diffraction

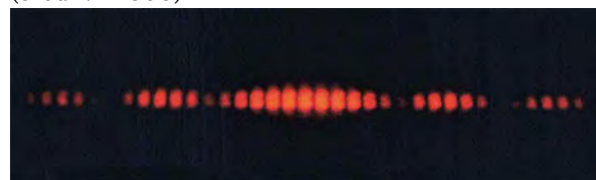
- X-rays are relatively short-wavelength EM radiation and can exhibit wave characteristics such as interference when interacting with correspondingly small objects.

4.7 Holography

- Holography is a technique based on wave interference to record and form three-dimensional images.
- Lasers offer a practical way to produce sharp holographic images because of their monochromatic and coherent light for pronounced interference patterns.

onto a double slit. The pattern is actually a combination of single- and double-slit interference. Note that the bright spots are evenly spaced. Is this a double- or single-slit characteristic? Note that some of the bright spots are dim on either side of the center. Is this a single- or double-slit characteristic? Which is smaller, the slit width or the separation between slits? Explain your responses.

(credit: PASCO)



4.5 Circular Apertures and Resolution

9. Is higher resolution obtained in a microscope with red or blue light? Explain your answer.
10. The resolving power of refracting telescope increases with the size of its objective lens. What other advantage is gained with a larger lens?
11. The distance between atoms in a molecule is about 10^{-8} cm. Can visible light be used to “see” molecules?
12. A beam of light always spreads out. Why can a beam not be created with parallel rays to prevent spreading? Why can lenses, mirrors, or apertures not be used to correct the spreading?

4.6 X-Ray Diffraction

13. Crystal lattices can be examined with X-rays but not UV. Why?

4.7 Holography

14. How can you tell that a hologram is a true three-dimensional image and that those in three-dimensional movies are not?

Problems

4.1 Single-Slit Diffraction

17. (a) At what angle is the first minimum for 550-nm light falling on a single slit of width $1.00\mu\text{m}$? (b) Will there be a second minimum?
18. (a) Calculate the angle at which a $2.00\text{-}\mu\text{m}$ -wide slit produces its first minimum for 410-nm violet light. (b) Where is the first minimum for 700-nm red light?
19. (a) How wide is a single slit that produces its first minimum for 633-nm light at an angle of 28.0° ? (b) At what angle will the second minimum be?
20. (a) What is the width of a single slit that produces its first minimum at 60.0° for 600-nm light? (b) Find the wavelength of light that has its first minimum at 62.0° .
21. Find the wavelength of light that has its third minimum at an angle of 48.6° when it falls on a single slit of width $3.00\mu\text{m}$.
22. (a) Sodium vapor light averaging 589 nm in wavelength falls on a single slit of width $7.50\mu\text{m}$. At what angle does it produce its second minimum? (b) What is the highest-order minimum produced?
23. Consider a single-slit diffraction pattern for $\lambda = 589\text{ nm}$, projected on a screen that is 1.00 m from a slit of width 0.25 mm. How far from the center of the pattern are the centers of the first and second dark fringes?
24. (a) Find the angle between the first minima for the two sodium vapor lines, which have wavelengths of 589.1 and 589.6 nm, when they fall upon a single slit of width $2.00\mu\text{m}$. (b) What is the distance between these minima if the diffraction pattern falls on a screen 1.00 m from the slit? (c) Discuss the ease or difficulty of measuring such a distance.
25. (a) What is the minimum width of a single slit (in multiples of λ) that will produce a first minimum for a wavelength λ ? (b) What is its minimum width if it produces 50 minima? (c) 1000 minima?
26. (a) If a single slit produces a first minimum at 14.5° , at what angle is the second-order minimum? (b) What is the angle of the third-order minimum? (c) Is there a fourth-order minimum? (d) Use your answers to illustrate how the angular width of the central maximum is about twice the angular width of the next maximum (which is the angle between the first and second minima).
27. If the separation between the first and the second minima of a single-slit diffraction pattern is 6.0 mm, what is the distance between the screen and the slit? The light wavelength is 500 nm and the slit width is 0.16 mm.
28. A water break at the entrance to a harbor consists of a rock barrier with a 50.0-m-wide opening. Ocean waves of 20.0-m wavelength approach the opening straight on. At what angles to the incident direction are the boats inside the harbor most protected against wave action?
29. An aircraft maintenance technician walks past a tall hangar door that acts like a single slit for sound entering the hangar. Outside the door, on a line perpendicular to the opening in the door, a jet engine makes a 600-Hz sound. At what angle with the door will the technician observe the first minimum in sound intensity if the vertical opening is 0.800 m wide and the speed of sound is 340 m/s?

4.2 Intensity in Single-Slit Diffraction

30. A single slit of width $3.0\mu\text{m}$ is illuminated by a sodium yellow light of wavelength 589 nm. Find the intensity at a 15° angle to the axis in terms of the intensity of the central maximum.
31. A single slit of width 0.1 mm is illuminated by a mercury light of wavelength 576 nm. Find the intensity at a 10° angle to the axis in terms of the intensity of the central maximum.

- 32.** The width of the central peak in a single-slit diffraction pattern is 5.0 mm. The wavelength of the light is 600 nm, and the screen is 2.0 m from the slit. (a) What is the width of the slit? (b) Determine the ratio of the intensity at 4.5 mm from the center of the pattern to the intensity at the center.
- 33.** Consider the single-slit diffraction pattern for $\lambda = 600$ nm, $a = 0.025$ mm, and $x = 2.0$ m. Find the intensity in terms of I_o at $\theta = 0.5^\circ$, 1.0° , 1.5° , 3.0° , and 10.0° .

4.3 Double-Slit Diffraction

- 34.** Two slits of width $2\ \mu\text{m}$, each in an opaque material, are separated by a center-to-center distance of $6\ \mu\text{m}$. A monochromatic light of wavelength 450 nm is incident on the double-slit. One finds a combined interference and diffraction pattern on the screen.
- (a) How many peaks of the interference will be observed in the central maximum of the diffraction pattern?
- (b) How many peaks of the interference will be observed if the slit width is doubled while keeping the distance between the slits same?
- (c) How many peaks of interference will be observed if the slits are separated by twice the distance, that is, $12\ \mu\text{m}$, while keeping the widths of the slits same?
- (d) What will happen in (a) if instead of 450-nm light another light of wavelength 680 nm is used?
- (e) What is the value of the ratio of the intensity of the central peak to the intensity of the next bright peak in (a)?
- (f) Does this ratio depend on the wavelength of the light?
- (g) Does this ratio depend on the width or separation of the slits?
- 35.** A double slit produces a diffraction pattern that is a combination of single- and double-slit interference. Find the ratio of the width of the slits to the separation between them, if the first minimum of the single-slit pattern falls on the fifth maximum of the double-slit pattern. (This will greatly reduce the intensity of the fifth maximum.)
- 36.** For a double-slit configuration where the slit separation is four times the slit width, how many interference fringes lie in the central peak of the diffraction pattern?
- 37.** Light of wavelength 500 nm falls normally on 50 slits that are 2.5×10^{-3} mm wide and spaced

5.0×10^{-3} mm apart. How many interference fringes lie in the central peak of the diffraction pattern?

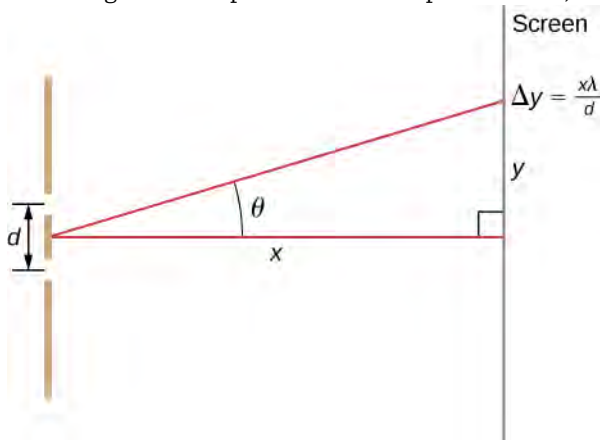
- 38.** A monochromatic light of wavelength 589 nm incident on a double slit with slit width $2.5\ \mu\text{m}$ and unknown separation results in a diffraction pattern containing nine interference peaks inside the central maximum. Find the separation of the slits.
- 39.** When a monochromatic light of wavelength 430 nm incident on a double slit of slit separation $5\ \mu\text{m}$, there are 11 interference fringes in its central maximum. How many interference fringes will be in the central maximum of a light of the same wavelength and slit widths, but a new slit separation of $4\ \mu\text{m}$?
- 40.** Determine the intensities of two interference peaks other than the central peak in the central maximum of the diffraction, if possible, when a light of wavelength 628 nm is incident on a double slit of width 500 nm and separation 1500 nm. Use the intensity of the central spot to be $1\ \text{mW}/\text{cm}^2$.

4.4 Diffraction Gratings

- 41.** A diffraction grating has 2000 lines per centimeter. At what angle will the first-order maximum be for 520-nm-wavelength green light?
- 42.** Find the angle for the third-order maximum for 580-nm-wavelength yellow light falling on a diffraction grating having 1500 lines per centimeter.
- 43.** How many lines per centimeter are there on a diffraction grating that gives a first-order maximum for 470-nm blue light at an angle of 25.0° ?
- 44.** What is the distance between lines on a diffraction grating that produces a second-order maximum for 760-nm red light at an angle of 60.0° ?
- 45.** Calculate the wavelength of light that has its second-order maximum at 45.0° when falling on a diffraction grating that has 5000 lines per centimeter.
- 46.** An electric current through hydrogen gas produces several distinct wavelengths of visible light. What are the wavelengths of the hydrogen spectrum, if they form first-order maxima at angles 24.2° , 25.7° , 29.1° , and 41.0° when projected on a diffraction grating having 10,000 lines per centimeter?
- 47.** (a) What do the four angles in the preceding

problem become if a 5000-line per centimeter diffraction grating is used? (b) Using this grating, what would the angles be for the second-order maxima? (c) Discuss the relationship between integral reductions in lines per centimeter and the new angles of various order maxima.

48. What is the spacing between structures in a feather that acts as a reflection grating, giving that they produce a first-order maximum for 525-nm light at a 30.0° angle?
49. An opal such as that shown in [Figure 4.15](#) acts like a reflection grating with rows separated by about $8\ \mu\text{m}$. If the opal is illuminated normally, (a) at what angle will red light be seen and (b) at what angle will blue light be seen?
50. At what angle does a diffraction grating produce a second-order maximum for light having a first-order maximum at 20.0° ?
51. (a) Find the maximum number of lines per centimeter a diffraction grating can have and produce a maximum for the smallest wavelength of visible light. (b) Would such a grating be useful for ultraviolet spectra? (c) For infrared spectra?
52. (a) Show that a 30,000 line per centimeter grating will not produce a maximum for visible light. (b) What is the longest wavelength for which it does produce a first-order maximum? (c) What is the greatest number of line per centimeter a diffraction grating can have and produce a complete second-order spectrum for visible light?
53. The analysis shown below also applies to diffraction gratings with lines separated by a distance d . What is the distance between fringes produced by a diffraction grating having 125 lines per centimeter for 600-nm light, if the screen is 1.50 m away? (*Hint:* The distance between adjacent fringes is $\Delta y = x\lambda/d$, assuming the slit separation d is comparable to λ .)



4.5 Circular Apertures and Resolution

54. The 305-m-diameter Arecibo radio telescope pictured in [Figure 4.20](#) detects radio waves with a 4.00-cm average wavelength. (a) What is the angle between two just-resolvable point sources for this telescope? (b) How close together could these point sources be at the 2 million light-year distance of the Andromeda Galaxy?
55. Assuming the angular resolution found for the Hubble Telescope in [Example 4.6](#), what is the smallest detail that could be observed on the moon?
56. Diffraction spreading for a flashlight is insignificant compared with other limitations in its optics, such as spherical aberrations in its mirror. To show this, calculate the minimum angular spreading of a flashlight beam that is originally 5.00 cm in diameter with an average wavelength of 600 nm.
57. (a) What is the minimum angular spread of a 633-nm wavelength He-Ne laser beam that is originally 1.00 mm in diameter? (b) If this laser is aimed at a mountain cliff 15.0 km away, how big will the illuminated spot be? (c) How big a spot would be illuminated on the moon, neglecting atmospheric effects? (This might be done to hit a corner reflector to measure the round-trip time and, hence, distance.)
58. A telescope can be used to enlarge the diameter of a laser beam and limit diffraction spreading. The laser beam is sent through the telescope in opposite the normal direction and can then be projected onto a satellite or the moon. (a) If this is done with the Mount Wilson telescope, producing a 2.54-m-diameter beam of 633-nm light, what is the minimum angular spread of the beam? (b) Neglecting atmospheric effects, what is the size of the spot this beam would make on the moon, assuming a lunar distance of $3.84 \times 10^8\ \text{m}$?
59. The limit to the eye's acuity is actually related to diffraction by the pupil. (a) What is the angle between two just-resolvable points of light for a 3.00-mm-diameter pupil, assuming an average wavelength of 550 nm? (b) Take your result to be the practical limit for the eye. What is the greatest possible distance a car can be from you if you can resolve its two headlights, given they are 1.30 m apart? (c) What is the distance between two just-resolvable points held at an arm's length (0.800 m) from your eye? (d) How does your answer to (c) compare to details you normally observe in everyday circumstances?

60. What is the minimum diameter mirror on a telescope that would allow you to see details as small as 5.00 km on the moon some 384,000 km away? Assume an average wavelength of 550 nm for the light received.
61. Find the radius of a star's image on the retina of an eye if its pupil is open to 0.65 cm and the distance from the pupil to the retina is 2.8 cm. Assume $\lambda = 550$ nm.
62. (a) The dwarf planet Pluto and its moon, Charon, are separated by 19,600 km. Neglecting atmospheric effects, should the 5.08-m-diameter Palomar Mountain telescope be able to resolve these bodies when they are 4.50×10^9 km from Earth? Assume an average wavelength of 550 nm. (b) In actuality, it is just barely possible to discern that Pluto and Charon are separate bodies using a ground-based telescope. What are the reasons for this?
63. A spy satellite orbits Earth at a height of 180 km. What is the minimum diameter of the objective lens in a telescope that must be used to resolve columns of troops marching 2.0 m apart? Assume $\lambda = 550$ nm.
64. What is the minimum angular separation of two stars that are just-resolvable by the 8.1-m Gemini South telescope, if atmospheric effects do not limit resolution? Use 550 nm for the wavelength of the light from the stars.
65. The headlights of a car are 1.3 m apart. What is the maximum distance at which the eye can resolve these two headlights? Take the pupil diameter to be 0.40 cm.
66. When dots are placed on a page from a laser printer, they must be close enough so that you do not see the individual dots of ink. To do this, the separation of the dots must be less than Rayleigh's criterion. Take the pupil of the eye to be 3.0 mm and the distance from the paper to the eye of 35 cm; find the minimum separation of two dots such that they cannot be resolved. How many dots per inch (dpi) does this correspond to?
67. Suppose you are looking down at a highway from a jetliner flying at an altitude of 6.0 km. How far apart must two cars be if you are able to distinguish them? Assume that $\lambda = 550$ nm and that the diameter of your pupils is 4.0 mm.
68. Can an astronaut orbiting Earth in a satellite at a distance of 180 km from the surface distinguish two skyscrapers that are 20 m apart? Assume that the pupils of the astronaut's eyes have a diameter of 5.0 mm and that most of the light is centered around 500 nm.
69. The characters of a stadium scoreboard are formed with closely spaced lightbulbs that radiate primarily yellow light. (Use $\lambda = 600$ nm.) How closely must the bulbs be spaced so that an observer 80 m away sees a display of continuous lines rather than the individual bulbs? Assume that the pupil of the observer's eye has a diameter of 5.0 mm.
70. If a microscope can accept light from objects at angles as large as $\alpha = 70^\circ$, what is the smallest structure that can be resolved when illuminated with light of wavelength 500 nm and (a) the specimen is in air? (b) When the specimen is immersed in oil, with index of refraction of 1.52?
71. A camera uses a lens with aperture 2.0 cm. What is the angular resolution of a photograph taken at 700 nm wavelength? Can it resolve the millimeter markings of a ruler placed 35 m away?

4.6 X-Ray Diffraction

72. X-rays of wavelength 0.103 nm reflect off a crystal and a second-order maximum is recorded at a Bragg angle of 25.5° . What is the spacing between the scattering planes in this crystal?
73. A first-order Bragg reflection maximum is observed when a monochromatic X-ray falls on a crystal at a 32.3° angle to a reflecting plane. What is the wavelength of this X-ray?
74. An X-ray scattering experiment is performed on a crystal whose atoms form planes separated by 0.440 nm. Using an X-ray source of wavelength 0.548 nm, what is the angle (with respect to the planes in question) at which the experimenter needs to illuminate the crystal in order to observe a first-order maximum?
75. The structure of the NaCl crystal forms reflecting planes 0.541 nm apart. What is the smallest angle, measured from these planes, at which X-ray diffraction can be observed, if X-rays of wavelength 0.085 nm are used?
76. On a certain crystal, a first-order X-ray diffraction maximum is observed at an angle of 27.1° relative to its surface, using an X-ray source of unknown wavelength. Additionally, when illuminated with a different, this time of known wavelength 0.137 nm, a second-order maximum is detected at 37.3° . Determine (a) the spacing between the reflecting planes, and (b) the unknown wavelength.

77. Calcite crystals contain scattering planes separated by 0.30 nm. What is the angular separation between first and second-order diffraction maxima when X-rays of 0.130 nm

wavelength are used?

78. The first-order Bragg angle for a certain crystal is 12.1° . What is the second-order angle?

Additional Problems

79. White light falls on two narrow slits separated by 0.40 mm. The interference pattern is observed on a screen 3.0 m away. (a) What is the separation between the first maxima for red light ($\lambda = 700$ nm) and violet light ($\lambda = 400$ nm)? (b) At what point nearest the central maximum will a maximum for yellow light ($\lambda = 600$ nm) coincide with a maximum for violet light? Identify the order for each maximum.
80. Microwaves of wavelength 10.0 mm fall normally on a metal plate that contains a slit 25 mm wide. (a) Where are the first minima of the diffraction pattern? (b) Would there be minima if the wavelength were 30.0 mm?
81. *Quasars*, or *quasi-stellar radio sources*, are astronomical objects discovered in 1960. They are distant but strong emitters of radio waves with angular size so small, they were originally unresolved, the same as stars. The quasar 3C405 is actually two discrete radio sources that subtend an angle of 82 arcsec. If this object is studied using radio emissions at a frequency of 410 MHz, what is the minimum diameter of a radio telescope that can resolve the two sources?
82. Two slits each of width 1800 nm and separated by the center-to-center distance of 1200 nm are illuminated by plane waves from a krypton ion laser-emitting at wavelength 461.9 nm. Find the number of interference peaks in the central diffraction peak.
83. A microwave of an unknown wavelength is incident on a single slit of width 6 cm. The angular width of the central peak is found to be 25° . Find the wavelength.
84. Red light (wavelength 632.8 nm in air) from a Helium-Neon laser is incident on a single slit of width 0.05 mm. The entire apparatus is immersed in water of refractive index 1.333. Determine the angular width of the central peak.
85. A light ray of wavelength 461.9 nm emerges from a 2-mm circular aperture of a krypton ion laser. Due to diffraction, the beam expands as it moves out. How large is the central bright spot at (a) 1 m, (b) 1 km, (c) 1000 km, and (d) at the surface of the moon at a distance of 400,000 km from Earth.
86. How far apart must two objects be on the moon to be distinguishable by eye if only the diffraction effects of the eye's pupil limit the resolution? Assume 550 nm for the wavelength of light, the pupil diameter 5.0 mm, and 400,000 km for the distance to the moon.
87. How far apart must two objects be on the moon to be resolvable by the 8.1-m-diameter Gemini North telescope at Mauna Kea, Hawaii, if only the diffraction effects of the telescope aperture limit the resolution? Assume 550 nm for the wavelength of light and 400,000 km for the distance to the moon.
88. A spy satellite is reputed to be able to resolve objects 10. cm apart while operating 197 km above the surface of Earth. What is the diameter of the aperture of the telescope if the resolution is only limited by the diffraction effects? Use 550 nm for light.
89. Monochromatic light of wavelength 530 nm passes through a horizontal single slit of width $1.5 \mu\text{m}$ in an opaque plate. A screen of dimensions $2.0 \text{ m} \times 2.0 \text{ m}$ is 1.2 m away from the slit. (a) Which way is the diffraction pattern spread out on the screen? (b) What are the angles of the minima with respect to the center? (c) What are the angles of the maxima? (d) How wide is the central bright fringe on the screen? (e) How wide is the next bright fringe on the screen?
90. A monochromatic light of unknown wavelength is incident on a slit of width $20 \mu\text{m}$. A diffraction pattern is seen at a screen 2.5 m away where the central maximum is spread over a distance of 10.0 cm. Find the wavelength.
91. A source of light having two wavelengths 550 nm and 600 nm of equal intensity is incident on a slit of width $1.8 \mu\text{m}$. Find the separation of the $m = 1$ bright spots of the two wavelengths on a screen 30.0 cm away.

92. A single slit of width 2100 nm is illuminated normally by a wave of wavelength 632.8 nm. Find the phase difference between waves from the top and one third from the bottom of the slit to a point on a screen at a horizontal distance of 2.0 m and vertical distance of 10.0 cm from the center.
93. A single slit of width $3.0 \mu\text{m}$ is illuminated by a sodium yellow light of wavelength 589 nm. Find the intensity at a 15° angle to the axis in terms of the intensity of the central maximum.
94. A single slit of width 0.10 mm is illuminated by a mercury lamp of wavelength 576 nm. Find the intensity at a 10° angle to the axis in terms of the intensity of the central maximum.
95. A diffraction grating produces a second maximum that is 89.7 cm from the central maximum on a screen 2.0 m away. If the grating has 600 lines per centimeter, what is the wavelength of the light that produces the diffraction pattern?
96. A grating with 4000 lines per centimeter is used to diffract light that contains all wavelengths between 400 and 650 nm. How wide is the first-order spectrum on a screen 3.0 m from the grating?
97. A diffraction grating with 2000 lines per centimeter is used to measure the wavelengths emitted by a hydrogen gas discharge tube. (a) At what angles will you find the maxima of the two first-order blue lines of wavelengths 410 and 434 nm? (b) The maxima of two other first-order lines are found at $\theta_1 = 0.097$ rad and $\theta_2 = 0.132$ rad. What are the wavelengths of these lines?
98. For white light ($400 \text{ nm} < \lambda < 700 \text{ nm}$) falling normally on a diffraction grating, show that the second and third-order spectra overlap no matter what the grating constant d is.
99. How many complete orders of the visible spectrum ($400 \text{ nm} < \lambda < 700 \text{ nm}$) can be produced with a diffraction grating that contains 5000 lines per centimeter?
100. Two lamps producing light of wavelength 589 nm are fixed 1.0 m apart on a wooden plank. What is the maximum distance an observer can be and still resolve the lamps as two separate sources of light, if the resolution is affected solely by the diffraction of light entering the eye? Assume light enters the eye through a pupil of diameter 4.5 mm.
101. On a bright clear day, you are at the top of a mountain and looking at a city 12 km away. There are two tall towers 20.0 m apart in the city. Can your eye resolve the two towers if the diameter of the pupil is 4.0 mm? If not, what should be the minimum magnification power of the telescope needed to resolve the two towers? In your calculations use 550 nm for the wavelength of the light.
102. Radio telescopes are telescopes used for the detection of radio emission from space. Because radio waves have much longer wavelengths than visible light, the diameter of a radio telescope must be very large to provide good resolution. For example, the radio telescope in Penticton, BC in Canada, has a diameter of 26 m and can be operated at frequencies as high as 6.6 GHz. (a) What is the wavelength corresponding to this frequency? (b) What is the angular separation of two radio sources that can be resolved by this telescope? (c) Compare the telescope's resolution with the angular size of the moon.



Figure 4.30 (credit: modification of work by Jason Nishiyama)

103. Calculate the wavelength of light that produces its first minimum at an angle of 36.9° when falling on a single slit of width $1.00 \mu\text{m}$.
104. (a) Find the angle of the third diffraction minimum for 633-nm light falling on a slit of width $20.0 \mu\text{m}$. (b) What slit width would place this minimum at 85.0° ?

- 105.** As an example of diffraction by apertures of everyday dimensions, consider a doorway of width 1.0 m. (a) What is the angular position of the first minimum in the diffraction pattern of 600-nm light? (b) Repeat this calculation for a musical note of frequency 440 Hz (A above middle C). Take the speed of sound to be 343 m/s.
- 106.** What are the angular positions of the first and second minima in a diffraction pattern produced by a slit of width 0.20 mm that is illuminated by 400 nm light? What is the angular width of the central peak?
- 107.** How far would you place a screen from the slit of the previous problem so that the second minimum is a distance of 2.5 mm from the center of the diffraction pattern?
- 108.** How narrow is a slit that produces a diffraction pattern on a screen 1.8 m away whose central peak is 1.0 m wide? Assume $\lambda = 589$ nm.
- 109.** Suppose that the central peak of a single-slit diffraction pattern is so wide that the first minima can be assumed to occur at angular positions of $\pm 90^\circ$. For this case, what is the ratio of the slit width to the wavelength of the light?
- 110.** The central diffraction peak of the double-slit interference pattern contains exactly nine fringes. What is the ratio of the slit separation to the slit width?
- 111.** Determine the intensities of three interference peaks other than the central peak in the central maximum of the diffraction, if possible, when a light of wavelength 500 nm is incident normally on a double slit of width 1000 nm and separation 1500 nm. Use the intensity of the central spot to be 1 mW/cm^2 .
- 112.** The yellow light from a sodium vapor lamp *seems* to be of pure wavelength, but it produces two first-order maxima at 36.093° and 36.129° when projected on a 10,000 line per centimeter diffraction grating. What are the two wavelengths to an accuracy of 0.1 nm?
- 113.** Structures on a bird feather act like a reflection grating having 8000 lines per centimeter. What is the angle of the first-order maximum for 600-nm light?
- 114.** If a diffraction grating produces a first-order maximum for the shortest wavelength of visible light at 30.0° , at what angle will the first-order maximum be for the largest wavelength of visible light?
- 115.** (a) What visible wavelength has its fourth-order maximum at an angle of 25.0° when projected on a 25,000-line per centimeter diffraction grating? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?
- 116.** Consider a spectrometer based on a diffraction grating. Construct a problem in which you calculate the distance between two wavelengths of electromagnetic radiation in your spectrometer. Among the things to be considered are the wavelengths you wish to be able to distinguish, the number of lines per meter on the diffraction grating, and the distance from the grating to the screen or detector. Discuss the practicality of the device in terms of being able to discern between wavelengths of interest.
- 117.** An amateur astronomer wants to build a telescope with a diffraction limit that will allow him to see if there are people on the moons of Jupiter. (a) What diameter mirror is needed to be able to see 1.00-m detail on a Jovian moon at a distance of 7.50×10^8 km from Earth? The wavelength of light averages 600 nm. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Challenge Problems

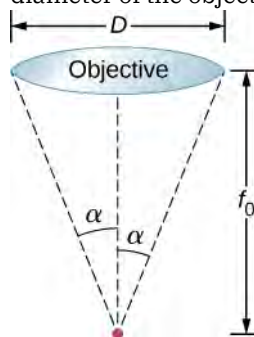
- 118.** Blue light of wavelength 450 nm falls on a slit of width 0.25 mm. A converging lens of focal length 20 cm is placed behind the slit and focuses the diffraction pattern on a screen. (a) How far is the screen from the lens? (b) What is the distance between the first and the third minima of the diffraction pattern?

- 119.** (a) Assume that the maxima are halfway between the minima of a single-slit diffraction pattern. Use the diameter and circumference of the phasor diagram, as described in [Intensity in Single-Slit Diffraction](#), to determine the intensities of the third and fourth maxima in terms of the intensity of the central maximum. (b) Do the same calculation, using [Equation 4.4](#).
- 120.** (a) By differentiating [Equation 4.4](#), show that the higher-order maxima of the single-slit diffraction pattern occur at values of β that satisfy $\tan \beta = \beta$. (b) Plot $y = \tan \beta$ and $y = \beta$ versus β and find the intersections of these two curves. What information do they give you about the locations of the maxima? (c) Convince yourself that these points do not appear exactly at $\beta = (n + \frac{1}{2})\pi$, where $n = 0, 1, 2, \dots$, but are quite close to these values.
- 121.** What is the maximum number of lines per centimeter a diffraction grating can have and produce a complete first-order spectrum for visible light?
- 122.** Show that a diffraction grating cannot produce a second-order maximum for a given wavelength of light unless the first-order maximum is at an angle less than 30.0° .

- 123.** A He-Ne laser beam is reflected from the surface of a CD onto a wall. The brightest spot is the reflected beam at an angle equal to the angle of incidence. However, fringes are also observed. If the wall is 1.50 m from the CD, and the first fringe is 0.600 m from the central maximum, what is the spacing of grooves on the CD?
- 124.** Objects viewed through a microscope are placed very close to the focal point of the objective lens. Show that the minimum separation x of two objects resolvable through the microscope is given by

$$x = \frac{1.22\lambda f_0}{D}, \quad 4.7$$

where f_0 is the focal length and D is the diameter of the objective lens as shown below.



CHAPTER 6

Photons and Matter Waves

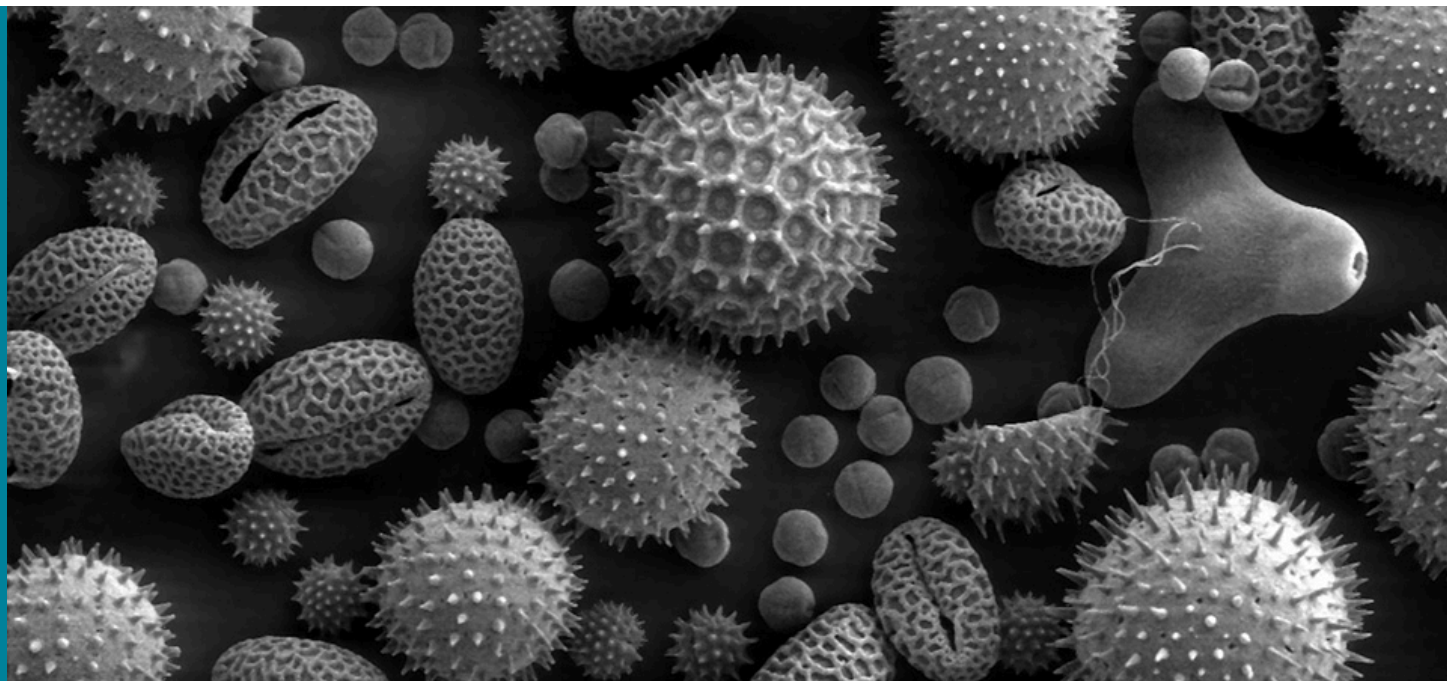


Figure 6.1 In this image of pollen taken with an electron microscope, the bean-shaped grains are about $50\mu\text{m}$ long. Electron microscopes can have a much higher resolving power than a conventional light microscope because electron wavelengths can be 100,000 times shorter than the wavelengths of visible-light photons. (credit: modification of work by Dartmouth College Electron Microscope Facility)

Chapter Outline

[6.1 Blackbody Radiation](#)

[6.2 Photoelectric Effect](#)

[6.3 The Compton Effect](#)

[6.4 Bohr's Model of the Hydrogen Atom](#)

[6.5 De Broglie's Matter Waves](#)

[6.6 Wave-Particle Duality](#)

INTRODUCTION Two of the most revolutionary concepts of the twentieth century were the description of light as a collection of particles, and the treatment of particles as waves. These wave properties of matter have led to the discovery of technologies such as electron microscopy, which allows us to examine submicroscopic objects such as grains of pollen, as shown above.

In this chapter, you will learn about the energy quantum, a concept that was introduced in 1900 by the German physicist Max Planck to explain blackbody radiation. We discuss how Albert Einstein extended Planck's

concept to a quantum of light (a “photon”) to explain the photoelectric effect. We also show how American physicist Arthur H. Compton used the photon concept in 1923 to explain wavelength shifts observed in X-rays. After a discussion of Bohr’s model of hydrogen, we describe how matter waves were postulated in 1924 by Louis-Victor de Broglie to justify Bohr’s model and we examine the experiments conducted in 1923–1927 by Clinton Davisson and Lester Germer that confirmed the existence of de Broglie’s matter waves.

6.1 Blackbody Radiation

Learning Objectives

By the end of this section, you will be able to:

- Apply Wien’s and Stefan’s laws to analyze radiation emitted by a blackbody
- Explain Planck’s hypothesis of energy quanta

All bodies emit electromagnetic radiation over a range of wavelengths. In an earlier chapter, we learned that a cooler body radiates less energy than a warmer body. We also know by observation that when a body is heated and its temperature rises, the perceived wavelength of its emitted radiation changes from infrared to red, and then from red to orange, and so forth. As its temperature rises, the body glows with the colors corresponding to ever-smaller wavelengths of the electromagnetic spectrum. This is the underlying principle of the incandescent light bulb: A hot metal filament glows red, and when heating continues, its glow eventually covers the entire visible portion of the electromagnetic spectrum. The temperature (T) of the object that emits radiation, or the **emitter**, determines the wavelength at which the radiated energy is at its maximum. For example, the Sun, whose surface temperature is in the range between 5000 K and 6000 K, radiates most strongly in a range of wavelengths about 560 nm in the visible part of the electromagnetic spectrum. Your body, when at its normal temperature of about 300 K, radiates most strongly in the infrared part of the spectrum.

Radiation that is incident on an object is partially absorbed and partially reflected. At thermodynamic equilibrium, the rate at which an object absorbs radiation is the same as the rate at which it emits it. Therefore, a good **absorber** of radiation (any object that absorbs radiation) is also a good emitter. A perfect absorber absorbs all electromagnetic radiation incident on it; such an object is called a **blackbody**.

Although the blackbody is an idealization, because no physical object absorbs 100% of incident radiation, we can construct a close realization of a blackbody in the form of a small hole in the wall of a sealed enclosure known as a cavity radiator, as shown in [Figure 6.2](#). The inside walls of a cavity radiator are rough and blackened so that any radiation that enters through a tiny hole in the cavity wall becomes trapped inside the cavity. At thermodynamic equilibrium (at temperature T), the cavity walls absorb exactly as much radiation as they emit. Furthermore, inside the cavity, the radiation entering the hole is balanced by the radiation leaving it. The emission spectrum of a blackbody can be obtained by analyzing the light radiating from the hole. Electromagnetic waves emitted by a blackbody are called **blackbody radiation**.

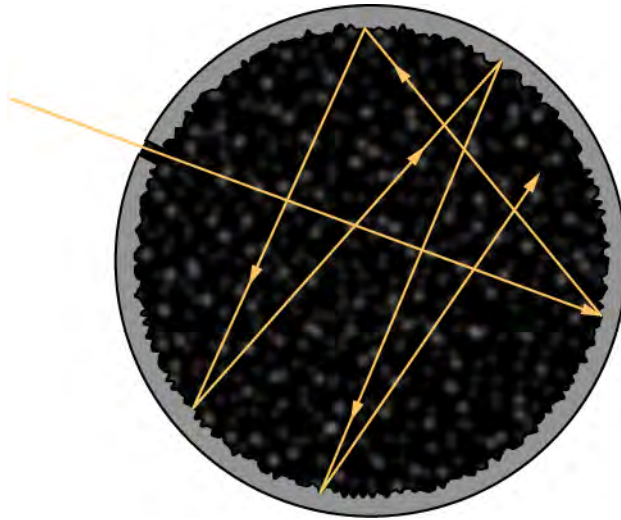


Figure 6.2 A blackbody is physically realized by a small hole in the wall of a cavity radiator.

The intensity $I(\lambda, T)$ of blackbody radiation depends on the wavelength λ of the emitted radiation and on the temperature T of the blackbody (Figure 6.3). The function $I(\lambda, T)$ is the **power intensity** that is radiated per unit wavelength; in other words, it is the power radiated per unit area of the hole in a cavity radiator per unit wavelength. According to this definition, $I(\lambda, T)d\lambda$ is the power per unit area that is emitted in the wavelength interval from λ to $\lambda + d\lambda$. The intensity distribution among wavelengths of radiation emitted by cavities was studied experimentally at the end of the nineteenth century. Generally, radiation emitted by materials only approximately follows the blackbody radiation curve (Figure 6.4); however, spectra of common stars do follow the blackbody radiation curve very closely.

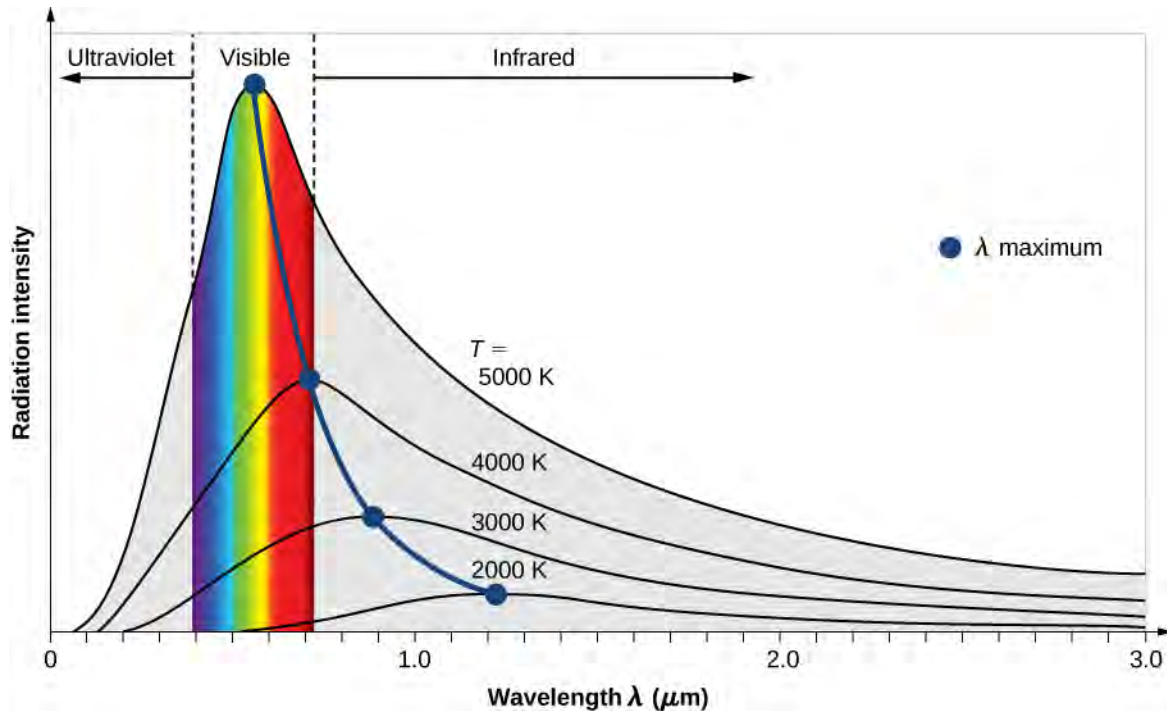


Figure 6.3 The intensity of blackbody radiation versus the wavelength of the emitted radiation. Each curve corresponds to a different blackbody temperature, starting with a low temperature (the lowest curve) to a high temperature (the highest curve).

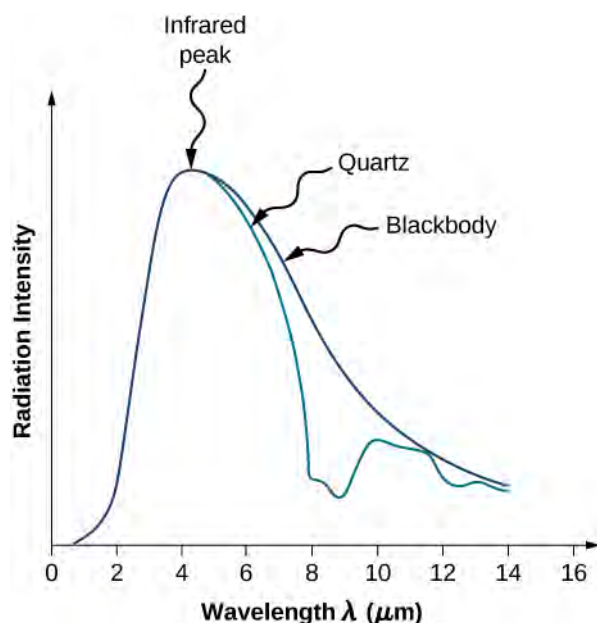


Figure 6.4 The spectrum of radiation emitted from a quartz surface (blue curve) and the blackbody radiation curve (black curve) at 600 K.

Two important laws summarize the experimental findings of blackbody radiation: *Wien's displacement law* and *Stefan's law*. Wien's displacement law is illustrated in [Figure 6.3](#) by the curve connecting the maxima on the intensity curves. In these curves, we see that the hotter the body, the shorter the wavelength corresponding to the emission peak in the radiation curve. Quantitatively, Wien's law reads

$$\lambda_{\text{max}} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} \quad 6.1$$

where λ_{max} is the position of the maximum in the radiation curve. In other words, λ_{max} is the wavelength at which a blackbody radiates most strongly at a given temperature T . Note that in [Equation 6.1](#), the temperature is in kelvins. Wien's displacement law allows us to estimate the temperatures of distant stars by measuring the wavelength of radiation they emit.



EXAMPLE 6.1

Temperatures of Distant Stars

On a clear evening during the winter months, if you happen to be in the Northern Hemisphere and look up at the sky, you can see the constellation Orion (The Hunter). One star in this constellation, Rigel, flickers in a blue color and another star, Betelgeuse, has a reddish color, as shown in [Figure 6.5](#). Which of these two stars is cooler, Betelgeuse or Rigel?

Strategy

We treat each star as a blackbody. Then according to Wien's law, its temperature is inversely proportional to the wavelength of its peak intensity. The wavelength $\lambda_{\text{max}}^{(\text{blue})}$ of blue light is shorter than the wavelength $\lambda_{\text{max}}^{(\text{red})}$ of red light. Even if we do not know the precise wavelengths, we can still set up a proportion.

Solution

Writing Wien's law for the blue star and for the red star, we have

$$\lambda_{\text{max}}^{(\text{red})} T_{(\text{red})} = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} = \lambda_{\text{max}}^{(\text{blue})} T_{(\text{blue})} \quad 6.2$$

When simplified, [Equation 6.2](#) gives

$$T_{(\text{red})} = \frac{\lambda_{\text{max}}^{(\text{blue})}}{\lambda_{\text{max}}^{(\text{red})}} T_{(\text{blue})} < T_{(\text{blue})} \quad 6.3$$

Therefore, Betelgeuse is cooler than Rigel.

Significance

Note that Wien's displacement law tells us that the higher the temperature of an emitting body, the shorter the wavelength of the radiation it emits. The qualitative analysis presented in this example is generally valid for any emitting body, whether it is a big object such as a star or a small object such as the glowing filament in an incandescent lightbulb.

✓ CHECK YOUR UNDERSTANDING 6.1

The flame of a peach-scented candle has a yellowish color and the flame of a Bunsen's burner in a chemistry lab has a bluish color. Which flame has a higher temperature?

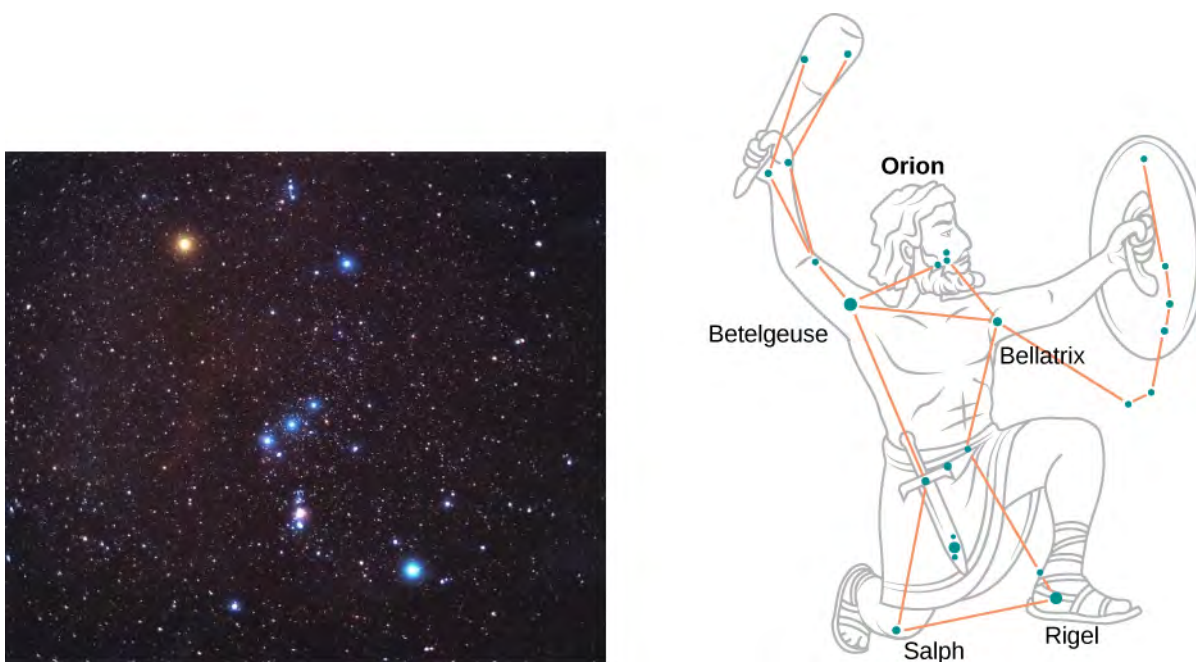


Figure 6.5 In the Orion constellation, the red star Betelgeuse, which usually takes on a yellowish tint, appears as the figure's right shoulder (in the upper left). The giant blue star on the bottom right is Rigel, which appears as the hunter's left foot. (credit left: modification of work by Matthew Spinelli, NASA APOD)

The second experimental relation is Stefan's law, which concerns the total power of blackbody radiation emitted across the entire spectrum of wavelengths at a given temperature. In [Figure 6.3](#), this total power is represented by the area under the blackbody radiation curve for a given T . As the temperature of a blackbody increases, the total emitted power also increases. Quantitatively, Stefan's law expresses this relation as

$$P(T) = \sigma AT^4 \quad 6.4$$

where A is the surface area of a blackbody, T is its temperature (in kelvins), and σ is the **Stefan–Boltzmann constant**, $\sigma = 5.670 \times 10^{-8} \text{ W}/(\text{m}^2 \cdot \text{K}^4)$. Stefan's law enables us to estimate how much energy a star is radiating by remotely measuring its temperature.

EXAMPLE 6.2

Power Radiated by Stars

A star such as our Sun will eventually evolve to a “red giant” star and then to a “white dwarf” star. A typical white dwarf is approximately the size of Earth, and its surface temperature is about $2.5 \times 10^4 \text{ K}$. A typical red giant has a surface temperature of $3.0 \times 10^3 \text{ K}$ and a radius $\sim 100,000$ times larger than that of a white dwarf. What is the average radiated power per unit area and the total power radiated by each of these types of stars? How do they compare?

Strategy

If we treat the star as a blackbody, then according to Stefan’s law, the total power that the star radiates is proportional to the fourth power of its temperature. To find the power radiated per unit area of the surface, we do not need to make any assumptions about the shape of the star because P/A depends only on temperature. However, to compute the total power, we need to make an assumption that the energy radiates through a spherical surface enclosing the star, so that the surface area is $A = 4\pi R^2$, where R is its radius.

Solution

A simple proportion based on Stefan’s law gives

$$\frac{P_{\text{dwarf}}/A_{\text{dwarf}}}{P_{\text{giant}}/A_{\text{giant}}} = \frac{\sigma T_{\text{dwarf}}^4}{\sigma T_{\text{giant}}^4} = \left(\frac{T_{\text{dwarf}}}{T_{\text{giant}}}\right)^4 = \left(\frac{2.5 \times 10^4}{3.0 \times 10^3}\right)^4 = 4820 \quad 6.5$$

The power emitted per unit area by a white dwarf is about 5000 times that the power emitted by a red giant. Denoting this ratio by $a = 4.8 \times 10^3$, Equation 6.5 gives

$$\frac{P_{\text{dwarf}}}{P_{\text{giant}}} = a \frac{A_{\text{dwarf}}}{A_{\text{giant}}} = a \frac{4\pi R_{\text{dwarf}}^2}{4\pi R_{\text{giant}}^2} = a \left(\frac{R_{\text{dwarf}}}{R_{\text{giant}}}\right)^2 = 4.8 \times 10^3 \left(\frac{R_{\text{dwarf}}}{10^5 R_{\text{dwarf}}}\right)^2 = 4.8 \times 10^{-7} \quad 6.6$$

We see that the total power emitted by a white dwarf is a tiny fraction of the total power emitted by a red giant. Despite its relatively lower temperature, the overall power radiated by a red giant far exceeds that of the white dwarf because the red giant has a much larger surface area. To estimate the absolute value of the emitted power per unit area, we again use Stefan’s law. For the white dwarf, we obtain

$$\frac{P_{\text{dwarf}}}{A_{\text{dwarf}}} = \sigma T_{\text{dwarf}}^4 = 5.670 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4} (2.5 \times 10^4 \text{ K})^4 = 2.2 \times 10^{10} \text{ W/m}^2 \quad 6.7$$

The analogous result for the red giant is obtained by scaling the result for a white dwarf:

$$\frac{P_{\text{giant}}}{A_{\text{giant}}} = \frac{2.2 \times 10^{10}}{4.82 \times 10^3} \frac{\text{W}}{\text{m}^2} = 4.56 \times 10^6 \frac{\text{W}}{\text{m}^2} \cong 4.6 \times 10^6 \frac{\text{W}}{\text{m}^2} \quad 6.8$$

Significance

To estimate the total power emitted by a white dwarf, in principle, we could use Equation 6.7. However, to find its surface area, we need to know the average radius, which is not given in this example. Therefore, the solution stops here. The same is also true for the red giant star.

CHECK YOUR UNDERSTANDING 6.2

An iron poker is being heated. As its temperature rises, the poker begins to glow—first dull red, then bright red, then orange, and then yellow. Use either the blackbody radiation curve or Wien’s law to explain these changes in the color of the glow.

CHECK YOUR UNDERSTANDING 6.3

Suppose that two stars, α and β , radiate exactly the same total power. If the radius of star α is three times that of star β , what is the ratio of the surface temperatures of these stars? Which one is hotter?

The term “blackbody” was coined by Gustav R. Kirchhoff in 1862. The blackbody radiation curve was known experimentally, but its shape eluded physical explanation until the year 1900. The physical model of a blackbody at temperature T is that of the electromagnetic waves enclosed in a cavity (see [Figure 6.2](#)) and at thermodynamic equilibrium with the cavity walls. The waves can exchange energy with the walls. The objective here is to find the energy density distribution among various modes of vibration at various wavelengths (or frequencies). In other words, we want to know how much energy is carried by a single wavelength or a band of wavelengths. Once we know the energy distribution, we can use standard statistical methods (similar to those studied in a previous chapter) to obtain the blackbody radiation curve, Stefan’s law, and Wien’s displacement law. When the physical model is correct, the theoretical predictions should be the same as the experimental curves.

In a classical approach to the blackbody radiation problem, in which radiation is treated as waves (as you have studied in previous chapters), the modes of electromagnetic waves trapped in the cavity are in equilibrium and continually exchange their energies with the cavity walls. There is no physical reason why a wave should do otherwise: Any amount of energy can be exchanged, either by being transferred from the wave to the material in the wall or by being received by the wave from the material in the wall. This classical picture is the basis of the model developed by Lord Rayleigh and, independently, by Sir James Jeans. The result of this classical model for blackbody radiation curves is known as the *Rayleigh–Jeans law*. However, as shown in [Figure 6.6](#), the Rayleigh–Jeans law fails to correctly reproduce experimental results. In the limit of short wavelengths, the Rayleigh–Jeans law predicts infinite radiation intensity, which is inconsistent with the experimental results in which radiation intensity has finite values in the ultraviolet region of the spectrum. This divergence between the results of classical theory and experiments, which came to be called the *ultraviolet catastrophe*, shows how classical physics fails to explain the mechanism of blackbody radiation.

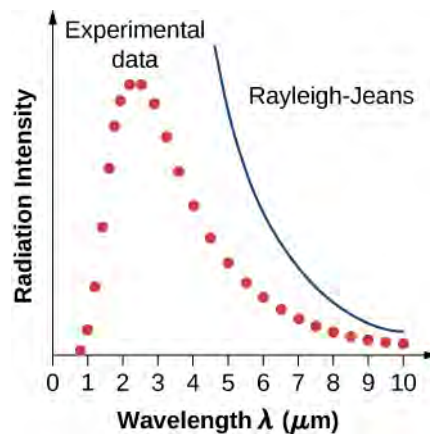


Figure 6.6 The ultraviolet catastrophe: The Rayleigh–Jeans law does not explain the observed blackbody emission spectrum.

The blackbody radiation problem was solved in 1900 by Max Planck. Planck used the same idea as the Rayleigh–Jeans model in the sense that he treated the electromagnetic waves between the walls inside the cavity classically, and assumed that the radiation is in equilibrium with the cavity walls. The innovative idea that Planck introduced in his model is the assumption that the cavity radiation originates from atomic oscillations inside the cavity walls, and that these oscillations can have only *discrete* values of energy. Therefore, the radiation trapped inside the cavity walls can exchange energy with the walls only in discrete amounts. Planck’s hypothesis of discrete energy values, which he called *quanta*, assumes that the oscillators inside the cavity walls have **quantized energies**. This was a brand new idea that went beyond the classical physics of the nineteenth century because, as you learned in a previous chapter, in the classical picture, the energy of an oscillator can take on any continuous value. Planck assumed that the energy of an oscillator (E_n) can have only discrete, or quantized, values:

$$E_n = nhf, \text{ where } n = 1, 2, 3, \dots \quad 6.9$$

In [Equation 6.9](#), f is the frequency of Planck's oscillator. The natural number n that enumerates these discrete energies is called a **quantum number**. The physical constant h is called *Planck's constant*:

$$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s} \quad 6.10$$

Each discrete energy value corresponds to a **quantum state of a Planck oscillator**. Quantum states are enumerated by quantum numbers. For example, when Planck's oscillator is in its first $n = 1$ quantum state, its energy is $E_1 = hf$; when it is in the $n = 2$ quantum state, its energy is $E_2 = 2hf$; when it is in the $n = 3$ quantum state, $E_3 = 3hf$; and so on.

Note that [Equation 6.9](#) shows that there are infinitely many quantum states, which can be represented as a sequence $\{hf, 2hf, 3hf, \dots, (n-1)hf, nhf, (n+1)hf, \dots\}$. Each two consecutive quantum states in this sequence are separated by an energy jump, $\Delta E = hf$. An oscillator in the wall can receive energy from the radiation in the cavity (absorption), or it can give away energy to the radiation in the cavity (emission). The absorption process sends the oscillator to a higher quantum state, and the emission process sends the oscillator to a lower quantum state. Whichever way this exchange of energy goes, the smallest amount of energy that can be exchanged is hf . There is no upper limit to how much energy can be exchanged, but whatever is exchanged must be an integer multiple of hf . If the energy packet does not have this exact amount, it is neither absorbed nor emitted at the wall of the blackbody.

Planck's Quantum Hypothesis

Planck's hypothesis of energy quanta states that the amount of energy emitted by the oscillator is carried by the quantum of radiation, ΔE :

$$\Delta E = hf$$

Recall that the frequency of electromagnetic radiation is related to its wavelength and to the speed of light by the fundamental relation $f\lambda = c$. This means that we can express [Equation 6.10](#) equivalently in terms of wavelength λ . When included in the computation of the energy density of a blackbody, Planck's hypothesis gives the following theoretical expression for the power intensity of emitted radiation per unit wavelength:

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \quad 6.11$$

where c is the speed of light in vacuum and k_B is Boltzmann's constant, $k_B = 1.380 \times 10^{-23} \text{ J/K}$. The theoretical formula expressed in [Equation 6.11](#) is called *Planck's blackbody radiation law*. This law is in agreement with the experimental blackbody radiation curve (see [Figure 6.7](#)). In addition, Wien's displacement law and Stefan's law can both be derived from [Equation 6.11](#). To derive Wien's displacement law, we use differential calculus to find the maximum of the radiation intensity curve $I(\lambda, T)$. To derive Stefan's law and find the value of the Stefan–Boltzmann constant, we use integral calculus and integrate $I(\lambda, T)$ to find the total power radiated by a blackbody at one temperature in the entire spectrum of wavelengths from $\lambda = 0$ to $\lambda = \infty$. This derivation is left as an exercise later in this chapter.

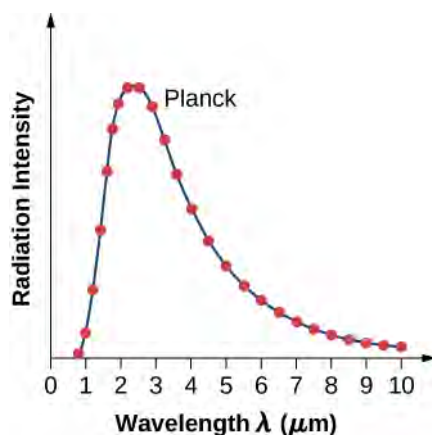


Figure 6.7 Planck's theoretical result (continuous curve) and the experimental blackbody radiation curve (dots).



EXAMPLE 6.3

Planck's Quantum Oscillator

A quantum oscillator in the cavity wall in [Figure 6.2](#) is vibrating at a frequency of 5.0×10^{14} Hz. Calculate the spacing between its energy levels.

Strategy

Energy states of a quantum oscillator are given by [Equation 6.9](#). The energy spacing ΔE is obtained by finding the energy difference between two adjacent quantum states for quantum numbers $n + 1$ and n .

Solution

We can substitute the given frequency and Planck's constant directly into the equation:

$$\Delta E = E_{n+1} - E_n = (n + 1)hf - nhf = hf = (6.626 \times 10^{-34} \text{ J} \cdot \text{s})(5.0 \times 10^{14} \text{ Hz}) = 3.3 \times 10^{-19} \text{ J}$$

Significance

Note that we do not specify what kind of material was used to build the cavity. Here, a quantum oscillator is a theoretical model of an atom or molecule of material in the wall.



CHECK YOUR UNDERSTANDING 6.4

A molecule is vibrating at a frequency of 5.0×10^{14} Hz. What is the smallest spacing between its vibrational energy levels?



EXAMPLE 6.4

Quantum Theory Applied to a Classical Oscillator

A 1.0-kg mass oscillates at the end of a spring with a spring constant of 1000 N/m. The amplitude of these oscillations is 0.10 m. Use the concept of quantization to find the energy spacing for this classical oscillator. Is the energy quantization significant for macroscopic systems, such as this oscillator?

Strategy

We use [Equation 6.10](#) as though the system were a quantum oscillator, but with the frequency f of the mass vibrating on a spring. To evaluate whether or not quantization has a significant effect, we compare the quantum energy spacing with the macroscopic total energy of this classical oscillator.

Solution

For the spring constant, $k = 1.0 \times 10^3 \text{ N/m}$, the frequency f of the mass, $m = 1.0 \text{ kg}$, is

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{1.0 \times 10^3 \text{ N/m}}{1.0 \text{ kg}}} \simeq 5.0 \text{ Hz}$$

The energy quantum that corresponds to this frequency is

$$\Delta E = hf = (6.626 \times 10^{-34} \text{ J} \cdot \text{s})(5.0 \text{ Hz}) = 3.3 \times 10^{-33} \text{ J}$$

When vibrations have amplitude $A = 0.10 \text{ m}$, the energy of oscillations is

$$E = \frac{1}{2} k A^2 = \frac{1}{2} (1000 \text{ N/m})(0.1 \text{ m})^2 = 5.0 \text{ J}$$

Significance

Thus, for a classical oscillator, we have $\Delta E/E \approx 10^{-34}$. We see that the separation of the energy levels is immeasurably small. Therefore, for all practical purposes, the energy of a classical oscillator takes on continuous values. This is why classical principles may be applied to macroscopic systems encountered in everyday life without loss of accuracy.

✓ CHECK YOUR UNDERSTANDING 6.5

Would the result in [Example 6.4](#) be different if the mass were not 1.0 kg but a tiny mass of 1.0 μg , and the amplitude of vibrations were 0.10 μm ?

When Planck first published his result, the hypothesis of energy quanta was not taken seriously by the physics community because it did not follow from any established physics theory at that time. It was perceived, even by Planck himself, as a useful mathematical trick that led to a good theoretical “fit” to the experimental curve. This perception was changed in 1905 when Einstein published his explanation of the photoelectric effect, in which he gave Planck’s energy quantum a new meaning: that of a particle of light.

6.2 Photoelectric Effect

Learning Objectives

By the end of this section, you will be able to:

- Describe physical characteristics of the photoelectric effect
- Explain why the photoelectric effect cannot be explained by classical physics
- Describe how Einstein’s idea of a particle of radiation explains the photoelectric effect

When a metal surface is exposed to a monochromatic electromagnetic wave of sufficiently short wavelength (or equivalently, above a threshold frequency), the incident radiation is absorbed and the exposed surface emits electrons. This phenomenon is known as the **photoelectric effect**. Electrons that are emitted in this process are called **photoelectrons**.

The experimental setup to study the photoelectric effect is shown schematically in [Figure 6.8](#). The target material serves as the cathode, which becomes the emitter of photoelectrons when it is illuminated by monochromatic radiation. We call this electrode the **photoelectrode**. Photoelectrons are collected at the anode, which is kept at a higher potential with respect to the cathode. The potential difference between the electrodes can be increased or decreased, or its polarity can be reversed. The electrodes are enclosed in an evacuated glass tube so that photoelectrons do not lose their kinetic energy on collisions with air molecules in the space between electrodes.

When the target material is not exposed to radiation, no current is registered in this circuit because the circuit is broken (note, there is a gap between the electrodes). But when the target material is connected to the negative terminal of a battery and exposed to radiation, a current is registered in this circuit; this current is

called the **photocurrent**. Suppose that we now reverse the potential difference between the electrodes so that the target material now connects with the positive terminal of a battery, and then we slowly increase the voltage. The photocurrent gradually dies out and eventually stops flowing completely at some value of this reversed voltage. The potential difference at which the photocurrent stops flowing is called the **stopping potential**.

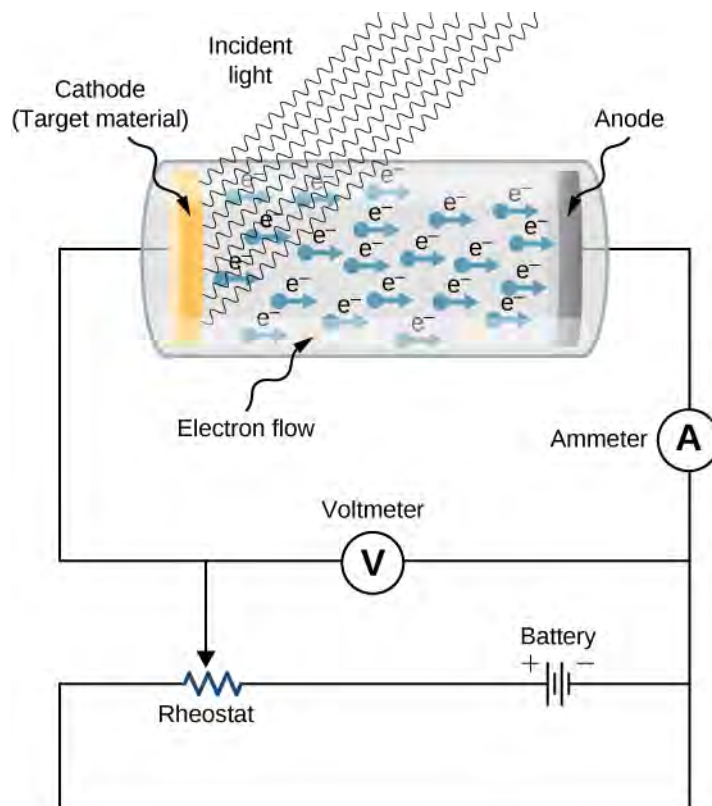


Figure 6.8 An experimental setup to study the photoelectric effect. The anode and cathode are enclosed in an evacuated glass tube. The voltmeter measures the electric potential difference between the electrodes, and the ammeter measures the photocurrent. The incident radiation is monochromatic.

Characteristics of the Photoelectric Effect

The photoelectric effect has three important characteristics that cannot be explained by classical physics: (1) the absence of a lag time, (2) the independence of the kinetic energy of photoelectrons on the intensity of incident radiation, and (3) the presence of a cut-off frequency. Let's examine each of these characteristics.

The absence of lag time

When radiation strikes the target material in the electrode, electrons are emitted almost instantaneously, even at very low intensities of incident radiation. This absence of lag time contradicts our understanding based on classical physics. Classical physics predicts that for low-energy radiation, it would take significant time before irradiated electrons could gain sufficient energy to leave the electrode surface; however, such an energy buildup is not observed.

The intensity of incident radiation and the kinetic energy of photoelectrons

Typical experimental curves are shown in [Figure 6.9](#), in which the photocurrent is plotted versus the applied potential difference between the electrodes. For the positive potential difference, the current steadily grows until it reaches a plateau. Furthering the potential increase beyond this point does not increase the photocurrent at all. A higher intensity of radiation produces a higher value of photocurrent. For the negative potential difference, as the absolute value of the potential difference increases, the value of the photocurrent decreases and becomes zero at the stopping potential. For any intensity of incident radiation, whether the intensity is high or low, the value of the stopping potential always stays at one value.

To understand why this result is unusual from the point of view of classical physics, we first have to analyze the energy of photoelectrons. A photoelectron that leaves the surface has kinetic energy K . It gained this energy from the incident electromagnetic wave. In the space between the electrodes, a photoelectron moves in the electric potential and its energy changes by the amount $q\Delta V$, where ΔV is the potential difference and $q = -e$. Because no forces are present but electric force, by applying the work-energy theorem, we obtain the energy balance $\Delta K - e\Delta V = 0$ for the photoelectron, where ΔK is the change in the photoelectron's kinetic energy. When the stopping potential $-\Delta V_s$ is applied, the photoelectron loses its initial kinetic energy K_i and comes to rest. Thus, its energy balance becomes $(0 - K_i) - e(-\Delta V_s) = 0$, so that $K_i = e\Delta V_s$. In the presence of the stopping potential, the largest kinetic energy $K_{\max}$ that a photoelectron can have is its initial kinetic energy, which it has at the surface of the photoelectrode. Therefore, the largest kinetic energy of photoelectrons can be directly measured by measuring the stopping potential:

$$K_{\max} = e\Delta V_s. \quad 6.12$$

At this point we can see where the classical theory is at odds with the experimental results. In classical theory, the photoelectron absorbs electromagnetic energy in a continuous way; this means that when the incident radiation has a high intensity, the kinetic energy in Equation 6.12 is expected to be high. Similarly, when the radiation has a low intensity, the kinetic energy is expected to be low. But the experiment shows that the maximum kinetic energy of photoelectrons is independent of the light intensity.

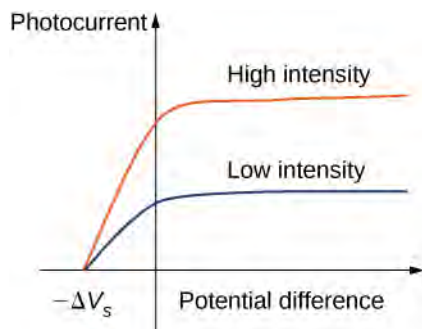


Figure 6.9 The detected photocurrent plotted versus the applied potential difference shows that for any intensity of incident radiation, whether the intensity is high (upper curve) or low (lower curve), the value of the stopping potential is always the same.

The presence of a cut-off frequency

For any metal surface, there is a minimum frequency of incident radiation below which photocurrent does not occur. The value of this **cut-off frequency** for the photoelectric effect is a physical property of the metal: Different materials have different values of cut-off frequency. Experimental data show a typical linear trend (see Figure 6.10). The kinetic energy of photoelectrons at the surface grows linearly with the increasing frequency of incident radiation. Measurements for all metal surfaces give linear plots with one slope. None of these observed phenomena is in accord with the classical understanding of nature. According to the classical description, the kinetic energy of photoelectrons should not depend on the frequency of incident radiation at all, and there should be no cut-off frequency. Instead, in the classical picture, electrons receive energy from the incident electromagnetic wave in a continuous way, and the amount of energy they receive depends only on the intensity of the incident light and nothing else. So in the classical understanding, as long as the light is shining, the photoelectric effect is expected to continue.

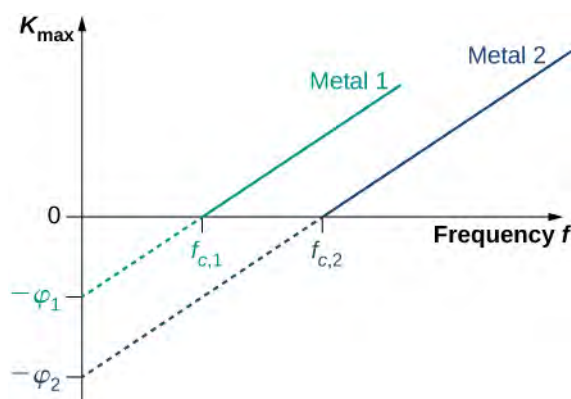


Figure 6.10 Kinetic energy of photoelectrons at the surface versus the frequency of incident radiation. The photoelectric effect can only occur above the cut-off frequency f_c . Measurements for all metal surfaces give linear plots with one slope. Each metal surface has its own cut-off frequency.

The Work Function

The photoelectric effect was explained in 1905 by A. Einstein. Einstein reasoned that if Planck's hypothesis about energy quanta was correct for describing the energy exchange between electromagnetic radiation and cavity walls, it should also work to describe energy absorption from electromagnetic radiation by the surface of a photoelectrode. He postulated that an electromagnetic wave carries its energy in discrete packets. Einstein's postulate goes beyond Planck's hypothesis because it states that the light itself consists of energy quanta. In other words, it states that electromagnetic waves are quantized.

In Einstein's approach, a beam of monochromatic light of frequency f is made of photons. A **photon** is a particle of light. Each photon moves at the speed of light and carries an energy quantum E_f . A photon's energy depends only on its frequency f . Explicitly, the **energy of a photon** is

$$E_f = hf \quad 6.13$$

where h is Planck's constant. In the photoelectric effect, photons arrive at the metal surface and each photon gives away *all* of its energy to only *one* electron on the metal surface. This transfer of energy from photon to electron is of the "all or nothing" type, and there are no fractional transfers in which a photon would lose only part of its energy and survive. The essence of a **quantum phenomenon** is either a photon transfers its entire energy and ceases to exist or there is no transfer at all. This is in contrast with the classical picture, where fractional energy transfers are permitted. Having this quantum understanding, the energy balance for an electron on the surface that receives the energy E_f from a photon is

$$E_f = K_{\max} + \phi$$

where $K_{\max}$ is the kinetic energy, given by Equation 6.12, that an electron has at the very instant it gets detached from the surface. In this energy balance equation, ϕ is the energy needed to detach a photoelectron from the surface. This energy ϕ is called the **work function** of the metal. Each metal has its characteristic work function, as illustrated in Table 6.1. To obtain the kinetic energy of photoelectrons at the surface, we simply invert the energy balance equation and use Equation 6.13 to express the energy of the absorbed photon. This gives us the expression for the kinetic energy of photoelectrons, which explicitly depends on the frequency of incident radiation:

$$K_{\max} = hf - \phi. \quad 6.14$$

This equation has a simple mathematical form but its physics is profound. We can now elaborate on the physical meaning behind Equation 6.14.

Typical Values of the Work Function for Some Common Metals

Metal	ϕ (eV)
Na	2.46
Al	4.08
Pb	4.14
Zn	4.31
Fe	4.50
Cu	4.70
Ag	4.73
Pt	6.35

Table 6.1

In Einstein's interpretation, interactions take place between individual electrons and individual photons. The absence of a lag time means that these one-on-one interactions occur instantaneously. This interaction time cannot be increased by lowering the light intensity. The light intensity corresponds to the number of photons arriving at the metal surface per unit time. Even at very low light intensities, the photoelectric effect still occurs because the interaction is between one electron and one photon. As long as there is at least one photon with enough energy to transfer it to a bound electron, a photoelectron will appear on the surface of the photoelectrode.

The existence of the cut-off frequency f_c for the photoelectric effect follows from [Equation 6.14](#) because the kinetic energy $K_{\max}$ of the photoelectron can take only positive values. This means that there must be some threshold frequency for which the kinetic energy is zero, $0 = hf_c - \phi$. In this way, we obtain the explicit formula for cut-off frequency:

$$f_c = \frac{\phi}{h}. \quad 6.15$$

Cut-off frequency depends only on the work function of the metal and is in direct proportion to it. When the work function is large (when electrons are bound fast to the metal surface), the energy of the threshold photon must be large to produce a photoelectron, and then the corresponding threshold frequency is large. Photons with frequencies larger than the threshold frequency f_c always produce photoelectrons because they have $K_{\max} > 0$. Photons with frequencies smaller than f_c do not have enough energy to produce photoelectrons. Therefore, when incident radiation has a frequency below the cut-off frequency, the photoelectric effect is not observed. Because frequency f and wavelength λ of electromagnetic waves are related by the fundamental relation $\lambda f = c$ (where c is the speed of light in vacuum), the cut-off frequency has its corresponding **cut-off wavelength** λ_c :

$$\lambda_c = \frac{c}{f_c} = \frac{c}{\phi/h} = \frac{hc}{\phi}. \quad 6.16$$

In this equation, $hc = 1240 \text{ eV} \cdot \text{nm}$. Our observations can be restated in the following equivalent way: When the incident radiation has wavelengths longer than the cut-off wavelength, the photoelectric effect does not occur.



EXAMPLE 6.5

Photoelectric Effect for Silver

Radiation with wavelength 300 nm is incident on a silver surface. Will photoelectrons be observed?

Strategy

Photoelectrons can be ejected from the metal surface only when the incident radiation has a shorter wavelength than the cut-off wavelength. The work function of silver is $\phi = 4.73$ eV (Table 6.1). To make the estimate, we use Equation 6.16.

Solution

The threshold wavelength for observing the photoelectric effect in silver is

$$\lambda_c = \frac{hc}{\phi} = \frac{1240 \text{ eV} \cdot \text{nm}}{4.73 \text{ eV}} = 262 \text{ nm}.$$

The incident radiation has wavelength 300 nm, which is longer than the cut-off wavelength; therefore, photoelectrons are not observed.

Significance

If the photoelectrode were made of sodium instead of silver, the cut-off wavelength would be 504 nm and photoelectrons would be observed.

Equation 6.14 in Einstein's model tells us that the maximum kinetic energy of photoelectrons is a linear function of the frequency of incident radiation, which is illustrated in Figure 6.10. For any metal, the slope of this plot has a value of Planck's constant. The intercept with the K_{max} -axis gives us a value of the work function that is characteristic for the metal. On the other hand, K_{max} can be directly measured in the experiment by measuring the value of the stopping potential ΔV_s (see Equation 6.12) at which the photocurrent stops. These direct measurements allow us to determine experimentally the value of Planck's constant, as well as work functions of materials.

Einstein's model also gives a straightforward explanation for the photocurrent values shown in Figure 6.9. For example, doubling the intensity of radiation translates to doubling the number of photons that strike the surface per unit time. The larger the number of photons, the larger is the number of photoelectrons, which leads to a larger photocurrent in the circuit. This is how radiation intensity affects the photocurrent. The photocurrent must reach a plateau at some value of potential difference because, in unit time, the number of photoelectrons is equal to the number of incident photons and the number of incident photons does not depend on the applied potential difference at all, but only on the intensity of incident radiation. The stopping potential does not change with the radiation intensity because the kinetic energy of photoelectrons (see Equation 6.14) does not depend on the radiation intensity.



EXAMPLE 6.6

Work Function and Cut-Off Frequency

When a 180-nm light is used in an experiment with an unknown metal, the measured photocurrent drops to zero at potential -0.80 V. Determine the work function of the metal and its cut-off frequency for the photoelectric effect.

Strategy

To find the cut-off frequency f_c , we use Equation 6.15, but first we must find the work function ϕ . To find ϕ , we use Equation 6.12 and Equation 6.14. Photocurrent drops to zero at the stopping value of potential, so we identify $\Delta V_s = 0.80$ V.

Solution

We use [Equation 6.12](#) to find the kinetic energy of the photoelectrons:

$$K_{\max} = e\Delta V_s = e(0.80\text{V}) = 0.80\text{ eV}.$$

Now we solve [Equation 6.14](#) for ϕ :

$$\phi = hf - K_{\max} = \frac{hc}{\lambda} - K_{\max} = \frac{1240\text{ eV} \cdot \text{nm}}{180\text{ nm}} - 0.80\text{ eV} = 6.09\text{ eV}.$$

Finally, we use [Equation 6.15](#) to find the cut-off frequency:

$$f_c = \frac{\phi}{h} = \frac{6.09\text{ eV}}{4.136 \times 10^{-15}\text{ eV} \cdot \text{s}} = 1.47 \times 10^{15}\text{ Hz}.$$

Significance

In calculations like the one shown in this example, it is convenient to use Planck's constant in the units of $\text{eV} \cdot \text{s}$ and express all energies in eV instead of joules.

**EXAMPLE 6.7****The Photon Energy and Kinetic Energy of Photoelectrons**

A 430-nm violet light is incident on a calcium photoelectrode with a work function of 2.71 eV.

Find the energy of the incident photons and the maximum kinetic energy of ejected electrons.

Strategy

The energy of the incident photon is $E_f = hf = hc/\lambda$, where we use $f\lambda = c$. To obtain the maximum energy of the ejected electrons, we use [Equation 6.16](#).

Solution

$$E_f = \frac{hc}{\lambda} = \frac{1240\text{ eV} \cdot \text{nm}}{430\text{ nm}} = 2.88\text{ eV}, \quad K_{\max} = E_f - \phi = 2.88\text{ eV} - 2.71\text{ eV} = 0.17\text{ eV}$$

Significance

In this experimental setup, photoelectrons stop flowing at the stopping potential of 0.17 V.

CHECK YOUR UNDERSTANDING 6.6

A yellow 589-nm light is incident on a surface whose work function is 1.20 eV. What is the stopping potential? What is the cut-off wavelength?

CHECK YOUR UNDERSTANDING 6.7

Cut-off frequency for the photoelectric effect in some materials is $8.0 \times 10^{13}\text{ Hz}$. When the incident light has a frequency of $1.2 \times 10^{14}\text{ Hz}$, the stopping potential is measured as -0.16 V . Estimate a value of Planck's constant from these data (in units $\text{J} \cdot \text{s}$ and $\text{eV} \cdot \text{s}$) and determine the percentage error of your estimation.

6.3 The Compton Effect

Learning Objectives

By the end of this section, you will be able to:

- Describe Compton's experiment
- Explain the Compton wavelength shift
- Describe how experiments with X-rays confirm the particle nature of radiation

Two of Einstein's influential ideas introduced in 1905 were the theory of special relativity and the concept of a light quantum, which we now call a photon. Beyond 1905, Einstein went further to suggest that freely propagating electromagnetic waves consisted of photons that are particles of light in the same sense that electrons or other massive particles are particles of matter. A beam of monochromatic light of wavelength λ (or equivalently, of frequency f) can be seen either as a classical wave or as a collection of photons that travel in a vacuum with one speed, c (the speed of light), and all carrying the same energy, $E_f = hf$. This idea proved useful for explaining the interactions of light with particles of matter.

Momentum of a Photon

Unlike a particle of matter that is characterized by its rest mass m_0 , a photon is massless. In a vacuum, unlike a particle of matter that may vary its speed but cannot reach the speed of light, a photon travels at only one speed, which is exactly the speed of light. From the point of view of Newtonian classical mechanics, these two characteristics imply that a photon should not exist at all. For example, how can we find the linear momentum or kinetic energy of a body whose mass is zero? This apparent paradox vanishes if we describe a photon as a relativistic particle. According to the theory of special relativity, any particle in nature obeys the relativistic energy equation

$$E^2 = p^2 c^2 + m_0^2 c^4. \quad 6.17$$

This relation can also be applied to a photon. In [Equation 6.17](#), E is the total energy of a particle, p is its linear momentum, and m_0 is its rest mass. For a photon, we simply set $m_0 = 0$ in this equation. This leads to the expression for the momentum p_f of a photon

$$p_f = \frac{E_f}{c}. \quad 6.18$$

Here the photon's energy E_f is the same as that of a light quantum of frequency f , which we introduced to explain the photoelectric effect:

$$E_f = hf = \frac{hc}{\lambda}. \quad 6.19$$

The wave relation that connects frequency f with wavelength λ and speed c also holds for photons:

$$\lambda f = c \quad 6.20$$

Therefore, a photon can be equivalently characterized by either its energy and wavelength, or its frequency and momentum. [Equation 6.19](#) and [Equation 6.20](#) can be combined into the explicit relation between a photon's momentum and its wavelength:

$$p_f = \frac{h}{\lambda}. \quad 6.21$$

Notice that this equation gives us only the magnitude of the photon's momentum and contains no information about the direction in which the photon is moving. To include the direction, it is customary to write the photon's momentum as a vector:

$$\vec{p}_f = \hbar \vec{k}. \quad 6.22$$

In [Equation 6.22](#), $\hbar = h/2\pi$ is the **reduced Planck's constant** (pronounced “h-bar”), which is just Planck's constant divided by the factor 2π . Vector $\vec{k}$ is called the “wave vector” or propagation vector (the direction in which a photon is moving). The **propagation vector** shows the direction of the photon's linear momentum vector. The magnitude of the wave vector is $k = |\vec{k}| = 2\pi/\lambda$ and is called the **wave number**. Notice that this

equation does not introduce any new physics. We can verify that the magnitude of the vector in [Equation 6.22](#) is the same as that given by [Equation 6.18](#).

The Compton Effect

The **Compton effect** is the term used for an unusual result observed when X-rays are scattered on some materials. By classical theory, when an electromagnetic wave is scattered off atoms, the wavelength of the scattered radiation is expected to be the same as the wavelength of the incident radiation. Contrary to this prediction of classical physics, observations show that when X-rays are scattered off some materials, such as graphite, the scattered X-rays have different wavelengths from the wavelength of the incident X-rays. This classically unexplainable phenomenon was studied experimentally by Arthur H. Compton and his collaborators, and Compton gave its explanation in 1923.

To explain the shift in wavelengths measured in the experiment, Compton used Einstein's idea of light as a particle. The Compton effect has a very important place in the history of physics because it shows that electromagnetic radiation cannot be explained as a purely wave phenomenon. The explanation of the Compton effect gave a convincing argument to the physics community that electromagnetic waves can indeed behave like a stream of photons, which placed the concept of a photon on firm ground.

The schematics of Compton's experimental setup are shown in [Figure 6.11](#). The idea of the experiment is straightforward: Monochromatic X-rays with wavelength λ are incident on a sample of graphite (the "target"), where they interact with atoms inside the sample; they later emerge as scattered X-rays with wavelength λ' . A detector placed behind the target can measure the intensity of radiation scattered in any direction θ with respect to the direction of the incident X-ray beam. This **scattering angle**, θ , is the angle between the direction of the scattered beam and the direction of the incident beam. In this experiment, we know the intensity and the wavelength λ of the incoming (incident) beam; and for a given scattering angle θ , we measure the intensity and the wavelength λ' of the outgoing (scattered) beam. Typical results of these measurements are shown in [Figure 6.12](#), where the x-axis is the wavelength of the scattered X-rays and the y-axis is the intensity of the scattered X-rays, measured for different scattering angles (indicated on the graphs). For all scattering angles (except for $\theta = 0^\circ$), we measure two intensity peaks. One peak is located at the wavelength λ , which is the wavelength of the incident beam. The other peak is located at some other wavelength, λ' . The two peaks are separated by $\Delta\lambda$, which depends on the scattering angle θ of the outgoing beam (in the direction of observation). The separation $\Delta\lambda$ is called the **Compton shift**.

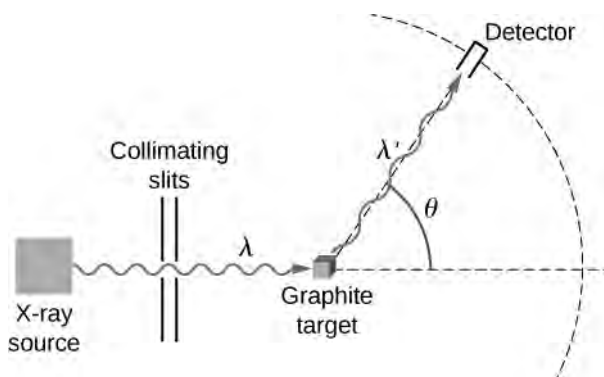


Figure 6.11 Experimental setup for studying Compton scattering.

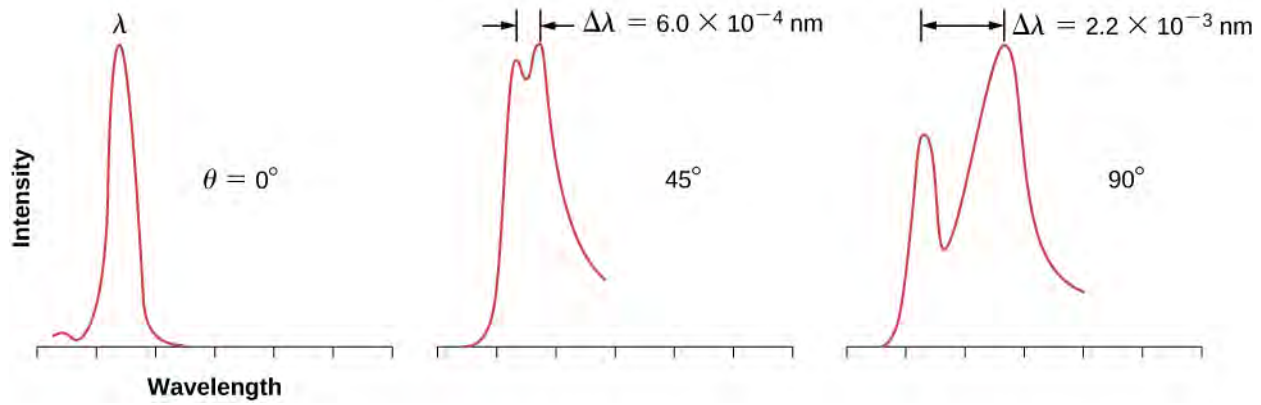


Figure 6.12 Experimental data show the Compton effect for X-rays scattering off graphite at various angles: The intensity of the scattered beam has two peaks. One peak appears at the wavelength λ of the incident radiation and the second peak appears at wavelength λ' . The separation $\Delta\lambda$ between the peaks depends on the scattering angle θ , which is the angular position of the detector in [Figure 6.11](#). The experimental data in this figure are plotted in arbitrary units so that the height of the profile reflects the intensity of the scattered beam above background noise.

Compton Shift

As given by Compton, the explanation of the Compton shift is that in the target material, graphite, valence electrons are loosely bound in the atoms and behave like free electrons. Compton assumed that the incident X-ray radiation is a stream of photons. An incoming photon in this stream collides with a valence electron in the graphite target. In the course of this collision, the incoming photon transfers some part of its energy and momentum to the target electron and leaves the scene as a scattered photon. This model explains in qualitative terms why the scattered radiation has a longer wavelength than the incident radiation. Put simply, a photon that has lost some of its energy emerges as a photon with a lower frequency, or equivalently, with a longer wavelength. To show that his model was correct, Compton used it to derive the expression for the Compton shift. In his derivation, he assumed that both photon and electron are relativistic particles and that the collision obeys two commonsense principles: (1) the conservation of linear momentum and (2) the conservation of total relativistic energy.

In the following derivation of the Compton shift, E_f and $\vec{p}_f$ denote the energy and momentum, respectively, of an incident photon with frequency f . The photon collides with a relativistic electron at rest, which means that immediately before the collision, the electron's energy is entirely its rest mass energy, m_0c^2 . Immediately after the collision, the electron has energy E and momentum $\vec{p}$, both of which satisfy [Equation 6.19](#).

Immediately after the collision, the outgoing photon has energy $\tilde{E}_f$, momentum $\vec{\tilde{p}}_f$, and frequency f' . The direction of the incident photon is horizontal from left to right, and the direction of the outgoing photon is at the angle θ , as illustrated in [Figure 6.11](#). The scattering angle θ is the angle between the momentum vectors $\vec{p}_f$ and $\vec{\tilde{p}}_f$, and we can write their scalar product:

$$\vec{p}_f \cdot \vec{\tilde{p}}_f = p_f \tilde{p}_f \cos\theta. \quad 6.23$$

Following Compton's argument, we assume that the colliding photon and electron form an isolated system. This assumption is valid for weakly bound electrons that, to a good approximation, can be treated as free particles. Our first equation is the conservation of energy for the photon-electron system:

$$E_f + m_0c^2 = \tilde{E}_f + E. \quad 6.24$$

The left side of this equation is the energy of the system at the instant immediately before the collision, and the right side of the equation is the energy of the system at the instant immediately after the collision. Our second equation is the conservation of linear momentum for the photon-electron system where the electron is at rest at the instant immediately before the collision:

$$\vec{p}_f = \vec{\tilde{p}}_f + \vec{p}. \quad 6.25$$

The left side of this equation is the momentum of the system right before the collision, and the right side of the equation is the momentum of the system right after collision. The entire physics of Compton scattering is contained in these three preceding equations—the remaining part is algebra. At this point, we could jump to the concluding formula for the Compton shift, but it is beneficial to highlight the main algebraic steps that lead to Compton's formula, which we give here as follows.

We start with rearranging the terms in [Equation 6.24](#) and squaring it:

$$\left[(E_f - \widetilde{E}_f) + m_0c^2\right]^2 = E^2.$$

In the next step, we substitute [Equation 6.19](#) for E^2 , simplify, and divide both sides by c^2 to obtain

$$(E_f/c - \widetilde{E}_f/c)^2 + 2m_0c(E_f/c - \widetilde{E}_f/c) = p^2.$$

Now we can use [Equation 6.21](#) to express this form of the energy equation in terms of momenta. The result is

$$(p_f - \widetilde{p}_f)^2 + 2m_0c(p_f - \widetilde{p}_f) = p^2. \quad 6.26$$

To eliminate p^2 , we turn to the momentum equation [Equation 6.25](#), rearrange its terms, and square it to obtain

$$(\vec{p}_f - \vec{\widetilde{p}}_f)^2 = p^2 \text{ and } (\vec{p}_f - \vec{\widetilde{p}}_f)^2 = p_f^2 + \widetilde{p}_f^2 - 2\vec{p}_f \cdot \vec{\widetilde{p}}_f.$$

The product of the momentum vectors is given by [Equation 6.23](#). When we substitute this result for p^2 in [Equation 6.26](#), we obtain the energy equation that contains the scattering angle θ :

$$(p_f - \widetilde{p}_f)^2 + 2m_0c(p_f - \widetilde{p}_f) = p_f^2 + \widetilde{p}_f^2 - 2p_f\widetilde{p}_f\cos\theta.$$

With further algebra, this result can be simplified to

$$\frac{1}{\widetilde{p}_f} - \frac{1}{p_f} = \frac{1}{m_0c}(1 - \cos\theta). \quad 6.27$$

Now recall [Equation 6.21](#) and write: $1/\widetilde{p}_f = \lambda'/h$ and $1/p_f = \lambda/h$. When these relations are substituted into [Equation 6.27](#), we obtain the relation for the Compton shift:

$$\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta). \quad 6.28$$

The factor h/m_0c is called the **Compton wavelength** of the electron:

$$\lambda_c = \frac{h}{m_0c} = 0.00243 \text{ nm} = 2.43 \text{ pm}. \quad 6.29$$

Denoting the shift as $\Delta\lambda = \lambda' - \lambda$, the concluding result can be rewritten as

$$\Delta\lambda = \lambda_c(1 - \cos\theta). \quad 6.30$$

This formula for the Compton shift describes outstandingly well the experimental results shown in [Figure 6.12](#). Scattering data measured for molybdenum, graphite, calcite, and many other target materials are in accord with this theoretical result. The nonshifted peak shown in [Figure 6.12](#) is due to photon collisions with tightly bound inner electrons in the target material. Photons that collide with the inner electrons of the target atoms in fact collide with the entire atom. In this extreme case, the rest mass in [Equation 6.29](#) must be changed to the rest mass of the atom. This type of shift is four orders of magnitude smaller than the shift caused by collisions with electrons and is so small that it can be neglected.

Compton scattering is an example of **inelastic scattering**, in which the scattered radiation has a longer wavelength than the wavelength of the incident radiation. In today's usage, the term "Compton scattering" is used for the inelastic scattering of photons by free, charged particles. In Compton scattering, treating photons as particles with momenta that can be transferred to charged particles provides the theoretical background to

explain the wavelength shifts measured in experiments; this is the evidence that radiation consists of photons.

EXAMPLE 6.8

Compton Scattering

An incident 71-pm X-ray is incident on a calcite target. Find the wavelength of the X-ray scattered at a 30° angle. What is the largest shift that can be expected in this experiment?

Strategy

To find the wavelength of the scattered X-ray, first we must find the Compton shift for the given scattering angle, $\theta = 30^\circ$. We use [Equation 6.30](#). Then we add this shift to the incident wavelength to obtain the scattered wavelength. The largest Compton shift occurs at the angle θ when $1 - \cos\theta$ has the largest value, which is for the angle $\theta = 180^\circ$.

Solution

The shift at $\theta = 30^\circ$ is

$$\Delta\lambda = \lambda_c(1 - \cos 30^\circ) = 0.134\lambda_c = (0.134)(2.43) \text{ pm} = 0.325 \text{ pm}.$$

This gives the scattered wavelength:

$$\lambda' = \lambda + \Delta\lambda = (71 + 0.325) \text{ pm} = 71.325 \text{ pm}.$$

The largest shift is

$$(\Delta\lambda)_{\max} = \lambda_c(1 - \cos 180^\circ) = 2(2.43 \text{ pm}) = 4.86 \text{ pm}.$$

Significance

The largest shift in wavelength is detected for the backscattered radiation; however, most of the photons from the incident beam pass through the target and only a small fraction of photons gets backscattered (typically, less than 5%). Therefore, these measurements require highly sensitive detectors.

CHECK YOUR UNDERSTANDING 6.8

An incident 71-pm X-ray is incident on a calcite target. Find the wavelength of the X-ray scattered at a 60° angle. What is the smallest shift that can be expected in this experiment?

6.4 Bohr's Model of the Hydrogen Atom

Learning Objectives

By the end of this section, you will be able to:

- Explain the difference between the absorption spectrum and the emission spectrum of radiation emitted by atoms
- Describe the Rutherford gold foil experiment and the discovery of the atomic nucleus
- Explain the atomic structure of hydrogen
- Describe the postulates of the early quantum theory for the hydrogen atom
- Summarize how Bohr's quantum model of the hydrogen atom explains the radiation spectrum of atomic hydrogen

Historically, Bohr's model of the hydrogen atom is the very first model of atomic structure that correctly explained the radiation spectra of atomic hydrogen. The model has a special place in the history of physics because it introduced an early quantum theory, which brought about new developments in scientific thought and later culminated in the development of quantum mechanics. To understand the specifics of Bohr's model, we must first review the nineteenth-century discoveries that prompted its formulation.

When we use a prism to analyze white light coming from the sun, several dark lines in the solar spectrum are

observed (Figure 6.13). Solar absorption lines are called **Fraunhofer lines** after Joseph von Fraunhofer, who accurately measured their wavelengths. During 1854–1861, Gustav Kirchhoff and Robert Bunsen discovered that for the various chemical elements, the line **emission spectrum** of an element exactly matches its line **absorption spectrum**. The difference between the absorption spectrum and the emission spectrum is explained in Figure 6.14. An absorption spectrum is observed when light passes through a gas. This spectrum appears as black lines that occur only at certain wavelengths on the background of the continuous spectrum of white light (Figure 6.13). The missing wavelengths tell us which wavelengths of the radiation are absorbed by the gas. The emission spectrum is observed when light is emitted by a gas. This spectrum is seen as colorful lines on the black background (see Figure 6.15 and Figure 6.16). Positions of the emission lines tell us which wavelengths of the radiation are emitted by the gas. Each chemical element has its own characteristic emission spectrum. For each element, the positions of its emission lines are exactly the same as the positions of its absorption lines. This means that atoms of a specific element absorb radiation only at specific wavelengths and radiation that does not have these wavelengths is not absorbed by the element at all. This also means that the radiation emitted by atoms of each element has exactly the same wavelengths as the radiation they absorb.

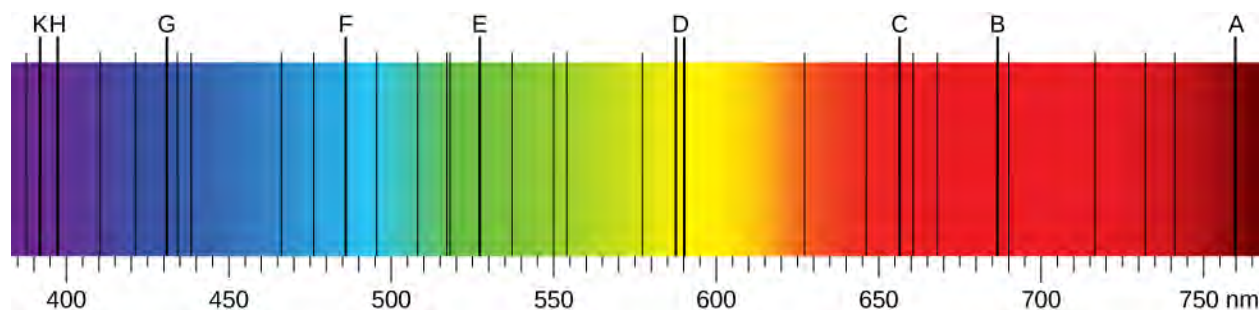


Figure 6.13 In the solar emission spectrum in the visible range from 380 nm to 710 nm, Fraunhofer lines are observed as vertical black lines at specific spectral positions in the continuous spectrum. Highly sensitive modern instruments observe thousands of such lines.

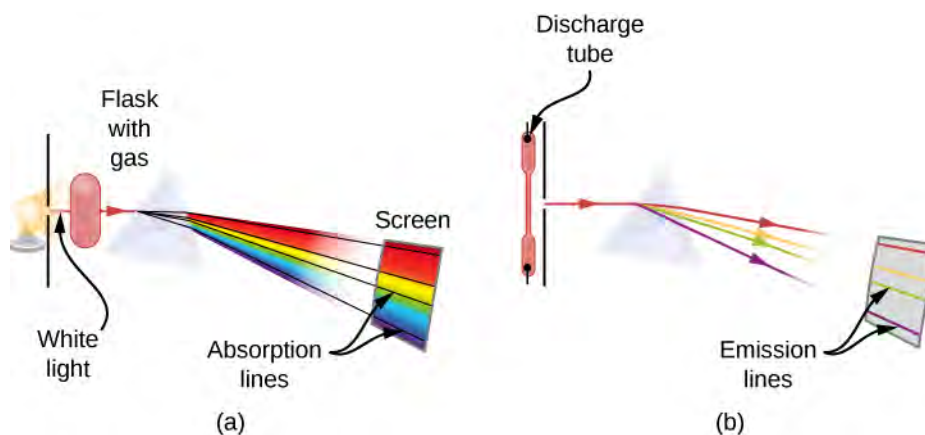


Figure 6.14 Observation of line spectra: (a) setup to observe absorption lines; (b) setup to observe emission lines. (a) White light passes through a cold gas that is contained in a glass flask. A prism is used to separate wavelengths of the passed light. In the spectrum of the passed light, some wavelengths are missing, which are seen as black absorption lines in the continuous spectrum on the viewing screen. (b) A gas is contained in a glass discharge tube that has electrodes at its ends. At a high potential difference between the electrodes, the gas glows and the light emitted from the gas passes through the prism that separates its wavelengths. In the spectrum of the emitted light, only specific wavelengths are present, which are seen as colorful emission lines on the screen.

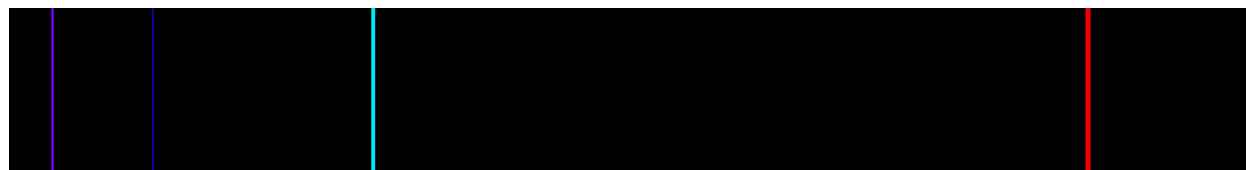


Figure 6.15 The emission spectrum of atomic hydrogen: The spectral positions of emission lines are characteristic for hydrogen atoms. (credit: “Merikanto”/Wikimedia Commons)

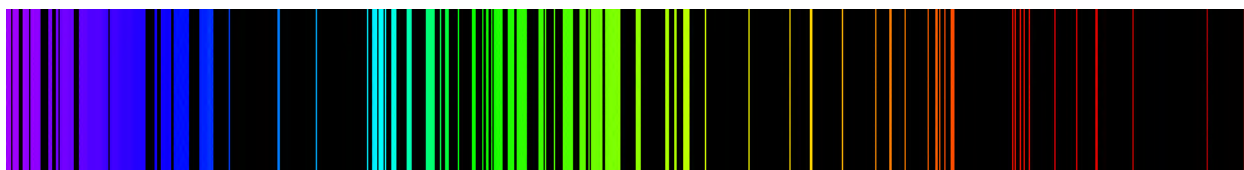


Figure 6.16 The emission spectrum of atomic iron: The spectral positions of emission lines are characteristic for iron atoms.

Emission spectra of the elements have complex structures; they become even more complex for elements with higher atomic numbers. The simplest spectrum, shown in [Figure 6.15](#), belongs to the hydrogen atom. Only four lines are visible to the human eye. As you read from right to left in [Figure 6.15](#), these lines are: red (656 nm), called the H- α line; aqua (486 nm), blue (434 nm), and violet (410 nm). The lines with wavelengths shorter than 400 nm appear in the ultraviolet part of the spectrum ([Figure 6.15](#), far left) and are invisible to the human eye. There are infinitely many invisible spectral lines in the series for hydrogen.

An empirical formula to describe the positions (wavelengths) λ of the hydrogen emission lines in this series was discovered in 1885 by Johann Balmer. It is known as the **Balmer formula**:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - \frac{1}{n^2} \right). \quad 6.31$$

The constant $R_H = 1.09737 \times 10^7 \text{ m}^{-1}$ is called the **Rydberg constant for hydrogen**. In [Equation 6.31](#), the positive integer n takes on values $n = 3, 4, 5, 6$ for the four visible lines in this series. The series of emission lines given by the Balmer formula is called the **Balmer series** for hydrogen. Other emission lines of hydrogen that were discovered in the twentieth century are described by the **Rydberg formula**, which summarizes all of the experimental data:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \text{ where } n_i = n_f + 1, n_f + 2, n_f + 3, \dots \quad 6.32$$

When $n_f = 1$, the series of spectral lines is called the **Lyman series**. When $n_f = 2$, the series is called the Balmer series, and in this case, the Rydberg formula coincides with the Balmer formula. When $n_f = 3$, the series is called the **Paschen series**. When $n_f = 4$, the series is called the **Brackett series**. When $n_f = 5$, the series is called the **Pfund series**. When $n_f = 6$, we have the **Humphreys series**. As you may guess, there are infinitely many such spectral bands in the spectrum of hydrogen because n_f can be any positive integer number.

The Rydberg formula for hydrogen gives the exact positions of the spectral lines as they are observed in a laboratory; however, at the beginning of the twentieth century, nobody could explain why it worked so well. The Rydberg formula remained unexplained until the first successful model of the hydrogen atom was proposed in 1913.

EXAMPLE 6.9

Limits of the Balmer Series

Calculate the longest and the shortest wavelengths in the Balmer series.

Strategy

We can use either the Balmer formula or the Rydberg formula. The longest wavelength is obtained when $1/n_i$ is largest, which is when $n_i = n_f + 1 = 3$, because $n_f = 2$ for the Balmer series. The smallest wavelength is obtained when $1/n_i$ is smallest, which is $1/n_i \rightarrow 0$ when $n_i \rightarrow \infty$.

Solution

The long-wave limit:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - \frac{1}{3^2} \right) = (1.09737 \times 10^7) \frac{1}{\text{m}} \left(\frac{1}{4} - \frac{1}{9} \right) \Rightarrow \lambda = 656.3 \text{ nm}$$

The short-wave limit:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - 0 \right) = (1.09737 \times 10^7) \frac{1}{\text{m}} \left(\frac{1}{4} \right) \Rightarrow \lambda = 364.6 \text{ nm}$$

Significance

Note that there are infinitely many spectral lines lying between these two limits.

 CHECK YOUR UNDERSTANDING 6.9

What are the limits of the Lyman series? Can you see these spectral lines?

The key to unlocking the mystery of atomic spectra is in understanding atomic structure. Scientists have long known that matter is made of atoms. According to nineteenth-century science, atoms are the smallest indivisible quantities of matter. This scientific belief was shattered by a series of groundbreaking experiments that proved the existence of subatomic particles, such as electrons, protons, and neutrons.

The electron was discovered and identified as the smallest quantity of electric charge by J.J. Thomson in 1897 in his cathode ray experiments, also known as β -ray experiments: A **β -ray** is a beam of electrons. In 1904, Thomson proposed the first model of atomic structure, known as the “plum pudding” model, in which an atom consisted of an unknown positively charged matter with negative electrons embedded in it like plums in a pudding. Around 1900, E. Rutherford, and independently, Paul Ulrich Villard, classified all radiation known at that time as **α -rays**, **β -rays**, and **γ -rays** (a γ -ray is a beam of highly energetic photons). In 1907, Rutherford and Thomas Royds used spectroscopy methods to show that positively charged particles of α -radiation (called **α -particles**) are in fact doubly ionized atoms of helium. In 1909, Rutherford, Ernest Marsden, and Hans Geiger used α -particles in their famous scattering experiment that disproved Thomson’s model (see [Linear Momentum and Collisions](#)).

In the **Rutherford gold foil experiment** (also known as the Geiger–Marsden experiment), α -particles were incident on a thin gold foil and were scattered by gold atoms inside the foil (see [Types of Collisions](#)). The outgoing particles were detected by a 360° scintillation screen surrounding the gold target (for a detailed description of the experimental setup, see [Linear Momentum and Collisions](#)). When a scattered particle struck the screen, a tiny flash of light (scintillation) was observed at that location. By counting the scintillations seen at various angles with respect to the direction of the incident beam, the scientists could determine what fraction of the incident particles were scattered and what fraction were not deflected at all. If the plum pudding model were correct, there would be no back-scattered α -particles. However, the results of the Rutherford experiment showed that, although a sizable fraction of α -particles emerged from the foil not scattered at all as though the foil were not in their way, a significant fraction of α -particles were back-scattered toward the source. This kind of result was possible only when most of the mass and the entire positive charge of the gold atom were concentrated in a tiny space inside the atom.

In 1911, Rutherford proposed a **nuclear model of the atom**. In Rutherford’s model, an atom contained a positively charged nucleus of negligible size, almost like a point, but included almost the entire mass of the atom. The atom also contained negative electrons that were located within the atom but relatively far away from the nucleus. Ten years later, Rutherford coined the name *proton* for the nucleus of hydrogen and the name *neutron* for a hypothetical electrically neutral particle that would mediate the binding of positive protons in the nucleus (the neutron was discovered in 1932 by James Chadwick). Rutherford is credited with the discovery of the atomic nucleus; however, the Rutherford model of atomic structure does not explain the Rydberg formula for the hydrogen emission lines.

Bohr's model of the hydrogen atom, proposed by Niels Bohr in 1913, was the first quantum model that correctly explained the hydrogen emission spectrum. Bohr's model combines the classical mechanics of planetary motion with the quantum concept of photons. Once Rutherford had established the existence of the atomic nucleus, Bohr's intuition that the negative electron in the hydrogen atom must revolve around the positive nucleus became a logical consequence of the inverse-square-distance law of electrostatic attraction. Recall that Coulomb's law describing the attraction between two opposite charges has a similar form to Newton's universal law of gravitation in the sense that the gravitational force and the electrostatic force are both decreasing as $1/r^2$, where r is the separation distance between the bodies. In the same way as Earth revolves around the sun, the negative electron in the hydrogen atom can revolve around the positive nucleus. However, an accelerating charge radiates its energy. Classically, if the electron moved around the nucleus in a planetary fashion, it would be undergoing centripetal acceleration, and thus would be radiating energy that would cause it to spiral down into the nucleus. Such a planetary hydrogen atom would not be stable, which is contrary to what we know about ordinary hydrogen atoms that do not disintegrate. Moreover, the classical motion of the electron is not able to explain the discrete emission spectrum of hydrogen.

To circumvent these two difficulties, Bohr proposed the following three **postulates of Bohr's model**:

1. The negative electron moves around the positive nucleus (proton) in a circular orbit. All electron orbits are centered at the nucleus. Not all classically possible orbits are available to an electron bound to the nucleus.
2. The allowed electron orbits satisfy the *first quantization condition*: In the n th orbit, the angular momentum L_n of the electron can take only discrete values:

$$L_n = n\hbar, \text{ where } n = 1, 2, 3, \dots \quad 6.33$$

This postulate says that the electron's angular momentum is quantized. Denoted by r_n and v_n , respectively, the radius of the n th orbit and the electron's speed in it, the first quantization condition can be expressed explicitly as

$$m_e v_n r_n = n\hbar. \quad 6.34$$

3. An electron is allowed to make transitions from one orbit where its energy is E_n to another orbit where its energy is E_m . When an atom absorbs a photon, the electron makes a transition to a higher-energy orbit. When an atom emits a photon, the electron transits to a lower-energy orbit. Electron transitions with the simultaneous photon absorption or photon emission take place *instantaneously*. The allowed electron transitions satisfy the *second quantization condition*:

$$hf = |E_n - E_m| \quad 6.35$$

where hf is the energy of either an emitted or an absorbed photon with frequency f . The second quantization condition states that an electron's change in energy in the hydrogen atom is quantized.

These three postulates of the early quantum theory of the hydrogen atom allow us to derive not only the Rydberg formula, but also the value of the Rydberg constant and other important properties of the hydrogen atom such as its energy levels, its ionization energy, and the sizes of electron orbits. Note that in Bohr's model, along with two nonclassical quantization postulates, we also have the classical description of the electron as a particle that is subjected to the Coulomb force, and its motion must obey Newton's laws of motion. The hydrogen atom, as an isolated system, must obey the laws of conservation of energy and momentum in the way we know from classical physics. Having this theoretical framework in mind, we are ready to proceed with our analysis.

Electron Orbits

To obtain the size r_n of the electron's n th orbit and the electron's speed v_n in it, we turn to Newtonian mechanics. As a charged particle, the electron experiences an electrostatic pull toward the positively charged nucleus in the center of its circular orbit. This electrostatic pull is the centripetal force that causes the electron

to move in a circle around the nucleus. Therefore, the magnitude of centripetal force is identified with the magnitude of the electrostatic force:

$$\frac{m_e v_n^2}{r_n} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_n^2}. \quad 6.36$$

Here, e denotes the value of the elementary charge. The negative electron and positive proton have the same value of charge, $|q| = e$. When [Equation 6.36](#) is combined with the first quantization condition given by [Equation 6.34](#), we can solve for the speed, v_n , and for the radius, r_n :

$$v_n = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar} \frac{1}{n} \quad 6.37$$

$$r_n = 4\pi\epsilon_0 \frac{\hbar^2}{m_e e^2} n^2. \quad 6.38$$

Note that these results tell us that the electron's speed as well as the radius of its orbit depend only on the index n that enumerates the orbit because all other quantities in the preceding equations are fundamental constants. We see from [Equation 6.38](#) that the size of the orbit grows as the square of n . This means that the second orbit is four times as large as the first orbit, and the third orbit is nine times as large as the first orbit, and so on. We also see from [Equation 6.37](#) that the electron's speed in the orbit decreases as the orbit size increases. The electron's speed is largest in the first Bohr orbit, for $n = 1$, which is the orbit closest to the nucleus. The radius of the first Bohr orbit is called the **Bohr radius of hydrogen**, denoted as a_0 . Its value is obtained by setting $n = 1$ in [Equation 6.38](#):

$$a_0 = 4\pi\epsilon_0 \frac{\hbar^2}{m_e e^2} = 5.29 \times 10^{-11} \text{ m} = 0.529 \text{ \AA}. \quad 6.39$$

We can substitute a_0 in [Equation 6.38](#) to express the radius of the n th orbit in terms of a_0 :

$$r_n = a_0 n^2. \quad 6.40$$

This result means that the electron orbits in hydrogen atom are *quantized* because the orbital radius takes on only specific values of $a_0, 4a_0, 9a_0, 16a_0, \dots$ given by [Equation 6.40](#), and no other values are allowed.

Electron Energies

The total energy E_n of an electron in the n th orbit is the sum of its kinetic energy K_n and its electrostatic potential energy U_n . Utilizing [Equation 6.37](#), we find that

$$K_n = \frac{1}{2} m_e v_n^2 = \frac{1}{32\pi^2\epsilon_0^2} \frac{m_e e^4}{\hbar^2} \frac{1}{n^2}. \quad 6.41$$

Recall that the electrostatic potential energy of interaction between two charges q_1 and q_2 that are separated by a distance r_{12} is $(1/4\pi\epsilon_0)q_1 q_2 / r_{12}$. Here, $q_1 = +e$ is the charge of the nucleus in the hydrogen atom (the charge of the proton), $q_2 = -e$ is the charge of the electron and $r_{12} = r_n$ is the radius of the n th orbit. Now we use [Equation 6.38](#) to find the potential energy of the electron:

$$U_n = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r_n} = -\frac{1}{16\pi^2\epsilon_0^2} \frac{m_e e^4}{\hbar^2} \frac{1}{n^2}. \quad 6.42$$

The total energy of the electron is the sum of [Equation 6.41](#) and [Equation 6.42](#):

$$E_n = K_n + U_n = -\frac{1}{32\pi^2\epsilon_0^2} \frac{m_e e^4}{\hbar^2} \frac{1}{n^2}. \quad 6.43$$

Note that the energy depends only on the index n because the remaining symbols in Equation 6.43 are physical constants. The value of the constant factor in Equation 6.43 is

$$E_0 = \frac{1}{32\pi^2\epsilon_0^2} \frac{m_e e^4}{\hbar^2} = \frac{1}{8\epsilon_0^2} \frac{m_e e^4}{h^2} = 2.17 \times 10^{-18} \text{ J} = 13.6 \text{ eV}. \quad 6.44$$

It is convenient to express the electron's energy in the n th orbit in terms of this energy, as

$$E_n = -E_0 \frac{1}{n^2}. \quad 6.45$$

Now we can see that the electron energies in the hydrogen atom are *quantized* because they can have only discrete values of $-E_0, -E_0/4, -E_0/9, -E_0/16, \dots$ given by Equation 6.45, and no other energy values are allowed. This set of allowed electron energies is called the **energy spectrum of hydrogen** (Figure 6.17). The index n that enumerates energy levels in Bohr's model is called the energy **quantum number**. We identify the energy of the electron inside the hydrogen atom with the energy of the hydrogen atom. Note that the smallest value of energy is obtained for $n = 1$, so the hydrogen atom cannot have energy smaller than that. This smallest value of the electron energy in the hydrogen atom is called the **ground state energy of the hydrogen atom** and its value is

$$E_1 = -E_0 = -13.6 \text{ eV}. \quad 6.46$$

The hydrogen atom may have other energies that are higher than the ground state. These higher energy states are known as **excited energy states of a hydrogen atom**.

There is only one ground state, but there are infinitely many excited states because there are infinitely many values of n in Equation 6.45. We say that the electron is in the “first excited state” when its energy is E_2 (when $n = 2$), the second excited state when its energy is E_3 (when $n = 3$) and, in general, in the n th excited state when its energy is E_{n+1} . There is no highest-of-all excited state; however, there is a limit to the sequence of excited states. If we keep increasing n in Equation 6.45, we find that the limit is $-\lim_{n \rightarrow \infty} E_0/n^2 = 0$. In this limit,

the electron is no longer bound to the nucleus but becomes a free electron. An electron remains bound in the hydrogen atom as long as its energy is negative. An electron that orbits the nucleus in the first Bohr orbit, closest to the nucleus, is in the ground state, where its energy has the smallest value. In the ground state, the electron is most strongly bound to the nucleus and its energy is given by Equation 6.46. If we want to remove this electron from the atom, we must supply it with enough energy, E_∞ , to at least balance out its ground state

energy E_1 :

$$E_\infty + E_1 = 0 \Rightarrow E_\infty = -E_1 = -(-E_0) = E_0 = 13.6 \text{ eV}. \quad 6.47$$

The energy that is needed to remove the electron from the atom is called the **ionization energy**. The ionization energy E_∞ that is needed to remove the electron from the first Bohr orbit is called the **ionization limit of the hydrogen atom**. The ionization limit in Equation 6.47 that we obtain in Bohr's model agrees with experimental value.

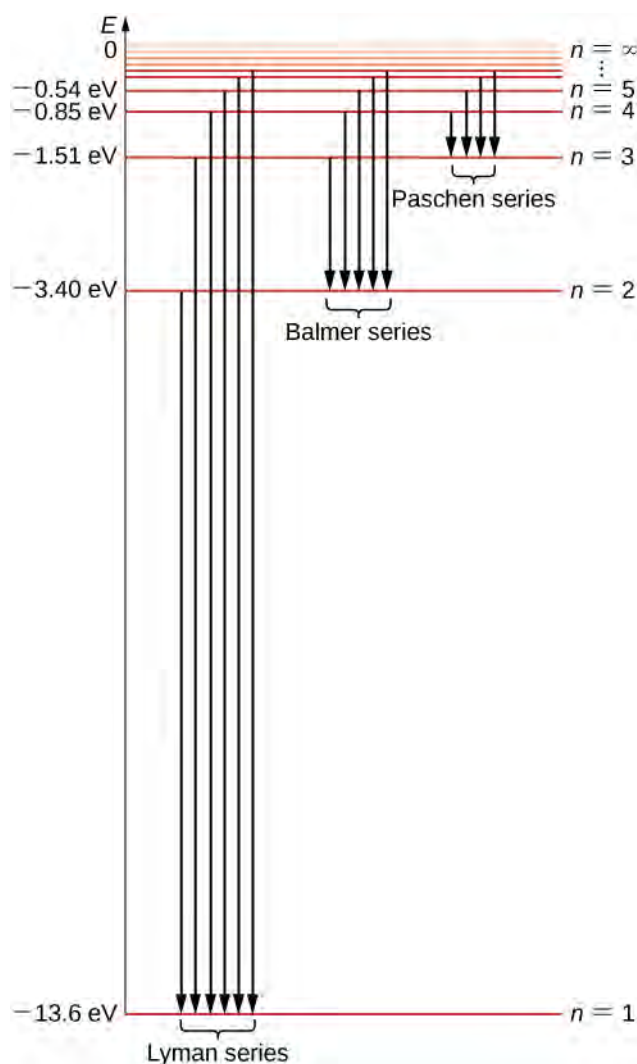


Figure 6.17 The energy spectrum of the hydrogen atom. Energy levels (horizontal lines) represent the bound states of an electron in the atom. There is only one ground state, $n = 1$, and infinite quantized excited states. The states are enumerated by the quantum number $n = 1, 2, 3, 4, \dots$. Vertical lines illustrate the allowed electron transitions between the states. Downward arrows illustrate transitions with an emission of a photon with a wavelength in the indicated spectral band.

Spectral Emission Lines of Hydrogen

To obtain the wavelengths of the emitted radiation when an electron makes a transition from the n th orbit to the m th orbit, we use the second of Bohr's quantization conditions and [Equation 6.45](#) for energies. The emission of energy from the atom can occur only when an electron makes a transition from an excited state to a lower-energy state. In the course of such a transition, the emitted photon carries away the difference of energies between the states involved in the transition. The transition cannot go in the other direction because the energy of a photon cannot be negative, which means that for emission we must have $E_n > E_m$ and $n > m$. Therefore, the third of Bohr's postulates gives

$$hf = |E_n - E_m| = E_n - E_m = -E_0 \frac{1}{n^2} + E_0 \frac{1}{m^2} = E_0 \left(\frac{1}{m^2} - \frac{1}{n^2} \right). \quad 6.48$$

Now we express the photon's energy in terms of its wavelength, $hf = hc/\lambda$, and divide both sides of [Equation 6.48](#) by hc . The result is

$$\frac{1}{\lambda} = \frac{E_0}{hc} \left(\frac{1}{m^2} - \frac{1}{n^2} \right). \quad 6.49$$

The value of the constant in this equation is

$$\frac{E_0}{hc} = \frac{13.6 \text{ eV}}{(4.136 \times 10^{-15} \text{ eV} \cdot \text{s})(2.997 \times 10^8 \text{ m/s})} = 1.097 \times 10^7 \frac{1}{\text{m}}. \quad 6.50$$

This value is exactly the Rydberg constant R_H in the Rydberg heuristic formula [Equation 6.32](#). In fact, [Equation 6.49](#) is identical to the Rydberg formula, because for a given m , we have $n = m + 1, m + 2, \dots$. In this way, the Bohr quantum model of the hydrogen atom allows us to derive the experimental Rydberg constant from first principles and to express it in terms of fundamental constants. Transitions between the allowed electron orbits are illustrated in [Figure 6.17](#).

We can repeat the same steps that led to [Equation 6.49](#) to obtain the wavelength of the absorbed radiation; this again gives [Equation 6.49](#) but this time for the positions of absorption lines in the absorption spectrum of hydrogen. The only difference is that for absorption, the quantum number m is the index of the orbit occupied by the electron before the transition (lower-energy orbit) and the quantum number n is the index of the orbit to which the electron makes the transition (higher-energy orbit). The difference between the electron energies in these two orbits is the energy of the absorbed photon.



EXAMPLE 6.10

Size and Ionization Energy of the Hydrogen Atom in an Excited State

If a hydrogen atom in the ground state absorbs a 93.7-nm photon, corresponding to a transition line in the Lyman series, how does this affect the atom's energy and size? How much energy is needed to ionize the atom when it is in this excited state? Give your answers in absolute units, and relative to the ground state.

Strategy

Before the absorption, the atom is in its ground state. This means that the electron transition takes place from the orbit $m = 1$ to some higher n th orbit. First, we must determine n for the absorbed wavelength $\lambda = 93.7 \text{ nm}$. Then, we can use [Equation 6.45](#) to find the energy E_n of the excited state and its ionization energy $E_{\infty, n}$, and use [Equation 6.40](#) to find the radius r_n of the atom in the excited state. To estimate n , we use [Equation 6.49](#).

Solution

Substitute $m = 1$ and $\lambda = 93.7 \text{ nm}$ in [Equation 6.49](#) and solve for n . You should not expect to obtain a perfect integer answer because of rounding errors, but your answer will be close to an integer, and you can estimate n by taking the integral part of your answer:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{1^2} - \frac{1}{n^2} \right) \Rightarrow n = \frac{1}{\sqrt{1 - \frac{1}{\lambda R_H}}} = \frac{1}{\sqrt{1 - \frac{1}{(93.7 \times 10^{-9} \text{ m})(1.097 \times 10^7 \text{ m}^{-1})}}} = 6.07 \Rightarrow n = 6.$$

The radius of the $n = 6$ orbit is

$$r_n = a_0 n^2 = a_0 6^2 = 36a_0 = 36(0.529 \times 10^{-10} \text{ m}) = 19.04 \times 10^{-10} \text{ m} \cong 19.0 \text{ Å}.$$

Thus, after absorbing the 93.7-nm photon, the size of the hydrogen atom in the excited $n = 6$ state is 36 times larger than before the absorption, when the atom was in the ground state. The energy of the fifth excited state ($n = 6$) is:

$$E_n = -\frac{E_0}{n^2} = -\frac{E_0}{6^2} = -\frac{E_0}{36} = -\frac{13.6 \text{ eV}}{36} \cong -0.378 \text{ eV}.$$

After absorbing the 93.7-nm photon, the energy of the hydrogen atom is larger than it was before the absorption. Ionization of the atom when it is in the fifth excited state ($n = 6$) requires 36 times less energy than is needed when the atom is in the ground state:

$$E_{\infty, 6} = -E_6 = -(-0.378 \text{ eV}) = 0.378 \text{ eV}.$$

Significance

We can analyze any spectral line in the spectrum of hydrogen in the same way. Thus, the experimental measurements of spectral lines provide us with information about the atomic structure of the hydrogen atom.

CHECK YOUR UNDERSTANDING 6.10

When an electron in a hydrogen atom is in the first excited state, what prediction does the Bohr model give about its orbital speed and kinetic energy? What is the magnitude of its orbital angular momentum?

Bohr's model of the hydrogen atom also correctly predicts the spectra of some hydrogen-like ions. **Hydrogen-like ions** are atoms of elements with an atomic number Z larger than one ($Z = 1$ for hydrogen) but with all electrons removed except one. For example, an electrically neutral helium atom has an atomic number $Z = 2$. This means it has two electrons orbiting the nucleus with a charge of $q = +Ze$. When one of the orbiting electrons is removed from the helium atom (we say, when the helium atom is singly ionized), what remains is a hydrogen-like atomic structure where the remaining electron orbits the nucleus with a charge of $q = +Ze$. This type of situation is described by the Bohr model. Assuming that the charge of the nucleus is not $+e$ but $+Ze$, we can repeat all steps, beginning with [Equation 6.36](#), to obtain the results for a hydrogen-like ion:

$$r_n = \frac{a_0}{Z} n^2 \quad 6.51$$

where a_0 is the Bohr orbit of hydrogen, and

$$E_n = -Z^2 E_0 \frac{1}{n^2} \quad 6.52$$

where E_0 is the ionization limit of a hydrogen atom. These equations are good approximations as long as the atomic number Z is not too large.

The Bohr model is important because it was the first model to postulate the quantization of electron orbits in atoms. Thus, it represents an early quantum theory that gave a start to developing modern quantum theory. It introduced the concept of a quantum number to describe atomic states. The limitation of the early quantum theory is that it cannot describe atoms in which the number of electrons orbiting the nucleus is larger than one. The Bohr model of hydrogen is a semi-classical model because it combines the classical concept of electron orbits with the new concept of quantization. The remarkable success of this model prompted many physicists to seek an explanation for why such a model should work at all, and to seek an understanding of the physics behind the postulates of early quantum theory. This search brought about the onset of an entirely new concept of “matter waves.”

6.5 De Broglie's Matter Waves

Learning Objectives

By the end of this section, you will be able to:

- Describe de Broglie's hypothesis of matter waves
- Explain how the de Broglie's hypothesis gives the rationale for the quantization of angular momentum in Bohr's quantum theory of the hydrogen atom
- Describe the Davisson–Germer experiment
- Interpret de Broglie's idea of matter waves and how they account for electron diffraction phenomena

Compton's formula established that an electromagnetic wave can behave like a particle of light when interacting with matter. In 1924, Louis de Broglie proposed a new speculative hypothesis that electrons and other particles of matter can behave like waves. Today, this idea is known as **de Broglie's hypothesis of matter waves**. In 1926, De Broglie's hypothesis, together with Bohr's early quantum theory, led to the development of a new theory of **wave quantum mechanics** to describe the physics of atoms and subatomic particles. Quantum

mechanics has paved the way for new engineering inventions and technologies, such as the laser and magnetic resonance imaging (MRI). These new technologies drive discoveries in other sciences such as biology and chemistry.

According to de Broglie's hypothesis, massless photons as well as massive particles must satisfy one common set of relations that connect the energy E with the frequency f , and the linear momentum p with the wavelength λ . We have discussed these relations for photons in the context of Compton's effect. We are recalling them now in a more general context. Any particle that has energy and momentum is a **de Broglie wave** of frequency f and wavelength λ :

$$E = hf \quad 6.53$$

$$\lambda = \frac{h}{p}. \quad 6.54$$

Here, E and p are, respectively, the relativistic energy and the momentum of a particle. De Broglie's relations are usually expressed in terms of the wave vector $\vec{k}$, $k = 2\pi/\lambda$, and the wave frequency $\omega = 2\pi f$, as we usually do for waves:

$$E = \hbar\omega \quad 6.55$$

$$\vec{p} = \hbar\vec{k}. \quad 6.56$$

Wave theory tells us that a wave carries its energy with the **group velocity**. For matter waves, this group velocity is the velocity u of the particle. Identifying the energy E and momentum p of a particle with its relativistic energy mc^2 and its relativistic momentum mu , respectively, it follows from de Broglie relations that matter waves satisfy the following relation:

$$\lambda f = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p} = \frac{mc^2}{mu} = \frac{c^2}{u} = \frac{c}{\beta} \quad 6.57$$

where $\beta = u/c$. When a particle is massless we have $u = c$ and [Equation 6.57](#) becomes $\lambda f = c$.



EXAMPLE 6.11

How Long Are de Broglie Matter Waves?

Calculate the de Broglie wavelength of: (a) a 0.65-kg basketball thrown at a speed of 10 m/s, (b) a nonrelativistic electron with a kinetic energy of 1.0 eV, and (c) a relativistic electron with a kinetic energy of 108 keV.

Strategy

We use [Equation 6.57](#) to find the de Broglie wavelength. When the problem involves a nonrelativistic object moving with a nonrelativistic speed u , such as in (a) when $\beta = u/c \ll 1$, we use nonrelativistic momentum p . When the nonrelativistic approximation cannot be used, such as in (c), we must use the relativistic momentum $p = mu = m_0\gamma u = E_0\gamma\beta/c$, where the rest mass energy of a particle is $E_0 = mc^2$ and γ is the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$. The total energy E of a particle is given by [Equation 6.53](#) and the kinetic energy is $K = E - E_0 = (\gamma - 1)E_0$. When the kinetic energy is known, we can invert [Equation 6.18](#) to find the momentum $p = \sqrt{(E^2 - E_0^2)/c^2} = \sqrt{K(K + 2E_0)}/c$ and substitute in [Equation 6.57](#) to obtain

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K + 2E_0)}}. \quad 6.58$$

Depending on the problem at hand, in this equation we can use the following values for hc :

$$hc = (6.626 \times 10^{-34} \text{ J} \cdot \text{s})(2.998 \times 10^8 \text{ m/s}) = 1.986 \times 10^{-25} \text{ J} \cdot \text{m} = 1.241 \text{ eV} \cdot \mu\text{m}$$

Solution

- a. For the basketball, the kinetic energy is

$$K = mu^2/2 = (0.65 \text{ kg})(10 \text{ m/s})^2/2 = 32.5 \text{ J}$$

and the rest mass energy is

$$E_0 = mc^2 = (0.65 \text{ kg})(2.998 \times 10^8 \text{ m/s})^2 = 5.84 \times 10^{16} \text{ J}.$$

We see that $K/(K + E_0) \ll 1$ and use $p = mu = (0.65 \text{ kg})(10 \text{ m/s}) = 6.5 \text{ J} \cdot \text{s/m}$:

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{6.5 \text{ J} \cdot \text{s/m}} = 1.02 \times 10^{-34} \text{ m}.$$

- b. For the nonrelativistic electron,

$$E_0 = mc^2 = (9.109 \times 10^{-31} \text{ kg})(2.998 \times 10^8 \text{ m/s})^2 = 511 \text{ keV}$$

and when $K = 1.0 \text{ eV}$, we have $K/(K + E_0) = (1/512) \times 10^{-3} \ll 1$, so we can use the nonrelativistic formula. However, it is simpler here to use [Equation 6.58](#):

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K + 2E_0)}} = \frac{1.241 \text{ eV} \cdot \mu\text{m}}{\sqrt{(1.0 \text{ eV})(1.0 \text{ eV} + 2(511 \text{ keV}))}} = 1.23 \text{ nm}.$$

If we use nonrelativistic momentum, we obtain the same result because 1 eV is much smaller than the rest mass of the electron.

- c. For a fast electron with $K = 108 \text{ keV}$, relativistic effects cannot be neglected because its total energy is $E = K + E_0 = 108 \text{ keV} + 511 \text{ keV} = 619 \text{ keV}$ and $K/E = 108/619$ is not negligible:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K + 2E_0)}} = \frac{1.241 \text{ eV} \cdot \mu\text{m}}{\sqrt{108 \text{ keV}[108 \text{ keV} + 2(511 \text{ keV})]}} = 3.55 \text{ pm}.$$

Significance

We see from these estimates that De Broglie's wavelengths of macroscopic objects such as a ball are immeasurably small. Therefore, even if they exist, they are not detectable and do not affect the motion of macroscopic objects.

✓ CHECK YOUR UNDERSTANDING 6.11

What is de Broglie's wavelength of a nonrelativistic proton with a kinetic energy of 1.0 eV ?

Using the concept of the electron matter wave, de Broglie provided a rationale for the quantization of the electron's angular momentum in the hydrogen atom, which was postulated in Bohr's quantum theory. The physical explanation for the first Bohr quantization condition comes naturally when we assume that an electron in a hydrogen atom behaves not like a particle but like a wave. To see it clearly, imagine a stretched guitar string that is clamped at both ends and vibrates in one of its normal modes. If the length of the string is l ([Figure 6.18](#)), the wavelengths of these vibrations cannot be arbitrary but must be such that an integer k number of half-wavelengths $\lambda/2$ fit exactly on the distance l between the ends. This is the condition $l = k\lambda/2$ for a standing wave on a string. Now suppose that instead of having the string clamped at the walls, we bend its length into a circle and fasten its ends to each other. This produces a circular string that vibrates in normal modes, satisfying the same standing-wave condition, but the number of half-wavelengths must now be an even number k , $k = 2n$, and the length l is now connected to the radius r_n of the circle. This means that the radii are not arbitrary but must satisfy the following standing-wave condition:

$$2\pi r_n = 2n \frac{\lambda}{2}. \quad 6.59$$

If an electron in the n th Bohr orbit moves as a wave, by [Equation 6.59](#) its wavelength must be equal to $\lambda = 2\pi r_n/n$. Assuming that [Equation 6.58](#) is valid, the electron wave of this wavelength corresponds to the

electron's linear momentum, $p = h/\lambda = nh/(2\pi r_n) = n\hbar/r_n$. In a circular orbit, therefore, the electron's angular momentum must be

$$L_n = r_n p = r_n \frac{n\hbar}{r_n} = n\hbar. \quad 6.60$$

This equation is the first of Bohr's quantization conditions, given by [Equation 6.36](#). Providing a physical explanation for Bohr's quantization condition is a convincing theoretical argument for the existence of matter waves.

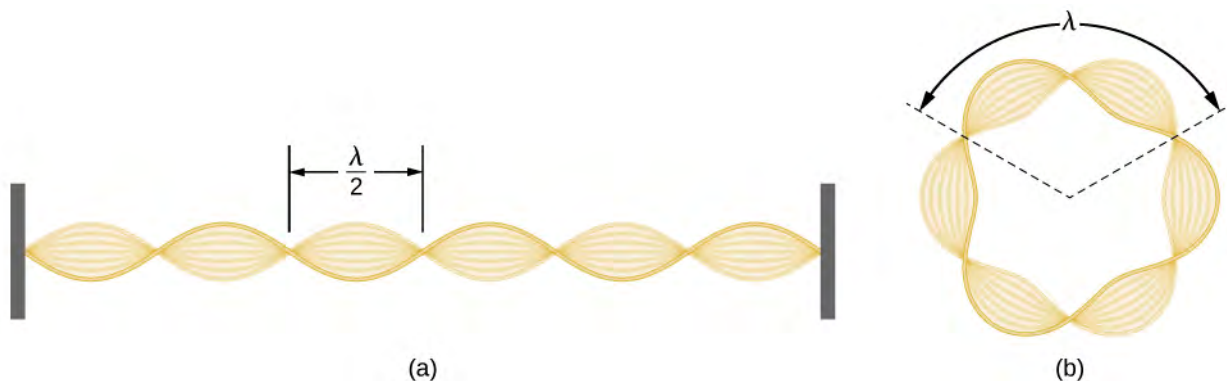


Figure 6.18 Standing-wave pattern: (a) a stretched string clamped at the walls; (b) an electron wave trapped in the third Bohr orbit in the hydrogen atom.



EXAMPLE 6.12

The Electron Wave in the Ground State of Hydrogen

Find the de Broglie wavelength of an electron in the ground state of hydrogen.

Strategy

We combine the first quantization condition in [Equation 6.60](#) with [Equation 6.36](#) and use [Equation 6.38](#) for the first Bohr radius with $n = 1$.

Solution

When $n = 1$ and $r_n = a_0 = 0.529 \text{ Å}$, the Bohr quantization condition gives $a_0 p = 1 \cdot \hbar \Rightarrow p = \hbar/a_0$. The electron wavelength is:

$$\lambda = h/p = h/\hbar/a_0 = 2\pi a_0 = 2\pi(0.529 \text{ Å}) = 3.324 \text{ Å}.$$

Significance

We obtain the same result when we use [Equation 6.58](#) directly.



CHECK YOUR UNDERSTANDING 6.12

Find the de Broglie wavelength of an electron in the third excited state of hydrogen.

Experimental confirmation of matter waves came in 1927 when C. Davisson and L. Germer performed a series of electron-scattering experiments that clearly showed that electrons do behave like waves. Davisson and Germer did not set up their experiment to confirm de Broglie's hypothesis: The confirmation came as a byproduct of their routine experimental studies of metal surfaces under electron bombardment.

In the particular experiment that provided the very first evidence of electron waves (known today as the **Davisson–Germer experiment**), they studied a surface of nickel. Their nickel sample was specially prepared

in a high-temperature oven to change its usual polycrystalline structure to a form in which large single-crystal domains occupy the volume. [Figure 6.19](#) shows the experimental setup. Thermal electrons are released from a heated element (usually made of tungsten) in the electron gun and accelerated through a potential difference ΔV , becoming a well-collimated beam of electrons produced by an electron gun. The kinetic energy K of the electrons is adjusted by selecting a value of the potential difference in the electron gun. This produces a beam of electrons with a set value of linear momentum, in accordance with the conservation of energy:

$$e\Delta V = K = \frac{p^2}{2m} \Rightarrow p = \sqrt{2me\Delta V}. \quad 6.61$$

The electron beam is incident on the nickel sample in the direction normal to its surface. At the surface, it scatters in various directions. The intensity of the beam scattered in a selected direction φ is measured by a highly sensitive detector. The detector's angular position with respect to the direction of the incident beam can be varied from $\varphi = 0^\circ$ to $\varphi = 90^\circ$. The entire setup is enclosed in a vacuum chamber to prevent electron collisions with air molecules, as such thermal collisions would change the electrons' kinetic energy and are not desirable.

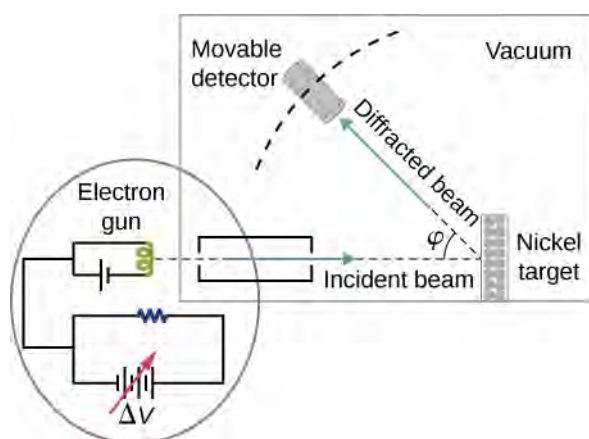


Figure 6.19 Schematics of the experimental setup of the Davisson–Germer diffraction experiment. A well-collimated beam of electrons is scattered off the nickel target. The kinetic energy of electrons in the incident beam is selected by adjusting a variable potential, ΔV , in the electron gun. Intensity of the scattered electron beam is measured for a range of scattering angles φ , whereas the distance between the detector and the target does not change.

When the nickel target has a polycrystalline form with many randomly oriented microscopic crystals, the incident electrons scatter off its surface in various random directions. As a result, the intensity of the scattered electron beam is much the same in any direction, resembling a diffuse reflection of light from a porous surface. However, when the nickel target has a regular crystalline structure, the intensity of the scattered electron beam shows a clear maximum at a specific angle and the results show a clear diffraction pattern (see [Figure 6.20](#)). Similar diffraction patterns formed by X-rays scattered by various crystalline solids were studied in 1912 by father-and-son physicists William H. Bragg and William L. Bragg. The Bragg law in X-ray crystallography provides a connection between the wavelength λ of the radiation incident on a crystalline lattice, the lattice spacing, and the position of the interference maximum in the diffracted radiation (see [Diffraction](#)).

The lattice spacing of the Davisson–Germer target, determined with X-ray crystallography, was measured to be $a = 2.15 \text{ \AA}$. Unlike X-ray crystallography in which X-rays penetrate the sample, in the original Davisson–Germer experiment, only the surface atoms interact with the incident electron beam. For the surface diffraction, the maximum intensity of the reflected electron beam is observed for scattering angles that satisfy the condition $n\lambda = a \sin \varphi$ (see [Figure 6.21](#)). The first-order maximum (for $n = 1$) is measured at a scattering angle of $\varphi \approx 50^\circ$ at $\Delta V \approx 54 \text{ V}$, which gives the wavelength of the incident radiation as $\lambda = (2.15 \text{ \AA}) \sin 50^\circ = 1.64 \text{ \AA}$. On the other hand, a 54-V potential accelerates the incident electrons to kinetic energies of $K = 54 \text{ eV}$. Their momentum, calculated from [Equation 6.61](#), is $p = 2.478 \times 10^{-5} \text{ eV} \cdot \text{s/m}$. When we substitute this result in [Equation 6.58](#), the de Broglie wavelength is obtained as

$$\lambda = \frac{h}{p} = \frac{4.136 \times 10^{-15} \text{ eV} \cdot \text{s}}{2.478 \times 10^{-5} \text{ eV} \cdot \text{s/m}} = 1.67 \text{ \AA}. \quad 6.62$$

The same result is obtained when we use $K = 54 \text{ eV}$ in Equation 6.61. The proximity of this theoretical result to the Davisson–Germer experimental value of $\lambda = 1.64 \text{ \AA}$ is a convincing argument for the existence of de Broglie matter waves.

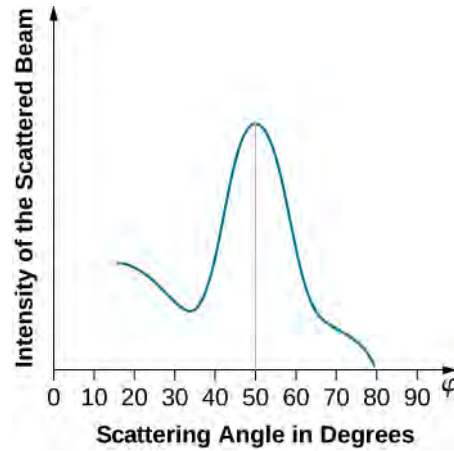


Figure 6.20 The experimental results of electron diffraction on a nickel target for the accelerating potential in the electron gun of about $\Delta V = 54 \text{ V}$: The intensity maximum is registered at the scattering angle of about $\varphi = 50^\circ$.

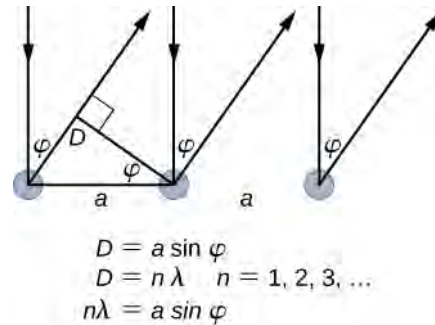


Figure 6.21 In the surface diffraction of a monochromatic electromagnetic wave on a crystalline lattice structure, the in-phase incident beams are reflected from atoms on the surface. A ray reflected from the left atom travels an additional distance $D = a \sin \varphi$ to the detector, where a is the lattice spacing. The reflected beams remain in-phase when D is an integer multiple of their wavelength λ . The intensity of the reflected waves has pronounced maxima for angles φ satisfying $n \lambda = a \sin \varphi$.

Diffraction lines measured with low-energy electrons, such as those used in the Davisson–Germer experiment, are quite broad (see Figure 6.20) because the incident electrons are scattered only from the surface. The resolution of diffraction images greatly improves when a higher-energy electron beam passes through a thin metal foil. This occurs because the diffraction image is created by scattering off many crystalline planes inside the volume, and the maxima produced in scattering at Bragg angles are sharp (see Figure 6.22).

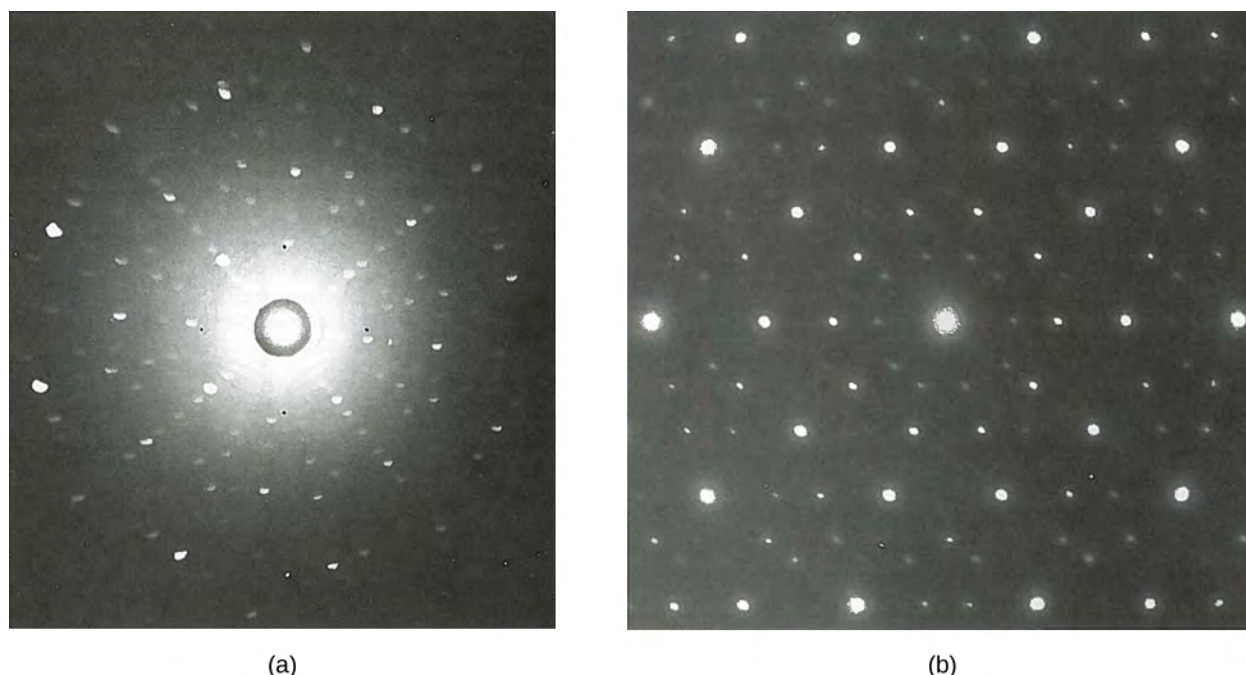


Figure 6.22 Diffraction patterns obtained in scattering on a crystalline solid: (a) with X-rays, and (b) with electrons. The observed pattern reflects the symmetry of the crystalline structure of the sample.

Since the work of Davisson and Germer, de Broglie's hypothesis has been extensively tested with various experimental techniques, and the existence of de Broglie waves has been confirmed for numerous elementary particles. Neutrons have been used in scattering experiments to determine crystalline structures of solids from interference patterns formed by neutron matter waves. The neutron has zero charge and its mass is comparable with the mass of a positively charged proton. Both neutrons and protons can be seen as matter waves. Therefore, the property of being a matter wave is not specific to electrically charged particles but is true of all particles in motion. Matter waves of molecules as large as carbon C_{60} have been measured. All physical objects, small or large, have an associated matter wave as long as they remain in motion. The universal character of de Broglie matter waves is firmly established.

EXAMPLE 6.13

Neutron Scattering

Suppose that a neutron beam is used in a diffraction experiment on a typical crystalline solid. Estimate the kinetic energy of a neutron (in eV) in the neutron beam and compare it with kinetic energy of an ideal gas in equilibrium at room temperature.

Strategy

We assume that a typical crystal spacing a is of the order of $1.0 \text{ \AA}$. To observe a diffraction pattern on such a lattice, the neutron wavelength λ must be on the same order of magnitude as the lattice spacing. We use [Equation 6.61](#) to find the momentum p and kinetic energy K . To compare this energy with the energy E_T of ideal gas in equilibrium at room temperature $T = 300\text{K}$, we use the relation $K = \frac{3}{2}k_B T$, where $k_B = 8.62 \times 10^{-5} \text{ eV/K}$ is the Boltzmann constant.

Solution

We evaluate pc to compare it with the neutron's rest mass energy $E_0 = 940 \text{ MeV}$:

$$p = \frac{h}{\lambda} \Rightarrow pc = \frac{hc}{\lambda} = \frac{1.241 \times 10^{-6} \text{ eV} \cdot \text{m}}{10^{-10} \text{ m}} = 12.41 \text{ keV}.$$

We see that $p^2 c^2 \ll E_0^2$ so $K \ll E_0$ and we can use the nonrelativistic kinetic energy:

$$K = \frac{p^2}{2m_n} = \frac{h^2}{2\lambda^2 m_n} = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(2 \times 10^{-20} \text{ m}^2)(1.66 \times 10^{-27} \text{ kg})} = 1.32 \times 10^{-20} \text{ J} = 82.7 \text{ meV}.$$

Kinetic energy of ideal gas in equilibrium at 300 K is:

$$K_T = \frac{3}{2} k_B T = \frac{3}{2} (8.62 \times 10^{-5} \text{ eV/K})(300 \text{ K}) = 38.8 \text{ MeV}.$$

We see that these energies are of the same order of magnitude.

Significance

Neutrons with energies in this range, which is typical for an ideal gas at room temperature, are called “thermal neutrons.”



EXAMPLE 6.14

Wavelength of a Relativistic Proton

In a supercollider at CERN, protons can be accelerated to velocities of $0.75c$. What are their de Broglie wavelengths at this speed? What are their kinetic energies?

Strategy

The rest mass energy of a proton is $E_0 = m_0 c^2 = (1.672 \times 10^{-27} \text{ kg})(2.998 \times 10^8 \text{ m/s})^2 = 938 \text{ MeV}$. When the proton's velocity is known, we have $\beta = 0.75$ and $\beta\gamma = 0.75 / \sqrt{1 - 0.75^2} = 1.134$. We obtain the wavelength λ and kinetic energy K from relativistic relations.

Solution

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{\beta\gamma E_0} = \frac{1.241 \text{ eV} \cdot \mu\text{m}}{1.134(938 \text{ MeV})} = 1.16 \text{ fm}$$

$$K = E_0(\gamma - 1) = 938 \text{ MeV}(1/\sqrt{1 - 0.75^2} - 1) = 480.1 \text{ MeV}$$

Significance

Notice that because a proton is 1835 times more massive than an electron, if this experiment were performed with electrons, a simple rescaling of these results would give us the electron's wavelength of $(1835)0.77 \text{ fm} = 1.4 \text{ pm}$ and its kinetic energy of $480.1 \text{ MeV} / 1835 = 261.6 \text{ keV}$.



CHECK YOUR UNDERSTANDING 6.13

Find the de Broglie wavelength and kinetic energy of a free electron that travels at a speed of $0.75c$.

6.6 Wave-Particle Duality

Learning Objectives

By the end of this section, you will be able to:

- Identify phenomena in which electromagnetic waves behave like a beam of photons and particles behave like waves
- Describe the physics principles behind electron microscopy
- Summarize the evolution of scientific thought that led to the development of quantum mechanics

The energy of radiation detected by a radio-signal receiving antenna comes as the energy of an electromagnetic wave. The same energy of radiation detected by a photocurrent in the photoelectric effect comes as the energy of individual photon particles. Therefore, the question arises about the nature of electromagnetic radiation: Is a photon a wave or is it a particle? Similar questions may be asked about other

known forms of energy. For example, an electron that forms part of an electric current in a circuit behaves like a particle moving in unison with other electrons inside the conductor. The same electron behaves as a wave when it passes through a solid crystalline structure and forms a diffraction image. Is an electron a wave or is it a particle? The same question can be extended to all particles of matter—elementary particles, as well as compound molecules—asking about their true physical nature. At our present state of knowledge, such questions about the true nature of things do not have conclusive answers. All we can say is that **wave-particle duality** exists in nature: Under some experimental conditions, a particle appears to act as a particle, and under different experimental conditions, a particle appears to act a wave. Conversely, under some physical circumstances electromagnetic radiation acts as a wave, and under other physical circumstances, radiation acts as a beam of photons.

This dualistic interpretation is not a new physics concept brought about by specific discoveries in the twentieth century. It was already present in a debate between Isaac Newton and Christiaan Huygens about the nature of light, beginning in the year 1670. According to Newton, a beam of light is a collection of corpuscles of light. According to Huygens, light is a wave. The corpuscular hypothesis failed in 1803, when Thomas Young announced his **double-slit interference experiment** with light (see [Figure 6.23](#)), which firmly established light as a wave. In James Clerk Maxwell's theory of electromagnetism (completed by the year 1873), light is an electromagnetic wave. Maxwell's classical view of radiation as an electromagnetic wave is still valid today; however, it is unable to explain blackbody radiation and the photoelectric effect, where light acts as a beam of photons.

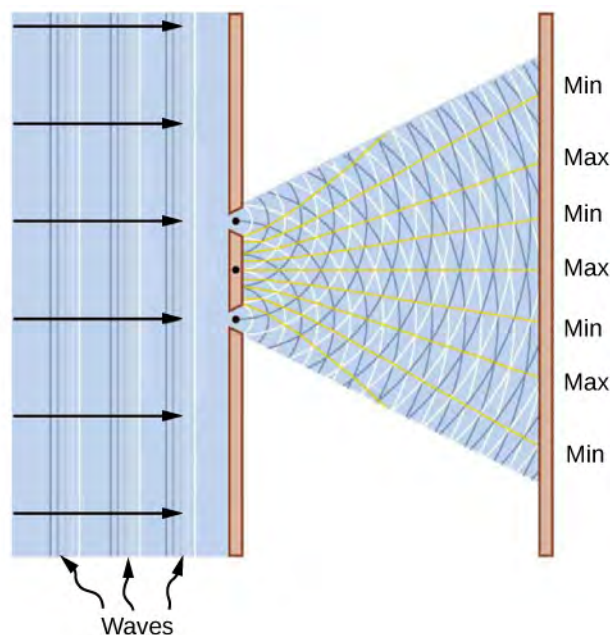


Figure 6.23 Young's double-slit experiment explains the interference of light by making an analogy with the interference of water waves. Two waves are generated at the positions of two slits in an opaque screen. The waves have the same wavelengths. They travel from their origins at the slits to the viewing screen placed to the right of the slits. The waves meet on the viewing screen. At the positions marked "Max" on the screen, the meeting waves are in-phase and the combined wave amplitude is enhanced. At positions marked "Min," the combined wave amplitude is zero. For light, this mechanism creates a bright-and-dark fringe pattern on the viewing screen.

A similar dichotomy existed in the interpretation of electricity. From Benjamin Franklin's observations of electricity in 1751 until J.J. Thomson's discovery of the electron in 1897, electric current was seen as a flow in a continuous electric medium. Within this theory of electric fluid, the present theory of electric circuits was developed, and electromagnetism and electromagnetic induction were discovered. Thomson's experiment showed that the unit of negative electric charge (an electron) can travel in a vacuum without any medium to carry the charge around, as in electric circuits. This discovery changed the way in which electricity is understood today and gave the electron its particle status. In Bohr's early quantum theory of the hydrogen atom, both the electron and the proton are particles of matter. Likewise, in the Compton scattering of X-rays on

electrons, the electron is a particle. On the other hand, in electron-scattering experiments on crystalline structures, the electron behaves as a wave.

A skeptic may raise a question that perhaps an electron might always be nothing more than a particle, and that the diffraction images obtained in electron-scattering experiments might be explained within some macroscopic model of a crystal and a macroscopic model of electrons coming at it like a rain of ping-pong balls. As a matter of fact, to investigate this question, we do not need a complex model of a crystal but just a couple of simple slits in a screen that is opaque to electrons. In other words, to gather convincing evidence about the nature of an electron, we need to repeat the Young double-slit experiment with electrons. If the electron is a wave, we should observe the formation of interference patterns typical for waves, such as those described in [Figure 6.23](#), even when electrons come through the slits one by one. However, if the electron is a not a wave but a particle, the interference fringes will not be formed.

The very first double-slit experiment with a beam of electrons, performed by Claus Jönsson in Germany in 1961, demonstrated that a beam of electrons indeed forms an interference pattern, which means that electrons collectively behave as a wave. The first double-slit experiments with *single* electrons passing through the slits one-by-one were performed by Giulio Pozzi in 1974 in Italy and by Akira Tonomura in 1989 in Japan. They show that interference fringes are formed gradually, even when electrons pass through the slits individually. This demonstrates conclusively that electron-diffraction images are formed because of the wave nature of electrons. The results seen in double-slit experiments with electrons are illustrated by the images of the interference pattern in [Figure 6.24](#).

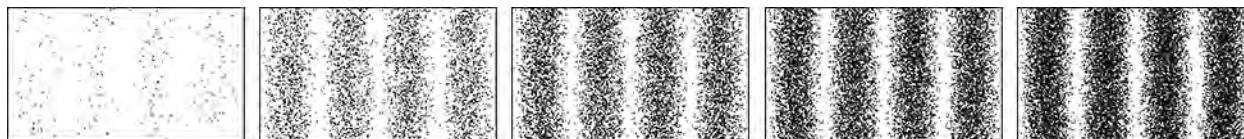


Figure 6.24 Computer-simulated interference fringes seen in the Young double-slit experiment with electrons. One pattern is gradually formed on the screen, regardless of whether the electrons come through the slits as a beam or individually one-by-one.



EXAMPLE 6.15

Double-Slit Experiment with Electrons

In one experimental setup for studying interference patterns of electron waves, two slits are created in a gold-coated silicon membrane. Each slit is 62-nm wide and 4- μm long, and the separation between the slits is 272 nm. The electron beam is created in an electron gun by heating a tungsten element and by accelerating the electrons across a 600-V potential. The beam is subsequently collimated using electromagnetic lenses, and the collimated beam of electrons is sent through the slits. Find the angular position of the first-order bright fringe on the viewing screen.

Strategy

Recall that the angular position θ of the n th order bright fringe that is formed in Young's two-slit interference pattern (discussed in a previous chapter) is related to the separation, d , between the slits and to the wavelength, λ , of the incident light by the equation $d\sin\theta = n\lambda$, where $n = 0, \pm 1, \pm 2, \dots$. The separation is given and is equal to $d = 272$ nm. For the first-order fringe, we take $n = 1$. The only thing we now need is the wavelength of the incident electron wave.

Since the electron has been accelerated from rest across a potential difference of $\Delta V = 600\text{V}$, its kinetic energy is $K = e\Delta V = 600\text{ eV}$. The rest-mass energy of the electron is $E_0 = 511\text{ keV}$.

We compute its de Broglie wavelength as that of a nonrelativistic electron because its kinetic energy K is much smaller than its rest energy E_0 , $K \ll E_0$.

Solution

The electron's wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e K}} = \frac{h}{\sqrt{2E_0/c^2 K}} = \frac{hc}{\sqrt{2E_0 K}} = \frac{1.241 \times 10^{-6} \text{ eV} \cdot \text{m}}{\sqrt{2(511 \text{ keV})(600 \text{ eV})}} = 0.050 \text{ nm}.$$

This λ is used to obtain the position of the first bright fringe:

$$\sin\theta = \frac{1 \cdot \lambda}{d} = \frac{0.050 \text{ nm}}{272 \text{ nm}} = 0.000184 \Rightarrow \theta = 0.010^\circ.$$

Significance

Notice that this is also the angular resolution between two consecutive bright fringes up to about $n = 1000$. For example, between the zero-order fringe and the first-order fringe, between the first-order fringe and the second-order fringe, and so on.

✓ CHECK YOUR UNDERSTANDING 6.14

For the situation described in [Example 6.15](#), find the angular position of the fifth-order bright fringe on the viewing screen.

The wave-particle dual nature of matter particles and of radiation is a declaration of our inability to describe physical reality within one unified classical theory because separately neither a classical particle approach nor a classical wave approach can fully explain the observed phenomena. This limitation of the classical approach was realized by the year 1928, and a foundation for a new statistical theory, called quantum mechanics, was put in place by Bohr, Edwin Schrödinger, Werner Heisenberg, and Paul Dirac. Quantum mechanics takes de Broglie's idea of matter waves to be the fundamental property of all particles and gives it a statistical interpretation. According to this interpretation, a wave that is associated with a particle carries information about the probable positions of the particle and about its other properties. A single particle is seen as a moving *wave packet* such as the one shown in [Figure 6.25](#). We can intuitively sense from this example that if a particle is a wave packet, we will not be able to measure its exact position in the same sense as we cannot pinpoint a location of a wave packet in a vibrating guitar string. The uncertainty, Δx , in measuring the particle's position is connected to the uncertainty, Δp , in the simultaneous measuring of its linear momentum by Heisenberg's uncertainty principle:

$$\Delta x \Delta p \geq \frac{1}{2} \hbar. \quad 6.63$$

Heisenberg's principle expresses the law of nature that, at the quantum level, our perception is limited. For example, if we know the exact position of a body (which means that $\Delta x = 0$ in [Equation 6.63](#)) at the same time we cannot know its momentum, because then the uncertainty in its momentum becomes infinite (because $\Delta p \geq 0.5\hbar/\Delta x$ in [Equation 6.63](#)). The **Heisenberg uncertainty principle** sets the limit on the precision of *simultaneous* measurements of position and momentum of a particle; it shows that the best precision we can obtain is when we have an equals sign (=) in [Equation 6.63](#), and we cannot do better than that, even with the best instruments of the future. Heisenberg's principle is a consequence of the wave nature of particles.



Figure 6.25 In this graphic, a particle is shown as a wave packet and its position does not have an exact value.

We routinely use many electronic devices that exploit wave-particle duality without even realizing the sophistication of the physics underlying their operation. One example of a technology based on the particle properties of photons and electrons is a charge-coupled device, which is used for light detection in any instrumentation where high-quality digital data are required, such as in digital cameras or in medical sensors. An example in which the wave properties of electrons is exploited is an electron microscope.

In 1931, physicist Ernst Ruska—building on the idea that magnetic fields can direct an electron beam just as lenses can direct a beam of light in an optical microscope—developed the first prototype of the electron microscope. This development originated the field of **electron microscopy**. In the transmission electron microscope (TEM), shown in [Figure 6.26](#), electrons are produced by a hot tungsten element and accelerated by a potential difference in an electron gun, which gives them up to 400 keV in kinetic energy. After leaving the electron gun, the electron beam is focused by electromagnetic lenses (a system of condensing lenses) and transmitted through a specimen sample to be viewed. The image of the sample is reconstructed from the transmitted electron beam. The magnified image may be viewed either directly on a fluorescent screen or indirectly by sending it, for example, to a digital camera or a computer monitor. The entire setup consisting of the electron gun, the lenses, the specimen, and the fluorescent screen are enclosed in a vacuum chamber to prevent the energy loss from the beam. Resolution of the TEM is limited only by spherical aberration (discussed in a previous chapter). Modern high-resolution models of a TEM can have resolving power greater than 0.5 Å and magnifications higher than 50 million times. For comparison, the best resolving power obtained with light microscopy is currently about 97 nm. A limitation of the TEM is that the samples must be about 100-nm thick and biological samples require a special preparation involving chemical “fixing” to stabilize them for ultrathin slicing.

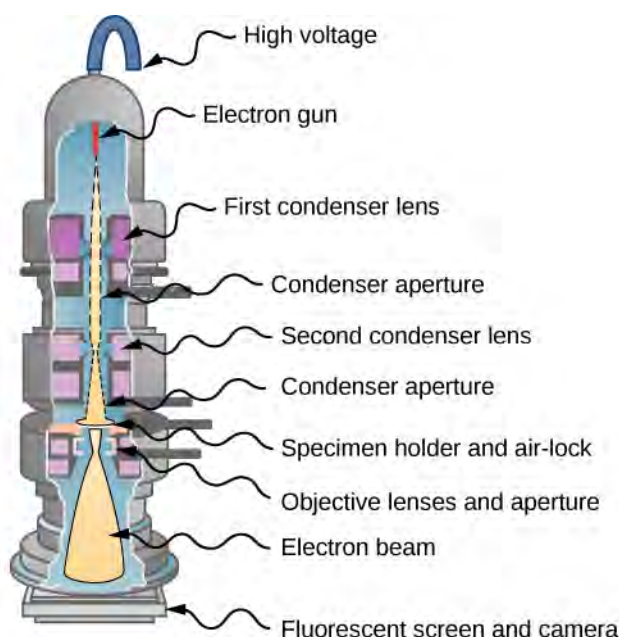


Figure 6.26 TEM: An electron beam produced by an electron gun is collimated by condenser lenses and passes through a specimen. The transmitted electrons are projected on a screen and the image is sent to a camera. (credit: modification of work by Dr. Graham Beards)

Such limitations do not appear in the scanning electron microscope (SEM), which was invented by Manfred von Ardenne in 1937. In an SEM, a typical energy of the electron beam is up to 40 keV and the beam is not transmitted through a sample but is scattered off its surface. Surface topography of the sample is reconstructed by analyzing back-scattered electrons, transmitted electrons, and the emitted radiation produced by electrons interacting with atoms in the sample. The resolving power of an SEM is better than 1 nm, and the magnification can be more than 250 times better than that obtained with a light microscope. The samples scanned by an SEM can be as large as several centimeters but they must be specially prepared, depending on electrical properties of the sample.

High magnifications of the TEM and SEM allow us to see individual molecules. High resolving powers of the TEM and SEM allow us to see fine details, such as those shown in the SEM micrograph of pollen at the beginning of this chapter ([Figure 6.1](#)).



EXAMPLE 6.16

Resolving Power of an Electron Microscope

If a 1.0-pm electron beam of a TEM passes through a 2.0- μm circular opening, what is the angle between the two just-resolvable point sources for this microscope?

Solution

We can directly use a formula for the resolving power, $\Delta\theta$, of a microscope (discussed in a previous chapter) when the wavelength of the incident radiation is $\lambda = 1.0 \text{ pm}$ and the diameter of the aperture is $D = 2.0 \mu\text{m}$:

$$\Delta\theta = 1.22 \frac{\lambda}{D} = 1.22 \frac{1.0 \text{ pm}}{2.0 \mu\text{m}} = 6.10 \times 10^{-7} \text{ rad} = 3.50 \times 10^{-5} \text{ degree.}$$

Significance

Note that if we used a conventional microscope with a 400-nm light, the resolving power would be only 14° , which means that all of the fine details in the image would be blurred.



CHECK YOUR UNDERSTANDING 6.15

Suppose that the diameter of the aperture in [Example 6.16](#) is halved. How does it affect the resolving power?

CHAPTER REVIEW

Key Terms

absorber any object that absorbs radiation

absorption spectrum wavelengths of absorbed radiation by atoms and molecules

Balmer formula describes the emission spectrum of a hydrogen atom in the visible-light range

Balmer series spectral lines corresponding to electron transitions to/from the $n = 2$ state of the hydrogen atom, described by the Balmer formula

blackbody perfect absorber/emitter

blackbody radiation radiation emitted by a blackbody

Bohr radius of hydrogen radius of the first Bohr's orbit

Bohr's model of the hydrogen atom first quantum model to explain emission spectra of hydrogen

Brackett series spectral lines corresponding to electron transitions to/from the $n = 4$ state

Compton effect the change in wavelength when an X-ray is scattered by its interaction with some materials

Compton shift difference between the wavelengths of the incident X-ray and the scattered X-ray

Compton wavelength physical constant with the value $\lambda_c = 2.43$ pm

cut-off frequency frequency of incident light below which the photoelectric effect does not occur

cut-off wavelength wavelength of incident light that corresponds to cut-off frequency

Davisson–Germer experiment historically first electron-diffraction experiment that revealed electron waves

de Broglie wave matter wave associated with any object that has mass and momentum

de Broglie's hypothesis of matter waves particles of matter can behave like waves

double-slit interference experiment Young's double-slit experiment, which shows the interference of waves

electron microscopy microscopy that uses electron waves to “see” fine details of nano-size objects

emission spectrum wavelengths of emitted radiation by atoms and molecules

emitter any object that emits radiation

energy of a photon quantum of radiant energy, depends only on a photon's frequency

energy spectrum of hydrogen set of allowed discrete energies of an electron in a hydrogen atom

excited energy states of the H atom energy state other than the ground state

Fraunhofer lines dark absorption lines in the continuum solar emission spectrum

ground state energy of the hydrogen atom energy of an electron in the first Bohr orbit of the hydrogen atom

group velocity velocity of a wave, energy travels with the group velocity

Heisenberg uncertainty principle sets the limits on precision in simultaneous measurements of momentum and position of a particle

Humphreys series spectral lines corresponding to electron transitions to/from the $n = 6$ state

hydrogen-like atom ionized atom with one electron remaining and nucleus with charge $+Ze$

inelastic scattering scattering effect where kinetic energy is not conserved but the total energy is conserved

ionization energy energy needed to remove an electron from an atom

ionization limit of the hydrogen atom ionization energy needed to remove an electron from the first Bohr orbit

Lyman series spectral lines corresponding to electron transitions to/from the ground state

nuclear model of the atom heavy positively charged nucleus at the center is surrounded by electrons, proposed by Rutherford

Paschen series spectral lines corresponding to electron transitions to/from the $n = 3$ state

Pfund series spectral lines corresponding to electron transitions to/from the $n = 5$ state

photocurrent in a circuit, current that flows when a photoelectrode is illuminated

photoelectric effect emission of electrons from a metal surface exposed to electromagnetic radiation of the proper frequency

photoelectrode in a circuit, an electrode that emits photoelectrons

photoelectron electron emitted from a metal surface in the presence of incident radiation

photon particle of light

Planck's hypothesis of energy quanta energy exchanges between the radiation and the walls take place only in the form of discrete energy quanta

postulates of Bohr's model three assumptions that set a frame for Bohr's model

power intensity energy that passes through a unit

surface per unit time	
propagation vector	vector with magnitude $2\pi/\lambda$ that has the direction of the photon's linear momentum
quantized energies	discrete energies; not continuous
quantum number	index that enumerates energy levels
quantum phenomenon	in interaction with matter, photon transfers either all its energy or nothing
quantum state of a Planck's oscillator	any mode of vibration of Planck's oscillator; enumerated by quantum number
reduced Planck's constant	Planck's constant divided by 2π
Rutherford's gold foil experiment	first experiment to demonstrate the existence of the atomic nucleus
Rydberg constant for hydrogen	physical constant in the Balmer formula
Rydberg formula	experimentally found positions of spectral lines of hydrogen atom
scattering angle	angle between the direction of the scattered beam and the direction of the incident beam
Stefan–Boltzmann constant	physical constant in Stefan's law
stopping potential	in a circuit, potential difference that stops photocurrent
wave number	magnitude of the propagation vector
wave quantum mechanics	theory that explains the physics of atoms and subatomic particles
wave-particle duality	particles can behave as waves and radiation can behave as particles
work function	energy needed to detach photoelectron from the metal surface
α-particle	doubly ionized helium atom
α-ray	beam of α -particles (alpha-particles)
β-ray	beam of electrons
γ-ray	beam of highly energetic photons

Key Equations

Wien's displacement law	$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$
Stefan's law	$P(T) = \sigma AT^4$
Planck's constant	$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s}$
Energy quantum of radiation	$\Delta E = hf$
Planck's blackbody radiation law	$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}$
Maximum kinetic energy of a photoelectron	$K_{\max} = e\Delta V_s$
Energy of a photon	$E_f = hf$
Energy balance for photoelectron	$K_{\max} = hf - \phi$
Cut-off frequency	$f_c = \frac{\phi}{h}$
Relativistic invariant energy equation	$E^2 = p^2 c^2 + m_0^2 c^4$
Energy-momentum relation for photon	$p_f = \frac{E_f}{c}$
Energy of a photon	$E_f = hf = \frac{hc}{\lambda}$

Magnitude of photon's momentum	$p_f = \frac{h}{\lambda}$
Photon's linear momentum vector	$\vec{p}_f = \hbar \vec{k}$
The Compton wavelength of an electron	$\lambda_c = \frac{h}{m_0 c} = 0.00243 \text{ nm}$
The Compton shift	$\Delta\lambda = \lambda_c(1 - \cos \theta)$
The Balmer formula	$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - \frac{1}{n^2} \right)$
The Rydberg formula	$\frac{1}{\lambda} = R_H \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), n_i = n_f + 1, n_f + 2, \dots$
Bohr's first quantization condition	$L_n = n\hbar, n = 1, 2, \dots$
Bohr's second quantization condition	$hf = E_n - E_m $
Bohr's radius of hydrogen	$a_0 = 4\pi\epsilon_0 \frac{\hbar^2}{m_e e^2} = 0.529 \text{ \AA}$
Bohr's radius of the n th orbit	$r_n = a_0 n^2$
Ground-state energy value, ionization limit	$E_0 = \frac{1}{8\epsilon_0^2} \frac{m_e e^4}{h^2} = 13.6 \text{ eV}$
Electron's energy in the n th orbit	$E_n = -E_0 \frac{1}{n^2}$
Ground state energy of hydrogen	$E_1 = -E_0 = -13.6 \text{ eV}$
The n th orbit of hydrogen-like ion	$r_n = \frac{a_0}{Z} n^2$
The n th energy of hydrogen-like ion	$E_n = -Z^2 E_0 \frac{1}{n^2}$
Energy of a matter wave	$E = hf$
The de Broglie wavelength	$\lambda = \frac{h}{p}$
The frequency-wavelength relation for matter waves	$\lambda f = \frac{c}{\beta}$
Heisenberg's uncertainty principle	$\Delta x \Delta p \geq \frac{1}{2} \hbar$

Summary

6.1 Blackbody Radiation

- All bodies radiate energy. The amount of radiation a body emits depends on its temperature. The experimental Wien's displacement law states that the hotter the body, the shorter the wavelength corresponding to the emission peak in the radiation curve. The experimental Stefan's law states that the total power of radiation emitted across the entire spectrum of wavelengths at a given temperature is proportional to the fourth power of the Kelvin temperature of the radiating body.
- Absorption and emission of radiation are studied within the model of a blackbody. In the classical approach, the exchange of energy between radiation and cavity walls is continuous. The classical approach does not explain the blackbody radiation curve.
- To explain the blackbody radiation curve, Planck assumed that the exchange of energy between radiation and cavity walls takes place only in discrete quanta of energy. Planck's hypothesis of energy quanta led to the theoretical Planck's radiation law, which agrees with the experimental blackbody radiation curve; it also explains Wien's and Stefan's laws.

6.2 Photoelectric Effect

- The photoelectric effect occurs when photoelectrons are ejected from a metal surface in response to monochromatic radiation incident on the surface. It has three characteristics: (1) it is instantaneous, (2) it occurs only when the radiation is above a cut-off frequency, and (3) kinetic energies of photoelectrons at the surface do not depend of the intensity of radiation. The photoelectric effect cannot be explained by classical theory.
- We can explain the photoelectric effect by assuming that radiation consists of photons (particles of light). Each photon carries a quantum of energy. The energy of a photon depends only on its frequency, which is the frequency of the radiation. At the surface, the entire energy of a photon is transferred to one photoelectron.
- The maximum kinetic energy of a photoelectron at the metal surface is the difference between the energy of the incident photon and the work function of the metal. The work function is the binding energy of electrons to the metal surface.

Each metal has its own characteristic work function.

6.3 The Compton Effect

- In the Compton effect, X-rays scattered off some materials have different wavelengths than the wavelength of the incident X-rays. This phenomenon does not have a classical explanation.
- The Compton effect is explained by assuming that radiation consists of photons that collide with weakly bound electrons in the target material. Both electron and photon are treated as relativistic particles. Conservation laws of the total energy and of momentum are obeyed in collisions.
- Treating the photon as a particle with momentum that can be transferred to an electron leads to a theoretical Compton shift that agrees with the wavelength shift measured in the experiment. This provides evidence that radiation consists of photons.
- Compton scattering is an inelastic scattering, in which scattered radiation has a longer wavelength than that of incident radiation.

6.4 Bohr's Model of the Hydrogen Atom

- Positions of absorption and emission lines in the spectrum of atomic hydrogen are given by the experimental Rydberg formula. Classical physics cannot explain the spectrum of atomic hydrogen.
- The Bohr model of hydrogen was the first model of atomic structure to correctly explain the radiation spectra of atomic hydrogen. It was preceded by the Rutherford nuclear model of the atom. In Rutherford's model, an atom consists of a positively charged point-like nucleus that contains almost the entire mass of the atom and of negative electrons that are located far away from the nucleus.
- Bohr's model of the hydrogen atom is based on three postulates: (1) an electron moves around the nucleus in a circular orbit, (2) an electron's angular momentum in the orbit is quantized, and (3) the change in an electron's energy as it makes a quantum jump from one orbit to another is always accompanied by the emission or absorption of a photon. Bohr's model is semi-classical because it combines the classical concept of electron orbit (postulate 1) with the new concept of quantization (postulates 2 and

- 3).
- Bohr's model of the hydrogen atom explains the emission and absorption spectra of atomic hydrogen and hydrogen-like ions with low atomic numbers. It was the first model to introduce the concept of a quantum number to describe atomic states and to postulate quantization of electron orbits in the atom. Bohr's model is an important step in the development of quantum mechanics, which deals with many-electron atoms.

6.5 De Broglie's Matter Waves

- De Broglie's hypothesis of matter waves postulates that any particle of matter that has linear momentum is also a wave. The wavelength of a matter wave associated with a particle is inversely proportional to the magnitude of the particle's linear momentum. The speed of the matter wave is the speed of the particle.
- De Broglie's concept of the electron matter wave provides a rationale for the quantization of the electron's angular momentum in Bohr's model of the hydrogen atom.
- In the Davisson–Germer experiment, electrons are scattered off a crystalline nickel surface. Diffraction patterns of electron matter waves are observed. They are the evidence for the existence of matter waves. Matter waves are observed in diffraction experiments with

various particles.

6.6 Wave-Particle Duality

- Wave-particle duality exists in nature: Under some experimental conditions, a particle acts as a particle; under other experimental conditions, a particle acts as a wave. Conversely, under some physical circumstances, electromagnetic radiation acts as a wave, and under other physical circumstances, radiation acts as a beam of photons.
- Modern-era double-slit experiments with electrons demonstrated conclusively that electron-diffraction images are formed because of the wave nature of electrons.
- The wave-particle dual nature of particles and of radiation has no classical explanation.
- Quantum theory takes the wave property to be the fundamental property of all particles. A particle is seen as a moving wave packet. The wave nature of particles imposes a limitation on the simultaneous measurement of the particle's position and momentum. Heisenberg's uncertainty principle sets the limits on precision in such simultaneous measurements.
- Wave-particle duality is exploited in many devices, such as charge-couple devices (used in digital cameras) or in the electron microscopy of the scanning electron microscope (SEM) and the transmission electron microscope (TEM).

Conceptual Questions

6.1 Blackbody Radiation

- Which surface has a higher temperature – the surface of a yellow star or that of a red star?
- Describe what you would see when looking at a body whose temperature is increased from 1000 K to 1,000,000 K.
- Explain the color changes in a hot body as its temperature is increased.
- Speculate as to why UV light causes sunburn, whereas visible light does not.
- Two cavity radiators are constructed with walls made of different metals. At the same temperature, how would their radiation spectra differ?
- Discuss why some bodies appear black, other bodies appear red, and still other bodies appear white.
- If everything radiates electromagnetic energy, why can we not see objects at room temperature

in a dark room?

- How much does the power radiated by a blackbody increase when its temperature (in K) is tripled?

6.2 Photoelectric Effect

- For the same monochromatic light source, would the photoelectric effect occur for all metals?
- In the interpretation of the photoelectric effect, how is it known that an electron does not absorb more than one photon?
- Explain how you can determine the work function from a plot of the stopping potential versus the frequency of the incident radiation in a photoelectric effect experiment. Can you determine the value of Planck's constant from this plot?
- Suppose that in the photoelectric-effect experiment we make a plot of the detected

current versus the applied potential difference. What information do we obtain from such a plot? Can we determine from it the value of Planck's constant? Can we determine the work function of the metal?

13. Speculate how increasing the temperature of a photoelectrode affects the outcomes of the photoelectric effect experiment.
14. Which aspects of the photoelectric effect cannot be explained by classical physics?
15. Is the photoelectric effect a consequence of the wave character of radiation or is it a consequence of the particle character of radiation? Explain briefly.
16. The metals sodium, iron, and molybdenum have work functions 2.5 eV, 3.9 eV, and 4.2 eV, respectively. Which of these metals will emit photoelectrons when illuminated with 400 nm light?

6.3 The Compton Effect

17. Discuss any similarities and differences between the photoelectric and the Compton effects.
18. Which has a greater momentum: an UV photon or an IR photon?
19. Does changing the intensity of a monochromatic light beam affect the momentum of the individual photons in the beam? Does such a change affect the net momentum of the beam?
20. Can the Compton effect occur with visible light? If so, will it be detectable?
21. Is it possible in the Compton experiment to observe scattered X-rays that have a shorter wavelength than the incident X-ray radiation?
22. Show that the Compton wavelength has the dimension of length.
23. At what scattering angle is the wavelength shift in the Compton effect equal to the Compton wavelength?

6.4 Bohr's Model of the Hydrogen Atom

24. Explain why the patterns of bright emission spectral lines have an identical spectral position to the pattern of dark absorption spectral lines for a given gaseous element.
25. Do the various spectral lines of the hydrogen atom overlap?
26. The Balmer series for hydrogen was discovered before either the Lyman or the Paschen series. Why?

27. When the absorption spectrum of hydrogen at room temperature is analyzed, absorption lines for the Lyman series are found, but none are found for the Balmer series. What does this tell us about the energy state of most hydrogen atoms at room temperature?
28. Hydrogen accounts for about 75% by mass of the matter at the surfaces of most stars. However, the absorption lines of hydrogen are strongest (of highest intensity) in the spectra of stars with a surface temperature of about 9000 K. They are weaker in the sun spectrum and are essentially nonexistent in very hot (temperatures above 25,000 K) or rather cool (temperatures below 3500 K) stars. Speculate as to why surface temperature affects the hydrogen absorption lines that we observe.
29. Discuss the similarities and differences between Thomson's model of the hydrogen atom and Bohr's model of the hydrogen atom.
30. Discuss the way in which Thomson's model is nonphysical. Support your argument with experimental evidence.
31. If, in a hydrogen atom, an electron moves to an orbit with a larger radius, does the energy of the hydrogen atom increase or decrease?
32. How is the energy conserved when an atom makes a transition from a higher to a lower energy state?
33. Suppose an electron in a hydrogen atom makes a transition from the $(n+1)$ th orbit to the n th orbit. Is the wavelength of the emitted photon longer for larger values of n , or for smaller values of n ?
34. Discuss why the allowed energies of the hydrogen atom are negative.
35. Can a hydrogen atom absorb a photon whose energy is greater than 13.6 eV?
36. Why can you see through glass but not through wood?
37. Do gravitational forces have a significant effect on atomic energy levels?
38. Show that Planck's constant has the dimensions of angular momentum.

6.5 De Broglie's Matter Waves

39. Which type of radiation is most suitable for the observation of diffraction patterns on crystalline solids; radio waves, visible light, or X-rays? Explain.
40. Speculate as to how the diffraction patterns of a typical crystal would be affected if γ -rays were used instead of X-rays.

41. If an electron and a proton are traveling at the same speed, which one has the shorter de Broglie wavelength?
42. If a particle is accelerating, how does this affect its de Broglie wavelength?
43. Why is the wave-like nature of matter not observed every day for macroscopic objects?
44. What is the wavelength of a neutron at rest? Explain.
45. Why does the setup of Davisson–Germer experiment need to be enclosed in a vacuum chamber? Discuss what result you expect when the chamber is not evacuated.
47. Discuss: How does the interference of water waves differ from the interference of electrons? How are they analogous?
48. Give at least one argument in support of the matter-wave hypothesis.
49. Give at least one argument in support of the particle-nature of radiation.
50. Explain the importance of the Young double-slit experiment.
51. Does the Heisenberg uncertainty principle allow a particle to be at rest in a designated region in space?
52. Can the de Broglie wavelength of a particle be known exactly?
53. Do the photons of red light produce better resolution in a microscope than blue light photons? Explain.
54. Discuss the main difference between an SEM and a TEM.

6.6 Wave-Particle Duality

46. Give an example of an experiment in which light behaves as waves. Give an example of an experiment in which light behaves as a stream of photons.

Problems

6.1 Blackbody Radiation

55. A 200-W heater emits a $1.5\text{-}\mu\text{m}$ radiation. (a) What value of the energy quantum does it emit? (b) Assuming that the specific heat of a 4.0-kg body is $0.83\text{kcal/kg} \cdot \text{K}$, how many of these photons must be absorbed by the body to increase its temperature by 2 K ? (c) How long does the heating process in (b) take, assuming that all radiation emitted by the heater gets absorbed by the body?
56. A 900-W microwave generator in an oven generates energy quanta of frequency 2560 MHz . (a) How many energy quanta does it emit per second? (b) How many energy quanta must be absorbed by a pasta dish placed in the radiation cavity to increase its temperature by 45.0 K ? Assume that the dish has a mass of 0.5 kg and that its specific heat is $0.9\text{ kcal/kg} \cdot \text{K}$. (c) Assume that all energy quanta emitted by the generator are absorbed by the pasta dish. How long must we wait until the dish in (b) is ready?
57. (a) For what temperature is the peak of blackbody radiation spectrum at 400 nm ? (b) If the temperature of a blackbody is 800 K , at what wavelength does it radiate the most energy?
58. The tungsten elements of incandescent light bulbs operate at 3200 K . At what wavelength does the filament radiate maximum energy?
59. Interstellar space is filled with radiation of wavelength $970\mu\text{m}$. This radiation is considered

to be a remnant of the “big bang.” What is the corresponding blackbody temperature of this radiation?

60. The radiant energy from the sun reaches its maximum at a wavelength of about 500.0 nm . What is the approximate temperature of the sun’s surface?

6.2 Photoelectric Effect

61. A photon has energy 20 keV . What are its frequency and wavelength?
62. The wavelengths of visible light range from approximately 400 to 750 nm . What is the corresponding range of photon energies for visible light?
63. What is the longest wavelength of radiation that can eject a photoelectron from silver? Is it in the visible range?
64. What is the longest wavelength of radiation that can eject a photoelectron from potassium, given the work function of potassium 2.24 eV ? Is it in the visible range?
65. Estimate the binding energy of electrons in magnesium, given that the wavelength of 337 nm is the longest wavelength that a photon may have to eject a photoelectron from magnesium photoelectrode.
66. The work function for potassium is 2.26 eV . What is the cutoff frequency when this metal is used as photoelectrode? What is the stopping potential when for the emitted electrons when

this photoelectrode is exposed to radiation of frequency 1200 THz?

67. Estimate the work function of aluminum, given that the wavelength of 304 nm is the longest wavelength that a photon may have to eject a photoelectron from aluminum photoelectrode.
68. What is the maximum kinetic energy of photoelectrons ejected from sodium by the incident radiation of wavelength 450 nm?
69. A 120-nm UV radiation illuminates a silver-plated electrode. What is the maximum kinetic energy of the ejected photoelectrons?
70. A 400-nm violet light ejects photoelectrons with a maximum kinetic energy of 0.860 eV from sodium photoelectrode. What is the work function of sodium?
71. A 600-nm light falls on a photoelectric surface and electrons with the maximum kinetic energy of 0.17 eV are emitted. Determine (a) the work function and (b) the cutoff frequency of the surface. (c) What is the stopping potential when the surface is illuminated with light of wavelength 400 nm?
72. The cutoff wavelength for the emission of photoelectrons from a particular surface is 500 nm. Find the maximum kinetic energy of the ejected photoelectrons when the surface is illuminated with light of wavelength 600 nm.
73. Find the wavelength of radiation that can eject 2.00-eV electrons from calcium electrode. The work function for calcium is 2.71 eV. In what range is this radiation?
74. Find the wavelength of radiation that can eject 0.10-eV electrons from potassium electrode. The work function for potassium is 2.24 eV. In what range is this radiation?
75. Find the maximum velocity of photoelectrons ejected by an 80-nm radiation, if the work function of photoelectrode is 4.73 eV.

6.3 The Compton Effect

76. What is the momentum of a 589-nm yellow photon?
77. What is the momentum of a 4-cm microwave photon?
78. In a beam of white light (wavelengths from 400 to 750 nm), what range of momentum can the photons have?
79. What is the energy of a photon whose momentum is $3.0 \times 10^{-24} \text{ kg} \cdot \text{m/s}$?
80. What is the wavelength of (a) a 12-keV X-ray photon; (b) a 2.0-MeV γ -ray photon?
81. Find the momentum and energy of a $1.0\text{-}\text{\AA}$

photon.

82. Find the wavelength and energy of a photon with momentum $5.00 \times 10^{-29} \text{ kg} \cdot \text{m/s}$.
83. A γ -ray photon has a momentum of $8.00 \times 10^{-21} \text{ kg} \cdot \text{m/s}$. Find its wavelength and energy.
84. (a) Calculate the momentum of a $2.5\text{-}\mu\text{m}$ photon. (b) Find the velocity of an electron with the same momentum. (c) What is the kinetic energy of the electron, and how does it compare to that of the photon?
85. Show that $p = h/\lambda$ and $E_f = hf$ are consistent with the relativistic formula $E^2 = p^2 c^2 + m_0^2 c^2$.
86. Show that the energy E in eV of a photon is given by $E = 1.241 \times 10^{-6} \text{ eV} \cdot \text{m}/\lambda$, where λ is its wavelength in meters.
87. For collisions with free electrons, compare the Compton shift of a photon scattered as an angle of 30° to that of a photon scattered at 45° .
88. X-rays of wavelength 12.5 pm are scattered from a block of carbon. What are the wavelengths of photons scattered at (a) 30° ; (b) 90° ; and, (c) 180° ?

6.4 Bohr's Model of the Hydrogen Atom

89. Calculate the wavelength of the first line in the Lyman series and show that this line lies in the ultraviolet part of the spectrum.
90. Calculate the wavelength of the fifth line in the Lyman series and show that this line lies in the ultraviolet part of the spectrum.
91. Calculate the energy changes corresponding to the transitions of the hydrogen atom: (a) from $n = 3$ to $n = 4$; (b) from $n = 2$ to $n = 1$; and (c) from $n = 3$ to $n = \infty$.
92. Determine the wavelength of the third Balmer line (transition from $n = 5$ to $n = 2$).
93. What is the frequency of the photon absorbed when the hydrogen atom makes the transition from the ground state to the $n = 4$ state?
94. When a hydrogen atom is in its ground state, what are the shortest and longest wavelengths of the photons it can absorb without being ionized?
95. When a hydrogen atom is in its third excited state, what are the shortest and longest wavelengths of the photons it can emit?
96. What is the longest wavelength that light can have if it is to be capable of ionizing the hydrogen atom in its ground state?
97. For an electron in a hydrogen atom in the $n = 2$ state, compute: (a) the angular momentum; (b)

the kinetic energy; (c) the potential energy; and (d) the total energy.

98. Find the ionization energy of a hydrogen atom in the fourth energy state.
99. It has been measured that it required 0.850 eV to remove an electron from the hydrogen atom. In what state was the atom before the ionization happened?
100. What is the radius of a hydrogen atom when the electron is in the first excited state?
101. Find the shortest wavelength in the Balmer series. In what part of the spectrum does this line lie?
102. Show that the entire Paschen series lies in the infrared part of the spectrum.
103. Do the Balmer series and the Lyman series overlap? Why? Why not? (Hint: calculate the shortest Balmer line and the longest Lyman line.)
104. (a) Which line in the Balmer series is the first one in the UV part of the spectrum? (b) How many Balmer lines lie in the visible part of the spectrum? (c) How many Balmer lines lie in the UV?
105. A 4.653- μm emission line of atomic hydrogen corresponds to transition between the states $n_f = 5$ and n_i . Find n_i .

6.5 De Broglie's Matter Waves

106. At what velocity will an electron have a wavelength of 1.00 m?
107. What is the de Broglie wavelength of an electron travelling at a speed of $5.0 \times 10^6 \text{ m/s}$?
108. What is the de Broglie wavelength of an electron that is accelerated from rest through a potential difference of 20 kV?
109. What is the de Broglie wavelength of a proton whose kinetic energy is 2.0 MeV? 10.0 MeV?
110. What is the de Broglie wavelength of a 10-kg football player running at a speed of 8.0 m/s?
111. (a) What is the energy of an electron whose de Broglie wavelength is that of a photon of yellow light with wavelength 590 nm? (b) What is the de Broglie wavelength of an electron whose energy is that of the photon of yellow light?
112. The de Broglie wavelength of a neutron is 0.01 nm. What is the speed and energy of this neutron?
113. What is the wavelength of an electron that is moving at a 3% of the speed of light?
114. At what velocity does a proton have a 6.0-fm wavelength (about the size of a nucleus)? Give your answer in units of c .
115. What is the velocity of a 0.400-kg billiard ball if its wavelength is 7.50 fm?
116. Find the wavelength of a proton that is moving at 1.00% of the speed of light (when $\beta = 0.01$).

6.6 Wave-Particle Duality

117. An AM radio transmitter radiates 500 kW at a frequency of 760 kHz. How many photons per second does the emitter emit?
118. Find the Lorentz factor γ and de Broglie's wavelength for a 50-GeV electron in a particle accelerator.
119. Find the Lorentz factor γ and de Broglie's wavelength for a 1.0-TeV proton in a particle accelerator.
120. What is the kinetic energy of a 0.01-nm electron in a TEM?
121. If electron is to be diffracted significantly by a crystal, its wavelength must be about equal to the spacing, d , of crystalline planes. Assuming $d = 0.250 \text{ nm}$, estimate the potential difference through which an electron must be accelerated from rest if it is to be diffracted by these planes.
122. X-rays form ionizing radiation that is dangerous to living tissue and undetectable to the human eye. Suppose that a student researcher working in an X-ray diffraction laboratory is accidentally exposed to a fatal dose of radiation. Calculate the temperature increase of the researcher under the following conditions: the energy of X-ray photons is 200 keV and the researcher absorbs 4×10^{13} photons per each kilogram of body weight during the exposure. Assume that the specific heat of the student's body is $0.83 \text{ kcal/kg} \cdot \text{K}$.
123. Solar wind (radiation) that is incident on the top of Earth's atmosphere has an average intensity of 1.3 kW/m^2 . Suppose that you are building a solar sail that is to propel a small toy spaceship with a mass of 0.1 kg in the space between the International Space Station and the moon. The sail is made from a very light material, which perfectly reflects the incident radiation. To assess whether such a project is feasible, answer the following questions, assuming that radiation photons are incident only in normal direction to the sail reflecting surface. (a) What is the radiation pressure (force per m^2) of the radiation falling on the mirror-like sail? (b) Given the radiation pressure computed in (a), what will be the

acceleration of the spaceship when the sail has of an area of 10.0m^2 ? (c) Given the acceleration estimate in (b), how fast will the spaceship be moving after 24 hours when it starts from rest?

- 124.** Treat the human body as a blackbody and determine the percentage increase in the total power of its radiation when its temperature increases from 98.6°F to 103°F .
- 125.** Show that Wien's displacement law results from Planck's radiation law. (*Hint:* substitute $x = hc/\lambda kT$ and write Planck's law in the form $I(x, T) = Ax^5/(e^x - 1)$, where $A = 2\pi(kT)^5/(h^4 c^3)$. Now, for fixed T , find

the position of the maximum in $I(x, T)$ by solving for x in the equation $dI(x, T)/dx = 0$.)

- 126.** Show that Stefan's law results from Planck's radiation law. *Hint:* To compute the total power of blackbody radiation emitted across the entire spectrum of wavelengths at a given temperature, integrate Planck's law over the

entire spectrum $P(T) = \int_0^\infty I(\lambda, T)d\lambda$. Use

the substitution $x = hc/\lambda kT$ and the tabulated value of the integral

$$\int_0^\infty dx x^3/(e^x - 1) = \pi^4/15.$$

Additional Problems

- 127.** Determine the power intensity of radiation per unit wavelength emitted at a wavelength of 500.0 nm by a blackbody at a temperature of $10,000\text{ K}$.
- 128.** The HCl molecule oscillates at a frequency of 87.0 THz . What is the difference (in eV) between its adjacent energy levels?
- 129.** A quantum mechanical oscillator vibrates at a frequency of 250.0 THz . What is the minimum energy of radiation it can emit?
- 130.** In about 5 billion years, the sun will evolve to a red giant. Assume that its surface temperature will decrease to about half its present value of 6000 K , while its present radius of $7.0 \times 10^8\text{ m}$ will increase to $1.5 \times 10^{11}\text{ m}$ (which is the current Earth-sun distance). Calculate the ratio of the total power emitted by the sun in its red giant stage to its present power.
- 131.** A sodium lamp emits 2.0 W of radiant energy, most of which has a wavelength of about 589 nm . Estimate the number of photons emitted per second by the lamp.
- 132.** Photoelectrons are ejected from a photoelectrode and are detected at a distance of 2.50 cm away from the photoelectrode. The work function of the photoelectrode is 2.71 eV and the incident radiation has a wavelength of 420 nm . How long does it take a photoelectron to travel to the detector?
- 133.** If the work function of a metal is 3.2 eV , what is the maximum wavelength that a photon can have to eject a photoelectron from this metal surface?
- 134.** The work function of a photoelectric surface is 2.00 eV . What is the maximum speed of the photoelectrons emitted from this surface when a 450-nm light falls on it?
- 135.** A 400-nm laser beam is projected onto a calcium electrode. The power of the laser beam is 2.00 mW and the work function of calcium is 2.31 eV . (a) How many photoelectrons per second are ejected? (b) What net power is carried away by photoelectrons?
- 136.** (a) Calculate the number of photoelectrons per second that are ejected from a 1.00-mm^2 area of sodium metal by a 500-nm radiation with intensity 1.30 kW/m^2 (the intensity of sunlight above Earth's atmosphere). (b) Given the work function of the metal as 2.28 eV , what power is carried away by these photoelectrons?
- 137.** A laser with a power output of 2.00 mW at a 400-nm wavelength is used to project a beam of light onto a calcium photoelectrode. (a) How many photoelectrons leave the calcium surface per second? (b) What power is carried away by ejected photoelectrons, given that the work function of calcium is 2.31 eV ? (c) Calculate the photocurrent. (d) If the photoelectrode suddenly becomes electrically insulated and the setup of two electrodes in the circuit suddenly starts to act like a 2.00-pF capacitor, how long will current flow before the capacitor voltage stops it?

- 138.** The work function for barium is 2.48 eV. Find the maximum kinetic energy of the ejected photoelectrons when the barium surface is illuminated with: (a) radiation emitted by a 100-kW radio station broadcasting at 800 kHz; (b) a 633-nm laser light emitted from a powerful He-Ne laser; and (c) a 434-nm blue light emitted by a small hydrogen gas discharge tube.
- 139.** (a) Calculate the wavelength of a photon that has the same momentum as a proton moving with 1% of the speed of light in a vacuum. (b) What is the energy of this photon in MeV? (c) What is the kinetic energy of the proton in MeV?
- 140.** (a) Find the momentum of a 100-keV X-ray photon. (b) Find the velocity of a neutron with the same momentum. (c) What is the neutron's kinetic energy in eV?
- 141.** The momentum of light, as it is for particles, is exactly reversed when a photon is reflected straight back from a mirror, assuming negligible recoil of the mirror. The change in momentum is twice the photon's incident momentum, as it is for the particles. Suppose that a beam of light has an intensity 1.0 kW/m^2 and falls on a -2.0-m^2 area of a mirror and reflects from it. (a) Calculate the energy reflected in 1.00 s. (b) What is the momentum imparted to the mirror? (c) Use Newton's second law to find the force on the mirror. (d) Does the assumption of no-recoil for the mirror seem reasonable?
- 142.** A photon of energy 5.0 keV collides with a stationary electron and is scattered at an angle of 60° . What is the energy acquired by the electron in the collision?
- 143.** A 0.75-nm photon is scattered by a stationary electron. The speed of the electron's recoil is $1.5 \times 10^6 \text{ m/s}$. (a) Find the wavelength shift of the photon. (b) Find the scattering angle of the photon.
- 144.** Find the maximum change in X-ray wavelength that can occur due to Compton scattering. Does this change depend on the wavelength of the incident beam?
- 145.** A photon of wavelength 700 nm is incident on a hydrogen atom. When this photon is absorbed, the atom becomes ionized. What is the lowest possible orbit that the electron could have occupied before being ionized?
- 146.** What is the maximum kinetic energy of an electron such that a collision between the electron and a stationary hydrogen atom in its ground state is definitely elastic?
- 147.** Singly ionized atomic helium He^{+1} is a hydrogen-like ion. (a) What is its ground-state radius? (b) Calculate the energies of its four lowest energy states. (c) Repeat the calculations for the Li^{2+} ion.
- 148.** A triply ionized atom of beryllium Be^{3+} is a hydrogen-like ion. When Be^{3+} is in one of its excited states, its radius in this n th state is exactly the same as the radius of the first Bohr orbit of hydrogen. Find n and compute the ionization energy for this state of Be^{3+} .
- 149.** In extreme-temperature environments, such as those existing in a solar corona, atoms may be ionized by undergoing collisions with other atoms. One example of such ionization in the solar corona is the presence of C^{5+} ions, detected in the Fraunhofer spectrum. (a) By what factor do the energies of the C^{5+} ion scale compare to the energy spectrum of a hydrogen atom? (b) What is the wavelength of the first line in the Paschen series of C^{5+} ? (c) In what part of the spectrum are these lines located?
- 150.** (a) Calculate the ionization energy for He^{+} . (b) What is the minimum frequency of a photon capable of ionizing He^{+} ?
- 151.** Experiments are performed with ultracold neutrons having velocities as small as 1.00 m/s. Find the wavelength of such an ultracold neutron and its kinetic energy.
- 152.** Find the velocity and kinetic energy of a 6.0-fm neutron. (Rest mass energy of neutron is $E_0 = 940 \text{ MeV}$.)
- 153.** The spacing between crystalline planes in the NaCl crystal is 0.281 nm, as determined by X-ray diffraction with X-rays of wavelength 0.170 nm. What is the energy of neutrons in the neutron beam that produces diffraction peaks at the same locations as the peaks obtained with the X-rays?
- 154.** What is the wavelength of an electron accelerated from rest in a 30.0-kV potential difference?
- 155.** Calculate the velocity of a 1.0- μm electron and a potential difference used to accelerate it from rest to this velocity.

- 156.** In a supercollider at CERN, protons are accelerated to velocities of $0.25c$. What are their wavelengths at this speed? What are their kinetic energies? If a beam of protons were to gain its kinetic energy in only one pass through a potential difference, how high would this potential difference have to be? (Rest mass energy of a proton is $E_0 = 938 \text{ MeV}$).
- 157.** Find the de Broglie wavelength of an electron accelerated from rest in an X-ray tube in the potential difference of 100 keV. (Rest mass energy of an electron is $E_0 = 511 \text{ keV}$.)
- 158.** The cutoff wavelength for the emission of photoelectrons from a particular surface is 500 nm. Find the maximum kinetic energy of the ejected photoelectrons when the surface is illuminated with light of wavelength 450 nm.
- 159.** Compare the wavelength shift of a photon scattered by a free electron to that of a photon scattered at the same angle by a free proton.
- 160.** The spectrometer used to measure the wavelengths of the scattered X-rays in the Compton experiment is accurate to $5.0 \times 10^{-4} \text{ nm}$. What is the minimum scattering angle for which the X-rays interacting with the free electrons can be distinguished from those interacting with the atoms?
- 161.** Consider a hydrogen-like ion where an electron is orbiting a nucleus that has charge $q = +Ze$. Derive the formulas for the energy E_n of the electron in n th orbit and the orbital radius r_n .
- 162.** Assume that a hydrogen atom exists in the $n = 2$ excited state for 10^{-8} s before decaying to the ground state. How many times does the electron orbit the proton nucleus during this time? How long does it take Earth to orbit the sun this many times?
- 163.** An atom can be formed when a negative muon is captured by a proton. The muon has the same charge as the electron and a mass 207 times that of the electron. Calculate the frequency of the photon emitted when this atom makes the transition from $n = 2$ to the $n = 1$ state. Assume that the muon is orbiting a stationary proton.

CHAPTER 7

Quantum Mechanics

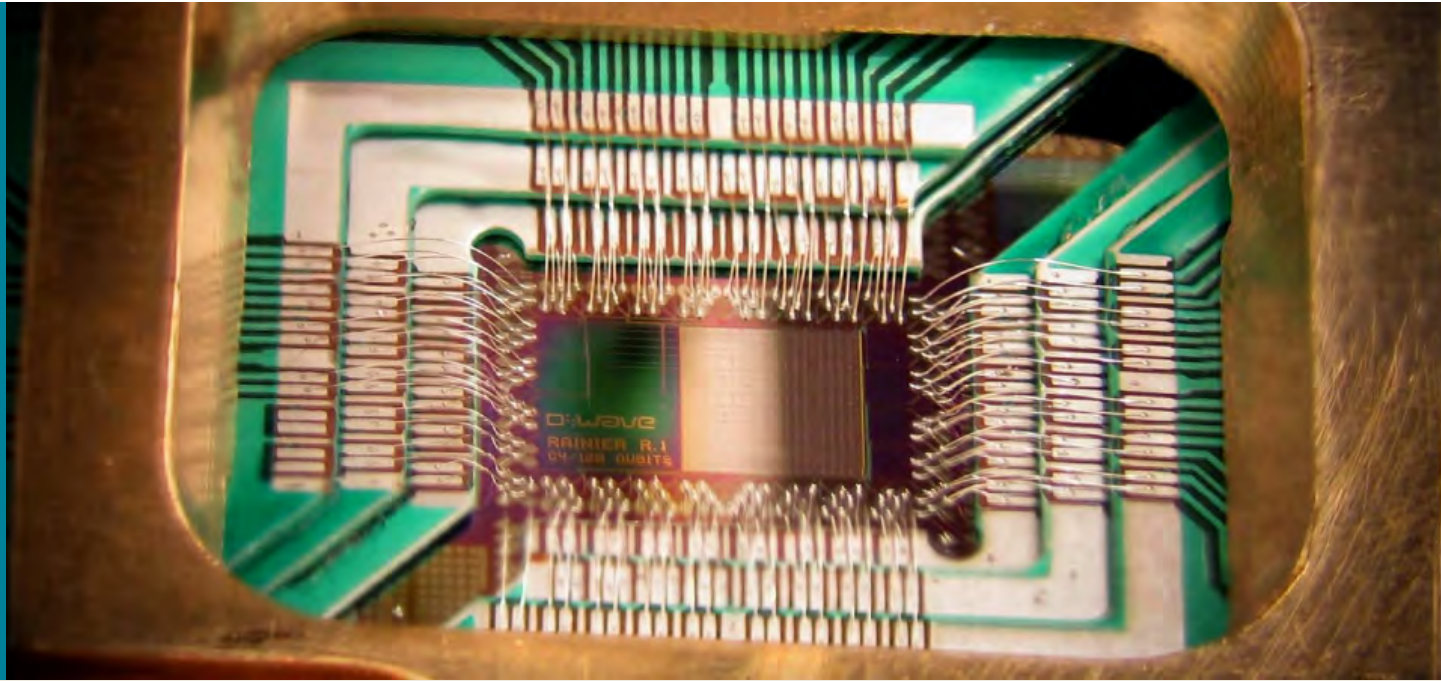


Figure 7.1 A D-wave qubit processor: The brain of a quantum computer that encodes information in quantum bits to perform complex calculations. (credit: modification of work by D-Wave Systems, Inc.)

Chapter Outline

[7.1 Wave Functions](#)

[7.2 The Heisenberg Uncertainty Principle](#)

[7.3 The Schrödinger Equation](#)

[7.4 The Quantum Particle in a Box](#)

[7.5 The Quantum Harmonic Oscillator](#)

[7.6 The Quantum Tunneling of Particles through Potential Barriers](#)

INTRODUCTION Quantum mechanics is a powerful framework for understanding the motions and interactions of particles at small scales, such as atoms and molecules. The ideas behind quantum mechanics often appear quite strange. In many ways, our everyday experience with the macroscopic physical world does not prepare us for the microscopic world of quantum mechanics. The purpose of this chapter is to introduce you to this exciting world.

Pictured above is a quantum-computer processor. This device is the “brain” of a quantum computer that operates at near-absolute zero temperatures. Unlike a digital computer, which encodes information in binary digits (definite states of either zero or one), a quantum computer encodes information in quantum bits or

qubits (mixed states of zero *and* one). Quantum computers are discussed in the first section of this chapter.

7.1 Wave Functions

Learning Objectives

By the end of this section, you will be able to:

- Describe the statistical interpretation of the wave function
- Use the wave function to determine probabilities
- Calculate expectation values of position, momentum, and kinetic energy

In the preceding chapter, we saw that particles act in some cases like particles and in other cases like waves. But what does it mean for a particle to “act like a wave”? What precisely is “waving”? What rules govern how this wave changes and propagates? How is the wave function used to make predictions? For example, if the amplitude of an electron wave is given by a function of position and time, $\Psi(x, t)$, defined for all x , *where* exactly is the electron? The purpose of this chapter is to answer these questions.

Using the Wave Function

A clue to the physical meaning of the wave function $\Psi(x, t)$ is provided by the two-slit interference of monochromatic light ([Figure 7.2](#)). (See also [Electromagnetic Waves](#) and [Interference](#).) The **wave function** of a light wave is given by $E(x, t)$, and its energy density is given by $|E|^2$, where E is the electric field strength. The energy of an individual photon depends only on the frequency of light, $\epsilon_{\text{photon}} = hf$, so $|E|^2$ is proportional to the number of photons. When light waves from S_1 interfere with light waves from S_2 at the viewing screen (a distance D away), an interference pattern is produced (part (a) of the figure). Bright fringes correspond to points of constructive interference of the light waves, and dark fringes correspond to points of destructive interference of the light waves (part (b)).

Suppose the screen is initially unexposed to light. If the screen is exposed to very weak light, the interference pattern appears gradually ([Figure 7.2\(c\)](#), left to right). Individual photon hits on the screen appear as dots. The dot density is expected to be large at locations where the interference pattern will be, ultimately, the most intense. In other words, the probability (per unit area) that a single photon will strike a particular spot on the screen is proportional to the square of the total electric field, $|E|^2$ at that point. Under the right conditions, the same interference pattern develops for matter particles, such as electrons.

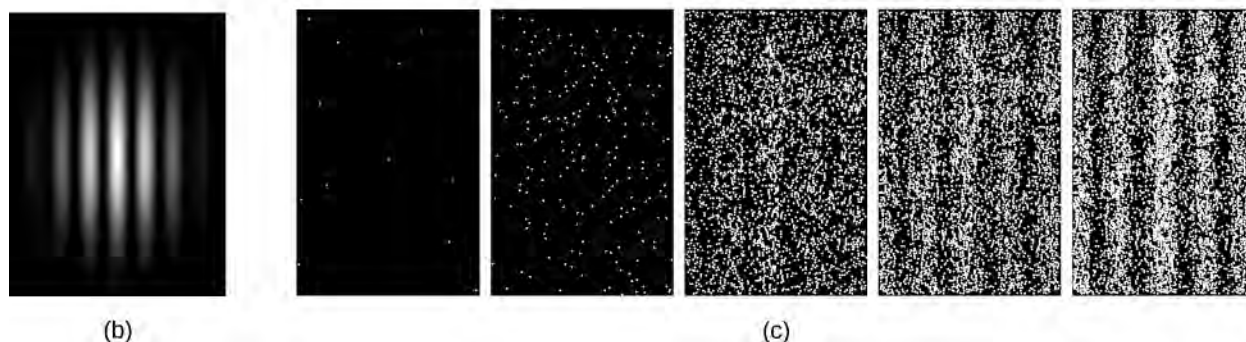
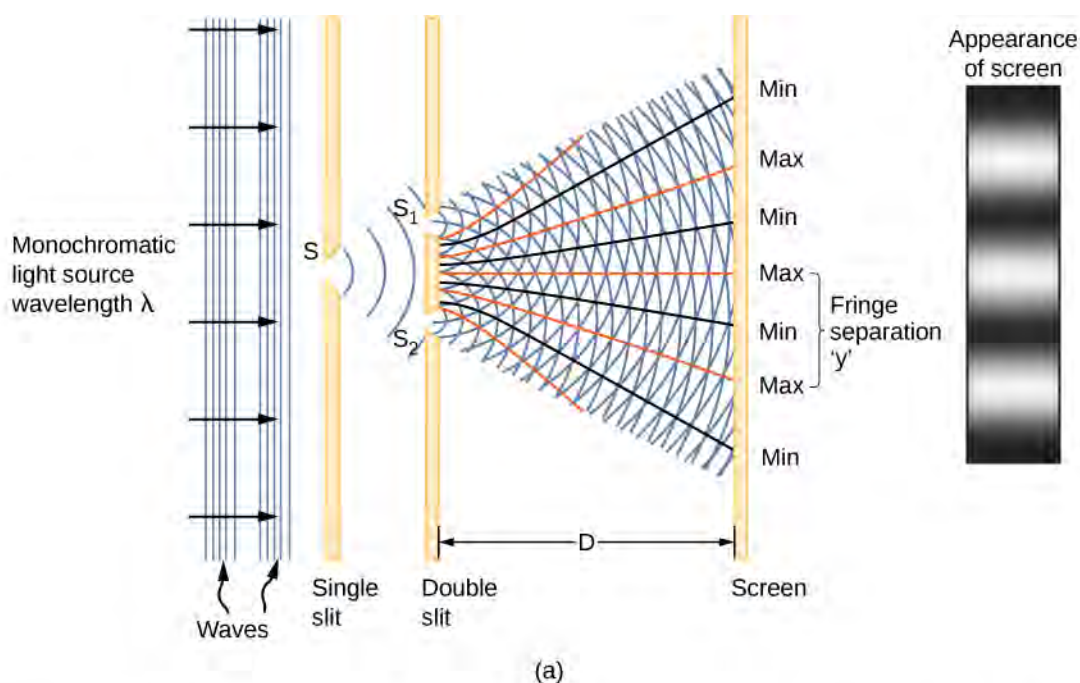


Figure 7.2 Two-slit interference of monochromatic light. (a) Schematic of two-slit interference; (b) light interference pattern; (c) interference pattern built up gradually under low-intensity light (left to right).

INTERACTIVE

Visit this [interactive simulation \(https://openstax.org/l/21intquawavint\)](https://openstax.org/l/21intquawavint) to learn more about quantum wave interference.

The square of the matter wave $|\Psi|^2$ in one dimension has a similar interpretation as the square of the electric field $|E|^2$. It gives the probability that a particle will be found at a particular position and time per unit length, also called the **probability density**. The probability (P) a particle is found in a narrow interval ($x, x + dx$) at time t is therefore

$$P(x, x + dx) = |\Psi(x, t)|^2 dx. \quad 7.1$$

(Later, we define the magnitude squared for the general case of a function with “imaginary parts.”) This probabilistic interpretation of the wave function is called the **Born interpretation**. Examples of wave functions and their squares for a particular time t are given in [Figure 7.3](#).

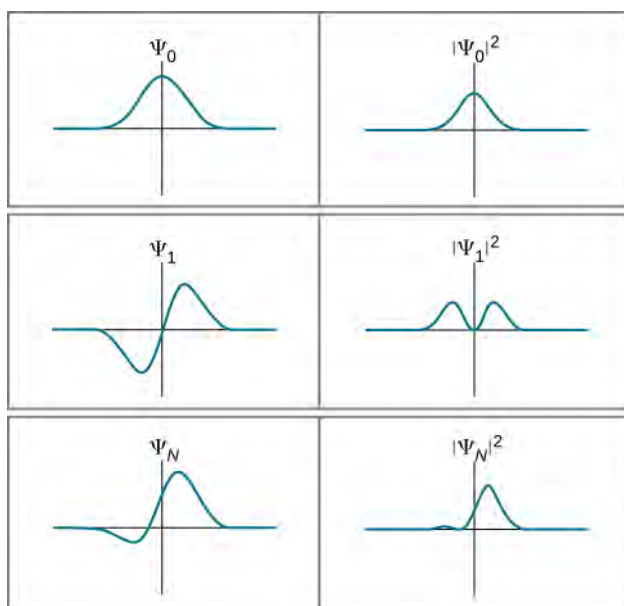


Figure 7.3 Several examples of wave functions and the corresponding square of their wave functions.

If the wave function varies slowly over the interval Δx , the probability a particle is found in the interval is approximately

$$P(x, x + \Delta x) \approx |\Psi(x, t)|^2 \Delta x. \quad 7.2$$

Notice that squaring the wave function ensures that the probability is positive. (This is analogous to squaring the electric field strength—which may be positive or negative—to obtain a positive value of intensity.) However, if the wave function does not vary slowly, we must integrate:

$$P(x, x + \Delta x) = \int_x^{x+\Delta x} |\Psi(x, t)|^2 dx. \quad 7.3$$

This probability is just the area under the function $|\Psi(x, t)|^2$ between x and $x + \Delta x$. The probability of finding the particle “somewhere” (the **normalization condition**) is

$$P(-\infty, +\infty) = \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1. \quad 7.4$$

For a particle in two dimensions, the integration is over an area and requires a double integral; for a particle in three dimensions, the integration is over a volume and requires a triple integral. For now, we stick to the simple one-dimensional case.



EXAMPLE 7.1

Where Is the Ball? (Part I)

A ball is constrained to move along a line inside a tube of length L . The ball is equally likely to be found anywhere in the tube at some time t . What is the probability of finding the ball in the left half of the tube at that

time? (The answer is 50%, of course, but how do we get this answer by using the probabilistic interpretation of the quantum mechanical wave function?)

Strategy

The first step is to write down the wave function. The ball is equally like to be found anywhere in the box, so one way to describe the ball with a *constant* wave function (Figure 7.4). The normalization condition can be used to find the value of the function and a simple integration over half of the box yields the final answer.

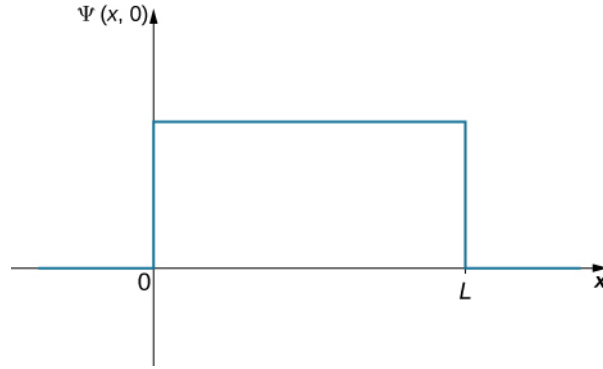


Figure 7.4 Wave function for a ball in a tube of length L .

Solution

The wave function of the ball can be written as $\Psi(x, t) = C(0 < x < L)$, where C is a constant, and $\Psi(x, t) = 0$ otherwise. We can determine the constant C by applying the normalization condition (we set $t = 0$ to simplify the notation):

$$P(x = -\infty, +\infty) = \int_{-\infty}^{+\infty} |\Psi(x, 0)|^2 dx = 1.$$

This integral can be broken into three parts: (1) negative infinity to zero, (2) zero to L , and (3) L to infinity. The particle is constrained to be in the tube, so $C = 0$ outside the tube and the first and last integrations are zero. The above equation can therefore be written

$$P(x = 0, L) = \int_0^L |C|^2 dx = 1.$$

The value C does not depend on x and can be taken out of the integral, so we obtain

$$|C|^2 \int_0^L dx = 1.$$

Integration gives

$$C = \sqrt{\frac{1}{L}}.$$

To determine the probability of finding the ball in the first half of the box ($0 < x < L/2$), we have

$$P(x = 0, L/2) = \int_0^{L/2} \left| \sqrt{\frac{1}{L}} \right|^2 dx = \left(\frac{1}{L} \right) \frac{L}{2} = 0.50.$$

Significance

The probability of finding the ball in the first half of the tube is 50%, as expected. Two observations are noteworthy. First, this result corresponds to the area under the constant function from $x = 0$ to $L/2$ (the area of a square left of $L/2$). Second, this calculation requires an integration of the *square* of the wave function. A common mistake in performing such calculations is to forget to square the wave function before integration.

EXAMPLE 7.2

Where Is the Ball? (Part II)

A ball is again constrained to move along a line inside a tube of length L . This time, the ball is found preferentially in the middle of the tube. One way to represent its wave function is with a simple cosine function (Figure 7.5). What is the probability of finding the ball in the last one-quarter of the tube?

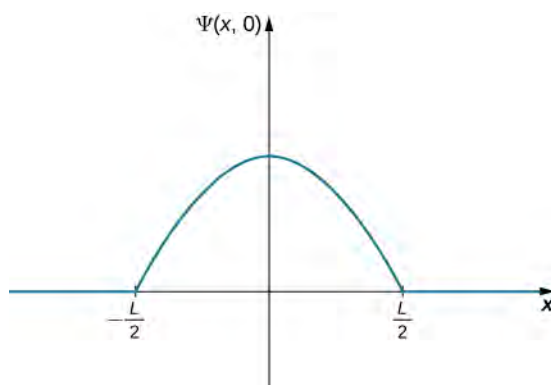


Figure 7.5 Wave function for a ball in a tube of length L , where the ball is preferentially in the middle of the tube.

Strategy

We use the same strategy as before. In this case, the wave function has two unknown constants: One is associated with the wavelength of the wave and the other is the amplitude of the wave. We determine the amplitude by using the boundary conditions of the problem, and we evaluate the wavelength by using the normalization condition. Integration of the square of the wave function over the last quarter of the tube yields the final answer. The calculation is simplified by centering our coordinate system on the peak of the wave function.

Solution

The wave function of the ball can be written

$$\Psi(x, 0) = A \cos(kx) \quad (-L/2 < x < L/2),$$

where A is the amplitude of the wave function and $k = 2\pi/\lambda$ is its wave number. Beyond this interval, the amplitude of the wave function is zero because the ball is confined to the tube. Requiring the wave function to terminate at the right end of the tube gives

$$\Psi\left(x = \frac{L}{2}, 0\right) = 0.$$

Evaluating the wave function at $x = L/2$ gives

$$A \cos(kL/2) = 0.$$

This equation is satisfied if the argument of the cosine is an integral multiple of $\pi/2$, $3\pi/2$, $5\pi/2$, and so on. In this case, we have

$$\frac{kL}{2} = \frac{\pi}{2},$$

or

$$k = \frac{\pi}{L}.$$

Applying the normalization condition gives $A = \sqrt{2/L}$, so the wave function of the ball is

$$\Psi(x, 0) = \sqrt{\frac{2}{L}} \cos(\pi x/L), \quad -L/2 < x < L/2.$$

To determine the probability of finding the ball in the last quarter of the tube, we square the function and integrate:

$$P(x = L/4, L/2) = \int_{L/4}^{L/2} \left| \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \right|^2 dx = 0.091.$$

Significance

The probability of finding the ball in the last quarter of the tube is 9.1%. The ball has a definite wavelength ($\lambda = 2L$). If the tube is of macroscopic length ($L = 1$ m), the momentum of the ball is

$$p = \frac{h}{\lambda} = \frac{h}{2L} \sim 10^{-36} \text{ m/s}.$$

This momentum is much too small to be measured by any human instrument.

An Interpretation of the Wave Function

We are now in position to begin to answer the questions posed at the beginning of this section. First, for a traveling particle described by $\Psi(x, t) = A \sin(kx - \omega t)$, what is “waving?” Based on the above discussion, the answer is a mathematical function that can, among other things, be used to determine where the particle is likely to be when a position measurement is performed. Second, how is the wave function used to make predictions? If it is necessary to find the probability that a particle will be found in a certain interval, square the wave function and integrate over the interval of interest. Soon, you will learn soon that the wave function can be used to make many other kinds of predictions, as well.

Third, if a matter wave is given by the wave function $\Psi(x, t)$, *where* exactly is the particle? Two answers exist: (1) when the observer *is not* looking (or the particle is not being otherwise detected), the particle is everywhere ($x = -\infty, +\infty$); and (2) when the observer *is* looking (the particle is being detected), the particle “jumps into” a particular position state ($x, x + dx$) with a probability given by $P(x, x + dx) = |\Psi(x, t)|^2 dx$ —a process called **state reduction** or **wave function collapse**. This answer is called the **Copenhagen interpretation** of the wave function, or of quantum mechanics.

To illustrate this interpretation, consider the simple case of a particle that can occupy a small container either at x_1 or x_2 (Figure 7.6). In classical physics, we assume the particle is located either at x_1 or x_2 when the observer is not looking. However, in quantum mechanics, the particle may exist in a state of indefinite position—that is, it may be located at x_1 *and* x_2 when the observer is not looking. The assumption that a particle can only have one value of position (when the observer is not looking) is abandoned. Similar comments can be made of other measurable quantities, such as momentum and energy.

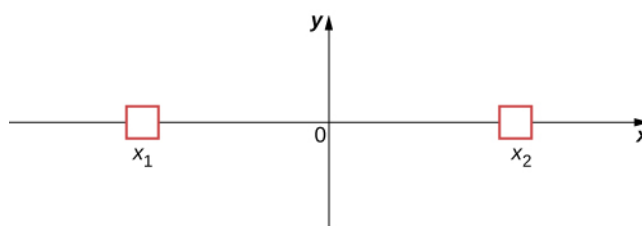


Figure 7.6 A two-state system of position of a particle.

The bizarre consequences of the Copenhagen interpretation of quantum mechanics are illustrated by a creative thought experiment first articulated by Erwin Schrödinger (*National Geographic*, 2013) ([Figure 7.7](#)):

“A cat is placed in a steel box along with a Geiger counter, a vial of poison, a hammer, and a radioactive substance. When the radioactive substance decays, the Geiger detects it and triggers the hammer to release the poison, which subsequently kills the cat. The radioactive decay is a random [probabilistic] process, and there is no way to predict when it will happen. Physicists say the atom exists in a state known as a superposition—both decayed and not decayed at the same time. Until the box is opened, an observer doesn’t know whether the cat is alive or dead—because the cat’s fate is intrinsically tied to whether or not the atom has decayed and the cat would [according to the Copenhagen interpretation] be “living and dead ... in equal parts” until it is observed.”

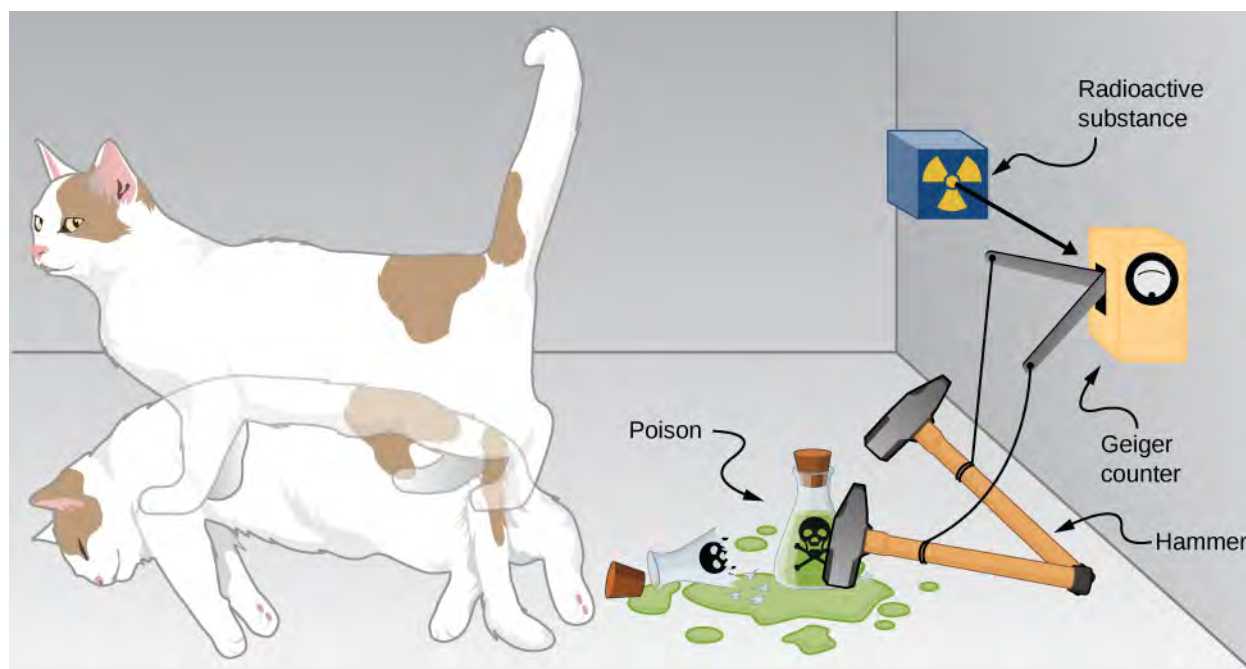


Figure 7.7 Schrödinger's cat.

Schrödinger took the absurd implications of this thought experiment (a cat simultaneously dead and alive) as an argument against the Copenhagen interpretation. However, this interpretation remains the most commonly taught view of quantum mechanics.

Two-state systems (left and right, atom decays and does not decay, and so on) are often used to illustrate the principles of quantum mechanics. These systems find many applications in nature, including electron spin and mixed states of particles, atoms, and even molecules. Two-state systems are also finding application in the quantum computer, as mentioned in the introduction of this chapter. Unlike a digital computer, which encodes information in binary digits (zeroes and ones), a quantum computer stores and manipulates data in the form of quantum bits, or qubits. In general, a qubit is not in a state of zero or one, but rather in a mixed state of zero *and* one. If a large number of qubits are placed in the same quantum state, the measurement of an individual qubit would produce a zero with a probability p , and a one with a probability $q = 1 - p$. Many scientists believe that quantum computers are the future of the computer industry.

Complex Conjugates

Later in this section, you will see how to use the wave function to describe particles that are “free” or bound by forces to other particles. The specific form of the wave function depends on the details of the physical system. A peculiarity of quantum theory is that these functions are usually **complex functions**. A complex function is one that contains one or more imaginary numbers ($i = \sqrt{-1}$). Experimental measurements produce real (nonimaginary) numbers only, so the above procedure to use the wave function must be slightly modified. In general, the probability that a particle is found in the narrow interval $(x, x + dx)$ at time t is given by

$$P(x, x + dx) = |\Psi(x, t)|^2 dx = \Psi^*(x, t) \Psi(x, t) dx, \quad 7.5$$

where $\Psi^*(x, t)$ is the complex conjugate of the wave function. The complex conjugate of a function is obtained by replacing every occurrence of $i = \sqrt{-1}$ in that function with $-i$. This procedure eliminates complex numbers in all predictions because the product $\Psi^*(x, t) \Psi(x, t)$ is always a real number.

✓ CHECK YOUR UNDERSTANDING 7.1

If $a = 3 + 4i$, what is the product $a^* a$?

Consider the motion of a free particle that moves along the x -direction. As the name suggests, a free particle experiences no forces and so moves with a constant velocity. As we will see in a later section of this chapter, a formal quantum mechanical treatment of a free particle indicates that its wave function has real *and* complex parts. In particular, the wave function is given by

$$\Psi(x, t) = A \cos(kx - \omega t) + iA \sin(kx - \omega t),$$

where A is the amplitude, k is the wave number, and ω is the angular frequency. Using Euler's formula, $e^{i\phi} = \cos(\phi) + i \sin(\phi)$, this equation can be written in the form

$$\Psi(x, t) = A e^{i(kx - \omega t)} = A e^{i\phi},$$

where ϕ is the phase angle. If the wave function varies slowly over the interval Δx , the probability of finding the particle in that interval is

$$P(x, x + \Delta x) \approx \Psi^*(x, t) \Psi(x, t) \Delta x = (A e^{-i\phi}) (A e^{i\phi}) \Delta x = (A^* A) \Delta x.$$

If A has real and complex parts ($a + ib$, where a and b are real constants), then

$$A^* A = (a + ib)(a - ib) = a^2 + b^2.$$

Notice that the complex numbers have vanished. Thus,

$$P(x, x + \Delta x) \approx |A|^2 \Delta x$$

is a real quantity. The interpretation of $\Psi^*(x, t) \Psi(x, t)$ as a probability density ensures that the predictions of quantum mechanics can be checked in the “real world.”

✓ CHECK YOUR UNDERSTANDING 7.2

Suppose that a particle with energy E is moving along the x -axis and is confined in the region between 0 and L . One possible wave function is

$$\psi(x, t) = \begin{cases} A e^{-iEt/\hbar} \sin \frac{\pi x}{L}, & \text{when } 0 \leq x \leq L \\ 0, & \text{otherwise} \end{cases}.$$

Determine the normalization constant.

Expectation Values

In classical mechanics, the solution to an equation of motion is a function of a measurable quantity, such as $x(t)$, where x is the position and t is the time. Note that the particle has one value of position for any time t . In quantum mechanics, however, the solution to an equation of motion is a wave function, $\Psi(x, t)$. The particle has many values of position for any time t , and only the probability density of finding the particle, $|\Psi(x, t)|^2$, can be known. The average value of position for a large number of particles with the same wave function is expected to be

$$\langle x \rangle = \int_{-\infty}^{\infty} x P(x, t) dx = \int_{-\infty}^{\infty} x \Psi^*(x, t) \Psi(x, t) dx. \quad 7.6$$

This is called the **expectation value** of the position. It is usually written

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t) x \Psi(x, t) dx, \quad 7.7$$

where the x is sandwiched between the wave functions. The reason for this will become apparent soon. Formally, x is called the **position operator**.

At this point, it is important to stress that a wave function can be written in terms of other quantities as well, such as velocity (v), momentum (p), and kinetic energy (K). The expectation value of momentum, for example, can be written

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(p, t) p \Psi(p, t) dp, \quad 7.8$$

Where dp is used instead of dx to indicate an infinitesimal interval in momentum. In some cases, we know the wave function in position, $\Psi(x, t)$, but seek the expectation of momentum. The procedure for doing this is

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t) \left(-i\hbar \frac{d}{dx} \right) \Psi(x, t) dx, \quad 7.9$$

where the quantity in parentheses, sandwiched between the wave functions, is called the **momentum operator** in the x -direction. [The momentum operator in [Equation 7.9](#) is said to be the position-space representation of the momentum operator.] The momentum operator must act (operate) on the wave function to the right, and then the result must be multiplied by the complex conjugate of the wave function on the left, before integration. The momentum operator in the x -direction is sometimes denoted

$$(p_x)_{\text{op}} = -i\hbar \frac{d}{dx}, \quad 7.10$$

Momentum operators for the y - and z -directions are defined similarly. This operator and many others are derived in a more advanced course in modern physics. In some cases, this derivation is relatively simple. For example, the kinetic energy operator is just

$$(K)_{\text{op}} = \frac{1}{2}m(v_x)_{\text{op}}^2 = \frac{(p_x)_{\text{op}}^2}{2m} = \frac{\left(-i\hbar \frac{d}{dx}\right)^2}{2m} = \frac{-\hbar^2}{2m} \left(\frac{d}{dx}\right) \left(\frac{d}{dx}\right). \quad 7.11$$

Thus, if we seek an expectation value of kinetic energy of a particle in one dimension, two successive ordinary derivatives of the wave function are required before integration.

Expectation-value calculations are often simplified by exploiting the symmetry of wave functions. Symmetric wave functions can be even or odd. An **even function** is a function that satisfies

$$\psi(x) = \psi(-x). \quad 7.12$$

In contrast, an **odd function** is a function that satisfies

$$\psi(x) = -\psi(-x). \quad 7.13$$

An example of even and odd functions is shown in [Figure 7.8](#). An even function is symmetric about the y -axis. This function is produced by reflecting $\psi(x)$ for $x > 0$ about the vertical y -axis. By comparison, an odd function is generated by reflecting the function about the y -axis and then about the x -axis. (An odd function is also referred to as an **anti-symmetric function**.)

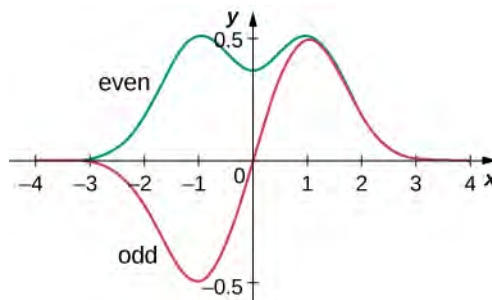


Figure 7.8 Examples of even and odd wave functions.

In general, an even function times an even function produces an even function. A simple example of an even function is the product $x^2 e^{-x^2}$ (even times even is even). Similarly, an odd function times an odd function produces an even function, such as $x \sin x$ (odd times odd is even). However, an odd function times an even function produces an odd function, such as $x e^{-x^2}$ (odd times even is odd). The integral over all space of an odd function is zero, because the total area of the function above the x -axis cancels the (negative) area below it. As the next example shows, this property of odd functions is very useful.

EXAMPLE 7.3

Expectation Value (Part I)

The normalized wave function of a particle is

$$\psi(x) = e^{-|x|/x_0} / \sqrt{x_0}.$$

Find the expectation value of position.

Strategy

Substitute the wave function into [Equation 7.7](#) and evaluate. The position operator introduces a multiplicative factor only, so the position operator need not be “sandwiched.”

Solution

First multiply, then integrate:

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx x |\psi(x)|^2 = \int_{-\infty}^{+\infty} dx x \left| \frac{e^{-|x|/x_0}}{\sqrt{x_0}} \right|^2 = \frac{1}{x_0} \int_{-\infty}^{+\infty} dx x e^{-2|x|/x_0} = 0.$$

Significance

The function in the integrand ($x e^{-2|x|/x_0}$) is odd since it is the product of an odd function (x) and an even function ($e^{-2|x|/x_0}$). The integral vanishes because the total area of the function about the x -axis cancels the (negative) area below it. The result ($\langle x \rangle = 0$) is not surprising since the probability density function is symmetric about $x = 0$.



EXAMPLE 7.4

Expectation Value (Part II)

The time-dependent wave function of a particle confined to a region between 0 and L is

$$\psi(x, t) = A e^{-i\omega t} \sin(\pi x/L)$$

where ω is angular frequency and E is the energy of the particle. (Note: The function varies as a sine because of the limits (0 to L). When $x = 0$, the sine factor is zero and the wave function is zero, consistent with the boundary conditions.) Calculate the expectation values of position, momentum, and kinetic energy.

Strategy

We must first normalize the wave function to find A . Then we use the operators to calculate the expectation values.

Solution

Computation of the normalization constant:

$$1 = \int_0^L dx \psi^*(x) \psi(x) = \int_0^L dx \left(A e^{+i\omega t} \sin \frac{\pi x}{L} \right) \left(A e^{-i\omega t} \sin \frac{\pi x}{L} \right) = A^2 \int_0^L dx \sin^2 \frac{\pi x}{L} = A^2 \frac{L}{2} \Rightarrow A = \sqrt{\frac{2}{L}}.$$

The expectation value of position is

$$\langle x \rangle = \int_0^L dx \psi^*(x) x \psi(x) = \int_0^L dx \left(A e^{+i\omega t} \sin \frac{\pi x}{L} \right) x \left(A e^{-i\omega t} \sin \frac{\pi x}{L} \right) = A^2 \int_0^L dx x \sin^2 \frac{\pi x}{L} = A^2 \frac{L^2}{4} = \frac{L}{2}.$$

The expectation value of momentum in the x -direction also requires an integral. To set this integral up, the associated operator must—by rule—act to the right on the wave function $\psi(x)$:

$$-i\hbar \frac{d}{dx} \psi(x) = -i\hbar \frac{d}{dx} A e^{-i\omega t} \sin \frac{\pi x}{L} = -i \frac{A\hbar}{2L} e^{-i\omega t} \cos \frac{\pi x}{L}.$$

Therefore, the expectation value of momentum is

$$\langle p \rangle = \int_0^L dx \left(A e^{+i\omega t} \sin \frac{\pi x}{L} \right) \left(-i \frac{A\hbar}{2L} e^{-i\omega t} \cos \frac{\pi x}{L} \right) = -i \frac{A^2 \hbar}{4L} \int_0^L dx \sin \frac{2\pi x}{L} = 0.$$

The function in the integral is a sine function with a wavelength equal to the width of the well, L —an odd function about $x = L/2$. As a result, the integral vanishes.

The expectation value of kinetic energy in the x -direction requires the associated operator to act on the wave

function:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} A e^{-i\omega t} \sin \frac{\pi x}{L} = -\frac{\hbar^2}{2m} A e^{-i\omega t} \frac{d^2}{dx^2} \sin \frac{\pi x}{L} = \frac{A \hbar^2}{8mL^2} e^{-i\omega t} \sin \frac{\pi x}{L}.$$

Thus, the expectation value of the kinetic energy is

$$\begin{aligned} \langle K \rangle &= \int_0^L dx \left(A e^{+i\omega t} \sin \frac{\pi x}{L} \right) \left(\frac{A \hbar^2}{8mL^2} e^{-i\omega t} \sin \frac{\pi x}{L} \right) \\ &= \frac{A^2 \hbar^2}{8mL^2} \int_0^L dx \sin^2 \frac{\pi x}{L} = \frac{A^2 \hbar^2}{8mL^2} \frac{L}{2} = \frac{\hbar^2}{8mL^2}. \end{aligned}$$

Significance

The average position of a large number of particles in this state is $L/2$. The average momentum of these particles is zero because a given particle is equally likely to be moving right or left. However, the particle is not at rest because its average kinetic energy is not zero. Finally, the probability density is

$$|\psi|^2 = (2/L) \sin^2(\pi x/L).$$

This probability density is largest at location $L/2$ and is zero at $x = 0$ and at $x = L$. Note that these conclusions do not depend explicitly on time.

✓ CHECK YOUR UNDERSTANDING 7.3

For the particle in the above example, find the probability of locating it between positions 0 and $L/4$

Quantum mechanics makes many surprising predictions. However, in 1920, Niels Bohr (founder of the Niels Bohr Institute in Copenhagen, from which we get the term “Copenhagen interpretation”) asserted that the predictions of quantum mechanics and classical mechanics must agree for all macroscopic systems, such as orbiting planets, bouncing balls, rocking chairs, and springs. This **correspondence principle** is now generally accepted. It suggests the rules of classical mechanics are an approximation of the rules of quantum mechanics for systems with very large energies. Quantum mechanics describes both the microscopic and macroscopic world, but classical mechanics describes only the latter.

7.2 The Heisenberg Uncertainty Principle

Learning Objectives

By the end of this section, you will be able to:

- Describe the physical meaning of the position-momentum uncertainty relation
- Explain the origins of the uncertainty principle in quantum theory
- Describe the physical meaning of the energy-time uncertainty relation

Heisenberg’s uncertainty principle is a key principle in quantum mechanics. Very roughly, it states that if we know *everything* about where a particle is located (the uncertainty of position is small), we know *nothing* about its momentum (the uncertainty of momentum is large), and vice versa. Versions of the uncertainty principle also exist for other quantities as well, such as energy and time. We discuss the momentum-position and energy-time uncertainty principles separately.

Momentum and Position

To illustrate the momentum-position uncertainty principle, consider a free particle that moves along the x -direction. The particle moves with a constant velocity u and momentum $p = mu$. According to de Broglie's relations, $p = \hbar k$ and $E = \hbar \omega$. As discussed in the previous section, the wave function for this particle is given by

$$\psi_k(x, t) = A[\cos(\omega t - kx) - i \sin(\omega t - kx)] = Ae^{-i(\omega t - kx)} = Ae^{-i\omega t} e^{ikx} \quad 7.14$$

and the probability density $|\psi_k(x, t)|^2 = A^2$ is *uniform* and independent of time. The particle is equally likely to be found anywhere along the x -axis but has definite values of wavelength and wave number, and therefore momentum. The uncertainty of position is infinite (we are completely uncertain about position) and the uncertainty of the momentum is zero (we are completely certain about momentum). This account of a free particle is consistent with Heisenberg's uncertainty principle.

Similar statements can be made of localized particles. In quantum theory, a localized particle is modeled by a linear superposition of free-particle (or plane-wave) states called a **wave packet**. An example of a wave packet is shown in [Figure 7.9](#). A wave packet contains many wavelengths and therefore by de Broglie's relations many momenta—possible in quantum mechanics! This particle also has many values of position, although the particle is confined mostly to the interval Δx . The particle can be better localized (Δx can be decreased) if more plane-wave states of different wavelengths or momenta are added together in the right way (Δp is increased). According to Heisenberg, these uncertainties obey the following relation.

The Heisenberg Uncertainty Principle

The product of the uncertainty in position of a particle and the uncertainty in its momentum can never be less than one-half of the reduced Planck constant:

$$\Delta x \Delta p \geq \hbar/2. \quad 7.15$$

This relation expresses Heisenberg's uncertainty principle. It places limits on what we can know about a particle from simultaneous measurements of position and momentum. If Δx is large, Δp is small, and vice versa. [Equation 7.15](#) can be derived in a more advanced course in modern physics. Reflecting on this relation in his work *The Physical Principles of the Quantum Theory*, Heisenberg wrote “Any use of the words ‘position’ and ‘velocity’ with accuracy exceeding that given by [the relation] is just as meaningless as the use of words whose sense is not defined.”

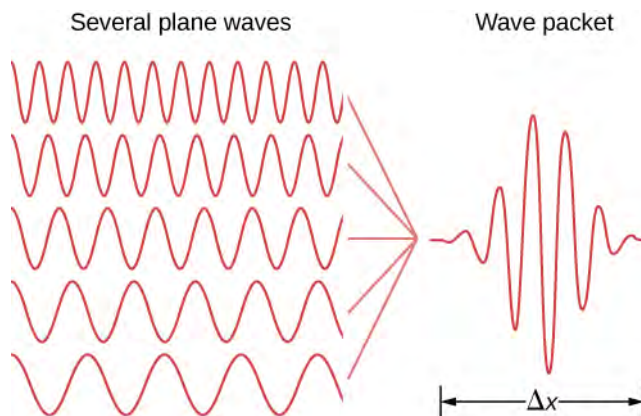


Figure 7.9 Adding together several plane waves of different wavelengths can produce a wave that is relatively localized.

Note that the uncertainty principle has nothing to do with the precision of an experimental apparatus. Even for perfect measuring devices, these uncertainties would remain because they originate in the wave-like nature of matter. The precise value of the product $\Delta x \Delta p$ depends on the specific form of the wave function. Interestingly, the Gaussian function (or bell-curve distribution) gives the minimum value of the uncertainty product: $\Delta x \Delta p = \hbar/2$.

EXAMPLE 7.5

The Uncertainty Principle Large and Small

Determine the minimum uncertainties in the positions of the following objects if their speeds are known with a precision of 1.0×10^{-3} m/s: (a) an electron and (b) a bowling ball of mass 6.0 kg.

Strategy

Given the uncertainty in speed $\Delta u = 1.0 \times 10^{-3}$ m/s, we have to first determine the uncertainty in momentum $\Delta p = m \Delta u$ and then invert [Equation 7.15](#) to find the uncertainty in position $\Delta x = \hbar/(2\Delta p)$.

Solution

a. For the electron:

$$\Delta p = m\Delta u = (9.1 \times 10^{-31} \text{ kg})(1.0 \times 10^{-3} \text{ m/s}) = 9.1 \times 10^{-34} \text{ kg} \cdot \text{m/s} ,$$

$$\Delta x = \frac{\hbar}{2 \Delta p} = 5.8 \text{ cm}.$$

b. For the bowling ball:

$$\Delta p = m\Delta u = (6.0 \text{ kg})(1.0 \times 10^{-3} \text{ m/s}) = 6.0 \times 10^{-3} \text{ kg} \cdot \text{m/s} ,$$

$$\Delta x = \frac{\hbar}{2 \Delta p} = 8.8 \times 10^{-33} \text{ m} .$$

Significance

Unlike the position uncertainty for the electron, the position uncertainty for the bowling ball is immeasurably small. Planck's constant is very small, so the limitations imposed by the uncertainty principle are not noticeable in macroscopic systems such as a bowling ball.

EXAMPLE 7.6

Uncertainty and the Hydrogen Atom

Estimate the ground-state energy of a hydrogen atom using Heisenberg's uncertainty principle. (*Hint:* According to early experiments, the size of a hydrogen atom is approximately 0.1 nm.)

Strategy

An electron bound to a hydrogen atom can be modeled by a particle bound to a one-dimensional box of length $L = 0.1$ nm. The ground-state wave function of this system is a half wave, like that given in [Example 7.1](#). This is the largest wavelength that can “fit” in the box, so the wave function corresponds to the lowest energy state. Note that this function is very similar in shape to a Gaussian (bell curve) function. We can take the average energy of a particle described by this function (E) as a good estimate of the ground state energy (E_0). This average energy of a particle is related to its average of the momentum squared, which is related to its momentum uncertainty.

Solution

To solve this problem, we must be specific about what is meant by “uncertainty of position” and “uncertainty

of momentum.” We identify the uncertainty of position (Δx) with the standard deviation of position (σ_x), and the uncertainty of momentum (Δp) with the standard deviation of momentum (σ_p). For the Gaussian function, the uncertainty product is

$$\sigma_x \sigma_p = \frac{\hbar}{2},$$

where

$$\sigma_x^2 = x^2 - \bar{x}^2 \quad \text{and} \quad \sigma_p^2 = p^2 - \bar{p}^2.$$

The particle is equally likely to be moving left as moving right, so $\bar{p} = 0$. Also, the uncertainty of position is comparable to the size of the box, so $\sigma_x = L$. The estimated ground state energy is therefore

$$E_0 = E_{\text{Gaussian}} = \frac{\bar{p}^2}{m} = \frac{\sigma_p^2}{2m} = \frac{1}{2m} \left(\frac{\hbar}{2\sigma_x} \right)^2 = \frac{1}{2m} \left(\frac{\hbar}{2L} \right)^2 = \frac{\hbar^2}{8mL^2}.$$

Multiplying numerator and denominator by c^2 gives

$$E_0 = \frac{(\hbar c)^2}{8(mc^2)L^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{8(0.511 \cdot 10^6 \text{ eV})(0.1 \text{ nm})^2} = 0.952 \text{ eV} \approx 1 \text{ eV}.$$

Significance

Based on early estimates of the size of a hydrogen atom and the uncertainty principle, the ground-state energy of a hydrogen atom is in the eV range. The ionization energy of an electron in the ground-state energy is approximately 10 eV, so this prediction is roughly confirmed. (Note: The product $\hbar c$ is often a useful value in performing calculations in quantum mechanics.)

Energy and Time

Another kind of uncertainty principle concerns uncertainties in simultaneous measurements of the energy of a quantum state and its lifetime,

$$\Delta E \Delta t \geq \frac{\hbar}{2}, \quad 7.16$$

where ΔE is the uncertainty in the energy measurement and Δt is the uncertainty in the lifetime measurement. The **energy-time uncertainty principle** does not result from a relation of the type expressed by Equation 7.15 for technical reasons beyond this discussion. Nevertheless, the general meaning of the energy-time principle is that a quantum state that exists for only a short time cannot have a definite energy. The reason is that the frequency of a state is inversely proportional to time and the frequency connects with the energy of the state, so to measure the energy with good precision, the state must be observed for many cycles.

To illustrate, consider the excited states of an atom. The finite lifetimes of these states can be deduced from the shapes of spectral lines observed in atomic emission spectra. Each time an excited state decays, the emitted energy is slightly different and, therefore, the emission line is characterized by a *distribution* of spectral frequencies (or wavelengths) of the emitted photons. As a result, all spectral lines are characterized by spectral widths. The average energy of the emitted photon corresponds to the theoretical energy of the excited state and gives the spectral location of the peak of the emission line. Short-lived states have broad spectral widths and long-lived states have narrow spectral widths.



EXAMPLE 7.7

Atomic Transitions

An atom typically exists in an excited state for about $\Delta t = 10^{-8}$ s. Estimate the uncertainty Δf in the frequency of emitted photons when an atom makes a transition from an excited state with the simultaneous emission of a photon with an average frequency of $f = 7.1 \times 10^{14}$ Hz. Is the emitted radiation monochromatic?

Strategy

We invert [Equation 7.16](#) to obtain the energy uncertainty $\Delta E \approx \hbar/2\Delta t$ and combine it with the photon energy $E = hf$ to obtain Δf . To estimate whether or not the emission is monochromatic, we evaluate $\Delta f/f$.

Solution

The spread in photon energies is $\Delta E = h\Delta f$. Therefore,

$$\Delta E \approx \frac{\hbar}{2\Delta t} \Rightarrow h\Delta f \approx \frac{\hbar}{2\Delta t} \Rightarrow \Delta f \approx \frac{1}{4\pi\Delta t} = \frac{1}{4\pi(10^{-8}\text{ s})} = 8.0 \times 10^6 \text{ Hz},$$

$$\frac{\Delta f}{f} = \frac{8.0 \times 10^6 \text{ Hz}}{7.1 \times 10^{14} \text{ Hz}} = 1.1 \times 10^{-8}.$$

Significance

Because the emitted photons have their frequencies within 1.1×10^{-6} percent of the average frequency, the emitted radiation can be considered monochromatic.



CHECK YOUR UNDERSTANDING 7.4

A sodium atom makes a transition from the first excited state to the ground state, emitting a 589.0-nm photon with energy 2.105 eV. If the lifetime of this excited state is 1.6×10^{-8} s, what is the uncertainty in energy of this excited state? What is the width of the corresponding spectral line?

7.3 The Schrödinger Equation

Learning Objectives

By the end of this section, you will be able to:

- Describe the role Schrödinger's equation plays in quantum mechanics
- Explain the difference between time-dependent and -independent Schrödinger's equations
- Interpret the solutions of Schrödinger's equation

In the preceding two sections, we described how to use a quantum mechanical wave function and discussed Heisenberg's uncertainty principle. In this section, we present a complete and formal theory of quantum mechanics that can be used to make predictions. In developing this theory, it is helpful to review the wave theory of light. For a light wave, the electric field $E(x, t)$ obeys the relation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}, \quad 7.17$$

where c is the speed of light and the symbol ∂ represents a *partial derivative*. (Recall from [Oscillations](#) that a partial derivative is closely related to an ordinary derivative, but involves functions of more than one variable. When taking the partial derivative of a function by a certain variable, all other variables are held constant.) A light wave consists of a very large number of photons, so the quantity $|E(x, t)|^2$ can be interpreted as a probability density of finding a single photon at a particular point in space (for example, on a viewing screen).

There are many solutions to this equation. One solution of particular importance is

$$E(x, t) = A \sin(kx - \omega t), \quad 7.18$$

where A is the amplitude of the electric field, k is the wave number, and ω is the angular frequency. Combining this equation with [Equation 7.17](#) gives

$$k^2 = \frac{\omega^2}{c^2}. \quad 7.19$$

According to de Broglie's equations, we have $p = \hbar k$ and $E = \hbar \omega$. Substituting these equations in [Equation 7.19](#) gives

$$p = \frac{E}{c}, \quad 7.20$$

or

$$E = pc. \quad 7.21$$

Therefore, according to Einstein's general energy-momentum equation ([Equation 5.11](#)), [Equation 7.17](#) describes a particle with a zero rest mass. This is consistent with our knowledge of a photon.

This process can be reversed. We can begin with the energy-momentum equation of a particle and then ask what wave equation corresponds to it. The energy-momentum equation of a nonrelativistic particle in one dimension is

$$E = \frac{p^2}{2m} + U(x, t), \quad 7.22$$

where p is the momentum, m is the mass, and U is the potential energy of the particle. The wave equation that goes with it turns out to be a key equation in quantum mechanics, called **Schrödinger's time-dependent equation**.

The Schrödinger Time-Dependent Equation

The equation describing the energy and momentum of a wave function is known as the Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + U(x, t) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}. \quad 7.23$$

As described in [Potential Energy and Conservation of Energy](#), the force on the particle described by this equation is given by

$$F = -\frac{\partial U(x, t)}{\partial x}. \quad 7.24$$

This equation plays a role in quantum mechanics similar to Newton's second law in classical mechanics. Once the potential energy of a particle is specified—or, equivalently, once the force on the particle is specified—we can solve this differential equation for the wave function. The solution to Newton's second law equation (also a differential equation) in one dimension is a function $x(t)$ that specifies where an object is at any time t . The solution to Schrödinger's time-dependent equation provides a tool—the wave function—that can be used to determine where the particle is *likely* to be. This equation can be also written in two or three dimensions. Solving Schrödinger's time-dependent equation often requires the aid of a computer.

Consider the special case of a free particle. A free particle experiences no force ($F = 0$). Based on [Equation 7.24](#), this requires only that

$$U(x, t) = U_0 = \text{constant}. \quad 7.25$$

For simplicity, we set $U_0 = 0$. Schrödinger's equation then reduces to

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} = i\hbar \frac{\partial \Psi(x, t)}{\partial t}. \quad 7.26$$

A valid solution to this equation is

$$\Psi(x, t) = Ae^{i(kx - \omega t)}. \quad 7.27$$

Not surprisingly, this solution contains an imaginary number ($i = \sqrt{-1}$) because the differential equation itself contains an imaginary number. As stressed before, however, quantum-mechanical predictions depend only on $|\Psi(x, t)|^2$, which yields completely real values. Notice that the real plane-wave solutions, $\Psi(x, t) = A \sin(kx - \omega t)$ and $\Psi(x, t) = A \cos(kx - \omega t)$, do not obey Schrödinger's equation. The temptation to think that a wave function can be seen, touched, and felt in nature is eliminated by the appearance of an imaginary number. In Schrödinger's theory of quantum mechanics, the wave function is merely a tool for calculating things.

If the potential energy function (U) does not depend on time, it is possible to show that

$$\Psi(x, t) = \psi(x) e^{-i\omega t} \quad 7.28$$

satisfies Schrödinger's time-dependent equation, where $\psi(x)$ is a *time-independent* function and $e^{-i\omega t}$ is a *space-independent* function. In other words, the wave function is *separable* into two parts: a space-only part and a time-only part. The factor $e^{-i\omega t}$ is sometimes referred to as a **time-modulation factor** since it modifies the space-only function. According to de Broglie, the energy of a matter wave is given by $E = \hbar\omega$, where E is its total energy. Thus, the above equation can also be written as

$$\Psi(x, t) = \psi(x) e^{-iEt/\hbar}. \quad 7.29$$

Any linear combination of such states (mixed state of energy or momentum) is also valid solution to this equation. Such states can, for example, describe a localized particle (see [Figure 7.9](#))

✓ CHECK YOUR UNDERSTANDING 7.5

A particle with mass m is moving along the x -axis in a potential given by the potential energy function $U(x) = 0.5m\omega^2 x^2$. Compute the product $\Psi(x, t)^* U(x) \Psi(x, t)$. Express your answer in terms of the time-independent wave function, $\psi(x)$.

Combining [Equation 7.23](#) and [Equation 7.28](#), Schrödinger's time-dependent equation reduces to

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + U(x) \psi(x) = E\psi(x), \quad 7.30$$

where E is the total energy of the particle (a real number). This equation is called **Schrödinger's time-independent equation**. Notice that we use “big psi” (Ψ) for the time-dependent wave function and “little psi” (ψ) for the time-independent wave function. The wave-function solution to this equation must be multiplied by the time-modulation factor to obtain the time-dependent wave function.

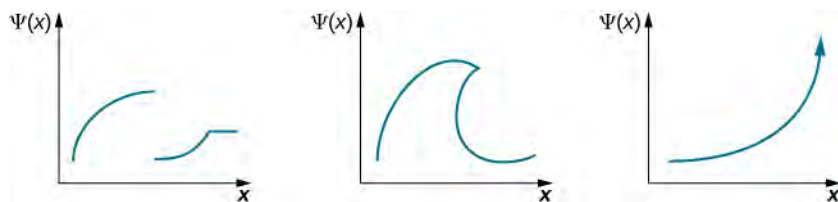
In the next sections, we solve Schrödinger's time-independent equation for three cases: a quantum particle in a box, a simple harmonic oscillator, and a quantum barrier. These cases provide important lessons that can be used to solve more complicated systems. The time-independent wave function $\psi(x)$ solutions must satisfy three conditions:

- $\psi(x)$ must be a continuous function.
- The first derivative of $\psi(x)$ with respect to space, $d\psi(x)/dx$, must be continuous, unless $V(x) = \infty$.
- $\psi(x)$ must not diverge (“blow up”) at $x = \pm\infty$.

The first condition avoids sudden jumps or gaps in the wave function. The second condition requires the wave function to be smooth at all points, except in special cases. (In a more advanced course on quantum mechanics, for example, potential spikes of infinite depth and height are used to model solids). The third condition requires the wave function be normalizable. This third condition follows from Born's interpretation of quantum mechanics. It ensures that $|\psi(x)|^2$ is a finite number so we can use it to calculate probabilities.

✓ CHECK YOUR UNDERSTANDING 7.6

Which of the following wave functions is a valid wave-function solution for Schrödinger's equation?



7.4 The Quantum Particle in a Box

Learning Objectives

By the end of this section, you will be able to:

- Describe how to set up a boundary-value problem for the stationary Schrödinger equation
- Explain why the energy of a quantum particle in a box is quantized
- Describe the physical meaning of stationary solutions to Schrödinger's equation and the connection of these solutions with time-dependent quantum states
- Explain the physical meaning of Bohr's correspondence principle

In this section, we apply Schrödinger's equation to a particle bound to a one-dimensional box. This special case provides lessons for understanding quantum mechanics in more complex systems. The energy of the particle is quantized as a consequence of a standing wave condition inside the box.

Consider a particle of mass m that is allowed to move only along the x -direction and its motion is confined to the region between hard and rigid walls located at $x = 0$ and at $x = L$ (Figure 7.10). Between the walls, the particle moves freely. This physical situation is called the **infinite square well**, described by the potential energy function

$$U(x) = \begin{cases} 0, & 0 \leq x \leq L, \\ \infty, & \text{otherwise.} \end{cases} \quad 7.31$$

Combining this equation with Schrödinger's time-independent wave equation gives

$$\frac{-\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x), \text{ for } 0 \leq x \leq L \quad 7.32$$

where E is the total energy of the particle. What types of solutions do we expect? The energy of the particle is a positive number, so if the value of the wave function is positive (right side of the equation), the curvature of the wave function is negative, or concave down (left side of the equation). Similarly, if the value of the wave function is negative (right side of the equation), the curvature of the wave function is positive or concave up (left side of equation). This condition is met by an oscillating wave function, such as a sine or cosine wave. Since these waves are confined to the box, we envision standing waves with fixed endpoints at $x = 0$ and $x = L$.

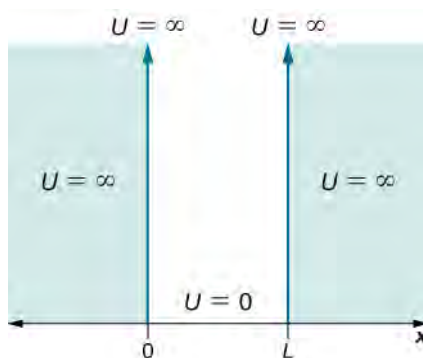


Figure 7.10 The potential energy function that confines the particle in a one-dimensional box.

Solutions $\psi(x)$ to this equation have a probabilistic interpretation. In particular, the square $|\psi(x)|^2$ represents the probability density of finding the particle at a particular location x . This function must be integrated to determine the probability of finding the particle in some interval of space. We are therefore looking for a normalizable solution that satisfies the following normalization condition:

$$\int_0^L dx |\psi(x)|^2 = 1. \quad 7.33$$

The walls are rigid and impenetrable, which means that the particle is never found beyond the wall. Mathematically, this means that the solution must vanish at the walls:

$$\psi(0) = \psi(L) = 0. \quad 7.34$$

We expect oscillating solutions, so the most general solution to this equation is

$$\psi_k(x) = A_k \cos kx + B_k \sin kx \quad 7.35$$

where k is the wave number, and A_k and B_k are constants. Applying the boundary condition expressed by [Equation 7.34](#) gives

$$\psi_k(0) = A_k \cos(k \cdot 0) + B_k \sin(k \cdot 0) = A_k = 0. \quad 7.36$$

Because we have $A_k = 0$, the solution must be

$$\psi_k(x) = B_k \sin kx. \quad 7.37$$

If B_k is zero, $\psi_k(x) = 0$ for all values of x and the normalization condition, [Equation 7.33](#), cannot be satisfied. Assuming $B_k \neq 0$, [Equation 7.34](#) for $x = L$ then gives

$$0 = B_k \sin(kL) \Rightarrow \sin(kL) = 0 \Rightarrow kL = n\pi, n = 1, 2, 3, \dots \quad 7.38$$

We discard the $n = 0$ solution because $\psi(x)$ for this quantum number would be zero everywhere—an un-normalizable and therefore unphysical solution. Substituting [Equation 7.37](#) into [Equation 7.32](#) gives

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} (B_k \sin(kx)) = E (B_k \sin(kx)). \quad 7.39$$

Computing these derivatives leads to

$$E = E_k = \frac{\hbar^2 k^2}{2m}. \quad 7.40$$

According to de Broglie, $p = \hbar k$, so this expression implies that the total energy is equal to the kinetic energy, consistent with our assumption that the “particle moves freely.” Combining the results of [Equation 7.38](#) and [Equation 7.40](#) gives

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2}, \quad n = 1, 2, 3, \dots \quad 7.41$$

Strange! A particle bound to a one-dimensional box can only have certain discrete (quantized) values of energy. Further, the particle cannot have a zero kinetic energy—it is impossible for a particle bound to a box to be “at rest.”

To evaluate the allowed wave functions that correspond to these energies, we must find the normalization constant B_n . We impose the normalization condition [Equation 7.33](#) on the wave function

$$\psi_n(x) = B_n \sin n\pi x/L \quad 7.42$$

$$1 = \int_0^L dx |\psi_n(x)|^2 = \int_0^L dx B_n^2 \sin^2 \frac{n\pi}{L} x = B_n^2 \int_0^L dx \sin^2 \frac{n\pi}{L} x = B_n^2 \frac{L}{2} \Rightarrow B_n = \sqrt{\frac{2}{L}}.$$

Hence, the wave functions that correspond to the energy values given in [Equation 7.41](#) are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots \quad 7.43$$

For the lowest energy state or **ground state energy**, we have

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2}, \quad \psi_1(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{\pi x}{L} \right). \quad 7.44$$

All other energy states can be expressed as

$$E_n = n^2 E_1, \quad \psi_n(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{n\pi x}{L} \right). \quad 7.45$$

The index n is called the **energy quantum number** or **principal quantum number**. The state for $n = 2$ is the first excited state, the state for $n = 3$ is the second excited state, and so on. The first three quantum states (for $n = 1, 2$, and 3) of a particle in a box are shown in [Figure 7.11](#).

The wave functions in [Equation 7.45](#) are sometimes referred to as the “states of definite energy.” Particles in these states are said to occupy **energy levels**, which are represented by the horizontal lines in [Figure 7.11](#). Energy levels are analogous to rungs of a ladder that the particle can “climb” as it gains or loses energy.

The wave functions in [Equation 7.45](#) are also called **stationary states** and **standing wave states**. These functions are “stationary,” because their probability density functions, $|\Psi(x, t)|^2$, do not vary in time, and “standing waves” because their real and imaginary parts oscillate up and down like a standing wave—like a rope waving between two children on a playground. Stationary states are states of definite energy [[Equation 7.45](#)], but linear combinations of these states, such as $\psi(x) = a\psi_1 + b\psi_2$ (also solutions to Schrödinger’s equation) are states of mixed energy.

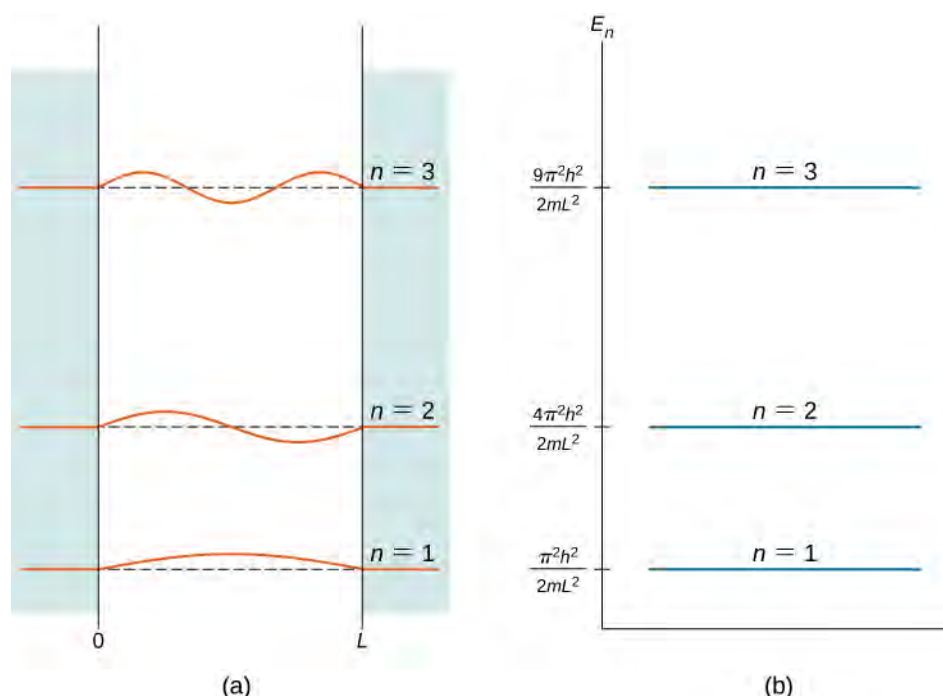


Figure 7.11 The first three quantum states of a quantum particle in a box for principal quantum numbers $n = 1, 2$, and 3 : (a) standing wave solutions and (b) allowed energy states.

Energy quantization is a consequence of the boundary conditions. If the particle is not confined to a box but wanders freely, the allowed energies are continuous. However, in this case, only certain energies ($E_1, 4E_1, 9E_1, \dots$) are allowed. The energy difference between adjacent energy levels is given by

$$\Delta E_{n+1,n} = E_{n+1} - E_n = (n+1)^2 E_1 - n^2 E_1 = (2n+1)E_1. \quad 7.46$$

Conservation of energy demands that if the energy of the system changes, the energy difference is carried in some other form of energy. For the special case of a charged particle confined to a small volume (for example, in an atom), energy changes are often carried away by photons. The frequencies of the emitted photons give us information about the energy differences (spacings) of the system and the volume of containment—the size of the “box” [see Equation 7.44].



EXAMPLE 7.8

A Simple Model of the Nucleus

Suppose a proton is confined to a box of width $L = 1.00 \times 10^{-14} \text{ m}$ (a typical nuclear radius). What are the energies of the ground and the first excited states? If the proton makes a transition from the first excited state to the ground state, what are the energy and the frequency of the emitted photon?

Strategy

If we assume that the proton confined in the nucleus can be modeled as a quantum particle in a box, all we need to do is to use Equation 7.41 to find its energies E_1 and E_2 . The mass of a proton is $m = 1.67 \times 10^{-27} \text{ kg}$. The emitted photon carries away the energy difference $\Delta E = E_2 - E_1$. We can use the relation $E_f = hf$ to find its frequency f .

Solution

The ground state:

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.05 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(1.67 \times 10^{-27} \text{ kg})(1.00 \times 10^{-14} \text{ m})^2} = 3.28 \times 10^{-13} \text{ J} = 2.05 \text{ MeV}.$$

The first excited state: $E_2 = 2^2 E_1 = 4(2.05 \text{ MeV}) = 8.20 \text{ MeV}$.

The energy of the emitted photon is $E_f = \Delta E = E_2 - E_1 = 8.20 \text{ MeV} - 2.05 \text{ MeV} = 6.15 \text{ MeV}$.

The frequency of the emitted photon is

$$f = \frac{E_f}{h} = \frac{6.15 \text{ MeV}}{4.14 \times 10^{-21} \text{ MeV} \cdot \text{s}} = 1.49 \times 10^{21} \text{ Hz}.$$

Significance

This is the typical frequency of a gamma ray emitted by a nucleus. The energy of this photon is about 10 million times greater than that of a visible light photon.

The expectation value of the position for a particle in a box is given by

$$\langle x \rangle = \int_0^L dx \psi_n^*(x) x \psi_n(x) = \int_0^L dx x |\psi_n^*(x)|^2 = \int_0^L dx x \frac{2}{L} \sin^2 \frac{n\pi x}{L} = \frac{L}{2}. \quad 7.47$$

We can also find the expectation value of the momentum or average momentum of a large number of particles in a given state:

$$\begin{aligned} \langle p \rangle &= \int_0^L dx \psi_n^*(x) \left[-i\hbar \frac{d}{dx} \psi_n(x) \right] \\ &= -i\hbar \int_0^L dx \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \left[\frac{d}{dx} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \right] = -i\frac{2\hbar}{L} \int_0^L dx \sin \frac{n\pi x}{L} \left[\frac{n\pi}{L} \cos \frac{n\pi x}{L} \right] \\ &= -i\frac{2n\pi\hbar}{L^2} \int_0^L dx \frac{1}{2} \sin \frac{2n\pi x}{L} = -i\frac{n\pi\hbar}{L^2} \frac{L}{2n\pi} \int_0^{2n\pi} d\varphi \sin \varphi = -i\frac{\hbar}{2L} \cdot 0 = 0. \end{aligned} \quad 7.48$$

Thus, for a particle in a state of definite energy, the average position is in the middle of the box and the average momentum of the particle is zero—as it would also be for a classical particle. Note that while the minimum energy of a classical particle can be zero (the particle can be at rest in the middle of the box), the minimum energy of a quantum particle is nonzero and given by [Equation 7.44](#). The average particle energy in the n th quantum state—its expectation value of energy—is

$$E_n = \langle E \rangle = n^2 \frac{\pi^2 \hbar^2}{2m}. \quad 7.49$$

The result is not surprising because the standing wave state is a state of definite energy. Any energy measurement of this system must return a value equal to one of these allowed energies.

Our analysis of the quantum particle in a box would not be complete without discussing Bohr's correspondence principle. This principle states that for large quantum numbers, the laws of quantum physics must give identical results as the laws of classical physics. To illustrate how this principle works for a quantum particle in a box, we plot the probability density distribution

$$|\psi_n(x)|^2 = \frac{2}{L} \sin^2(n\pi x/L) \quad 7.50$$

for finding the particle around location x between the walls when the particle is in quantum state ψ_n . [Figure 7.12](#) shows these probability distributions for the ground state, for the first excited state, and for a highly excited state that corresponds to a large quantum number. We see from these plots that when a quantum particle is in the ground state, it is most likely to be found around the middle of the box, where the probability

distribution has the largest value. This is not so when the particle is in the first excited state because now the probability distribution has the zero value in the middle of the box, so there is no chance of finding the particle there. When a quantum particle is in the first excited state, the probability distribution has two maxima, and the best chance of finding the particle is at positions close to the locations of these maxima. This quantum picture is unlike the classical picture.

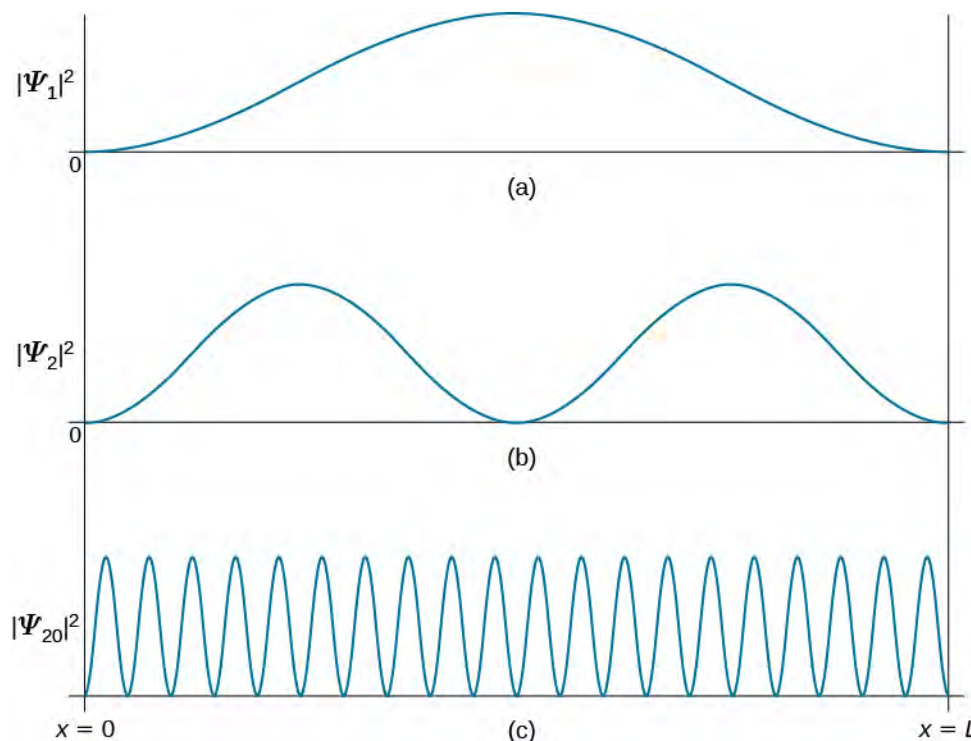


Figure 7.12 The probability density distribution $|\psi_n(x)|^2$ for a quantum particle in a box for: (a) the ground state, $n = 1$; (b) the first excited state, $n = 2$; and, (c) the nineteenth excited state, $n = 20$.

The probability density of finding a classical particle between x and $x + \Delta x$ depends on how much time Δt the particle spends in this region. Assuming that its speed u is constant, this time is $\Delta t = \Delta x/u$, which is also constant for any location between the walls. Therefore, the probability density of finding the classical particle at x is uniform throughout the box, and there is no preferable location for finding a classical particle. This classical picture is matched in the limit of large quantum numbers. For example, when a quantum particle is in a highly excited state, shown in [Figure 7.12](#), the probability density is characterized by rapid fluctuations and then the probability of finding the quantum particle in the interval Δx does not depend on where this interval is located between the walls.



EXAMPLE 7.9

A Classical Particle in a Box

A small 0.40-kg cart is moving back and forth along an air track between two bumpers located 2.0 m apart. We assume no friction; collisions with the bumpers are perfectly elastic so that between the bumpers, the cart maintains a constant speed of 0.50 m/s. Treating the cart as a quantum particle, estimate the value of the principal quantum number that corresponds to its classical energy.

Strategy

We find the kinetic energy K of the cart and its ground state energy E_1 as though it were a quantum particle. The energy of the cart is completely kinetic, so $K = n^2 E_1$ ([Equation 7.45](#)). Solving for n gives $n = (K/E_1)^{1/2}$.

Solution

The kinetic energy of the cart is

$$K = \frac{1}{2}mu^2 = \frac{1}{2}(0.40 \text{ kg})(0.50 \text{ m/s})^2 = 0.050 \text{ J}.$$

The ground state of the cart, treated as a quantum particle, is

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.05 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2(0.40 \text{ kg})(2.0 \text{ m})^2} = 1.700 \times 10^{-68} \text{ J}.$$

Therefore, $n = (K/E_1)^{1/2} = (0.050/1.700 \times 10^{-68})^{1/2} = 1.2 \times 10^{33}$.

Significance

We see from this example that the energy of a classical system is characterized by a very large quantum number. Bohr's correspondence principle concerns this kind of situation. We can apply the formalism of quantum mechanics to any kind of system, quantum or classical, and the results are correct in each case. In the limit of high quantum numbers, there is no advantage in using quantum formalism because we can obtain the same results with the less complicated formalism of classical mechanics. However, we cannot apply classical formalism to a quantum system in a low-number energy state.

✓ CHECK YOUR UNDERSTANDING 7.7

(a) Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. What is the probability of finding a quantum particle in its ground state somewhere between $x = 0$ and $x = L/4$? (b) Repeat question (a) for a classical particle.

Having found the stationary states $\psi_n(x)$ and the energies E_n by solving the time-independent Schrödinger equation [Equation 7.32](#), we use [Equation 7.28](#) to write wave functions $\Psi_n(x, t)$ that are solutions of the time-dependent Schrödinger's equation given by [Equation 7.23](#). For a particle in a box this gives

$$\Psi_n(x, t) = e^{-i\omega_n t} \psi_n(x) = \sqrt{\frac{2}{L}} e^{-iE_n t/\hbar} \sin \frac{n\pi x}{L}, n = 1, 2, 3, \dots \quad 7.51$$

where the energies are given by [Equation 7.41](#).

The quantum particle in a box model has practical applications in a relatively newly emerged field of optoelectronics, which deals with devices that convert electrical signals into optical signals. This model also deals with nanoscale physical phenomena, such as a nanoparticle trapped in a low electric potential bounded by high-potential barriers.

7.5 The Quantum Harmonic Oscillator

Learning Objectives

By the end of this section, you will be able to:

- Describe the model of the quantum harmonic oscillator
- Identify differences between the classical and quantum models of the harmonic oscillator
- Explain physical situations where the classical and the quantum models coincide

Oscillations are found throughout nature, in such things as electromagnetic waves, vibrating molecules, and the gentle back-and-forth sway of a tree branch. In previous chapters, we used Newtonian mechanics to study macroscopic oscillations, such as a block on a spring and a simple pendulum. In this chapter, we begin to study oscillating systems using quantum mechanics. We begin with a review of the classic harmonic oscillator.

The Classic Harmonic Oscillator

A simple harmonic oscillator is a particle or system that undergoes harmonic motion about an equilibrium position, such as an object with mass vibrating on a spring. In this section, we consider oscillations in one-dimension only. Suppose a mass moves back-and-forth along the

CHAPTER REVIEW

Key Terms

anti-symmetric function odd function

Born interpretation states that the square of a wave function is the probability density

complex function function containing both real and imaginary parts

Copenhagen interpretation states that when an observer *is not* looking or when a measurement is not being made, the particle has many values of measurable quantities, such as position

correspondence principle in the limit of large energies, the predictions of quantum mechanics agree with the predictions of classical mechanics

energy levels states of definite energy, often represented by horizontal lines in an energy “ladder” diagram

energy quantum number index that labels the allowed energy states

energy-time uncertainty principle energy-time relation for uncertainties in the simultaneous measurements of the energy of a quantum state and of its lifetime

even function in one dimension, a function symmetric with the origin of the coordinate system

expectation value average value of the physical quantity assuming a large number of particles with the same wave function

field emission electron emission from conductor surfaces when a strong external electric field is applied in normal direction to conductor’s surface

ground state energy lowest energy state in the energy spectrum

Heisenberg’s uncertainty principle places limits on what can be known from a simultaneous measurements of position and momentum; states that if the uncertainty on position is small then the uncertainty on momentum is large, and vice versa

infinite square well potential function that is zero in a fixed range and infinitely beyond this range

momentum operator operator that corresponds to the momentum of a particle

nanotechnology technology that is based on manipulation of nanostructures such as molecules or individual atoms to produce nano-devices such as integrated circuits

normalization condition requires that the probability density integrated over the entire physical space results in the number one

odd function in one dimension, a function antisymmetric with the origin of the coordinate system

position operator operator that corresponds to the position of a particle

potential barrier potential function that rises and falls with increasing values of position

principal quantum number energy quantum number

probability density square of the particle’s wave function

quantum dot small region of a semiconductor nanocrystal embedded in another semiconductor nanocrystal, acting as a potential well for electrons

quantum tunneling phenomenon where particles penetrate through a potential energy barrier with a height greater than the total energy of the particles

resonant tunneling tunneling of electrons through a finite-height potential well that occurs only when electron energies match an energy level in the well, occurs in quantum dots

resonant-tunneling diode quantum dot with an applied voltage bias across it

scanning tunneling microscope (STM) device that utilizes quantum-tunneling phenomenon at metallic surfaces to obtain images of nanoscale structures

Schrödinger’s time-dependent equation equation in space and time that allows us to determine wave functions of a quantum particle

Schrödinger’s time-independent equation equation in space that allows us to determine wave functions of a quantum particle; this wave function must be multiplied by a time-modulation factor to obtain the time-dependent wave function

standing wave state stationary state for which the real and imaginary parts of $\Psi(x, t)$ oscillate up and down like a standing wave (often modeled with sine and cosine functions)

state reduction hypothetical process in which an observed or detected particle “jumps into” a definite state, often described in terms of the collapse of the particle’s wave function

stationary state state for which the probability density function, $|\Psi(x, t)|^2$, does not vary in time

time-modulation factor factor $e^{-i\omega t}$ that multiplies the time-independent wave function

when the potential energy of the particle is time independent

transmission probability also called tunneling probability, the probability that a particle will tunnel through a potential barrier

tunnel diode electron tunneling-junction between two different semiconductors

tunneling probability also called transmission probability, the probability that a particle will

tunnel through a potential barrier

wave function function that represents the quantum state of a particle (quantum system)

wave function collapse equivalent to state reduction

wave packet superposition of many plane matter waves that can be used to represent a localized particle

Key Equations

Normalization condition in one dimension

$$P(x = -\infty, +\infty) = \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1$$

Probability of finding a particle in a narrow interval of position in one dimension ($x, x + dx$)

$$P(x, x + dx) = \Psi^*(x, t) \Psi(x, t) dx$$

Expectation value of position in one dimension

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t) x \Psi(x, t) dx$$

Heisenberg's position-momentum uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Heisenberg's energy-time uncertainty principle

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Schrödinger's time-dependent equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + U(x, t) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

General form of the wave function for a time-independent potential in one dimension

$$\Psi(x, t) = \psi(x) e^{-i\omega t}$$

Schrödinger's time-independent equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + U(x) \psi(x) = E \psi(x)$$

Schrödinger's equation (free particle)

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = E \psi(x)$$

Allowed energies (particle in box of length L)

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2}, n = 1, 2, 3, \dots$$

Stationary states (particle in a box of length L)

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, n = 1, 2, 3, \dots$$

Potential-energy function of a harmonic oscillator

$$U(x) = \frac{1}{2}m\omega^2 x^2$$

Schrödinger equation (harmonic oscillator)

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x)$$

The energy spectrum

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega, n = 0, 1, 2, 3, \dots$$

The energy wave functions

$$\psi_n(x) = N_n e^{-\beta^2 x^2/2} H_n(\beta x), n = 0, 1, 2, 3, \dots$$

Potential barrier

$$U(x) = \begin{cases} 0, & \text{when } x < 0 \\ U_0, & \text{when } 0 \leq x \leq L \\ 0, & \text{when } x > L \end{cases}$$

Definition of the transmission coefficient

$$T(L, E) = \frac{|\psi_{\text{tra}}(x)|^2}{|\psi_{\text{in}}(x)|^2}$$

A parameter in the transmission coefficient

$$\beta^2 = \frac{2m}{\hbar^2}(U_0 - E)$$

Transmission coefficient, exact

$$T(L, E) = \frac{1}{\cosh^2 \beta L + (\gamma/2)^2 \sinh^2 \beta L}$$

Transmission coefficient, approximate

$$T(L, E) = 16 \frac{E}{U_0} \left(1 - \frac{E}{U_0}\right) e^{-2\beta L}$$

Summary

7.1 Wave Functions

- In quantum mechanics, the state of a physical system is represented by a wave function.
- In Born's interpretation, the square of the particle's wave function represents the probability density of finding the particle around a specific location in space.
- Wave functions must first be normalized before using them to make predictions.
- The expectation value is the average value of a quantity that requires a wave function and an integration.

7.2 The Heisenberg Uncertainty Principle

- The Heisenberg uncertainty principle states that it is impossible to simultaneously measure the x-components of position and of momentum of a particle with an arbitrarily high precision. The product of experimental uncertainties is always larger than or equal to $\hbar/2$.
- The limitations of this principle have nothing to do with the quality of the experimental

apparatus but originate in the wave-like nature of matter.

- The energy-time uncertainty principle expresses the experimental observation that a quantum state that exists only for a short time cannot have a definite energy.

7.3 The Schrödinger Equation

- The Schrödinger equation is the fundamental equation of wave quantum mechanics. It allows us to make predictions about wave functions.
- When a particle moves in a time-independent potential, a solution of the time-dependent Schrödinger equation is a product of a time-independent wave function and a time-modulation factor.
- The Schrödinger equation can be applied to many physical situations.

7.4 The Quantum Particle in a Box

- Energy states of a quantum particle in a box are found by solving the time-independent Schrödinger equation.

- To solve the time-independent Schrödinger equation for a particle in a box and find the stationary states and allowed energies, we require that the wave function terminate at the box wall.
- Energy states of a particle in a box are quantized and indexed by principal quantum number.
- The quantum picture differs significantly from the classical picture when a particle is in a low-energy state of a low quantum number.
- In the limit of high quantum numbers, when the quantum particle is in a highly excited state, the quantum description of a particle in a box coincides with the classical description, in the spirit of Bohr's correspondence principle.

7.5 The Quantum Harmonic Oscillator

- The quantum harmonic oscillator is a model built in analogy with the model of a classical harmonic oscillator. It models the behavior of many physical systems, such as molecular vibrations or wave packets in quantum optics.
- The allowed energies of a quantum oscillator are discrete and evenly spaced. The energy spacing is equal to Planck's energy quantum.
- The ground state energy is larger than zero. This means that, unlike a classical oscillator, a quantum oscillator is never at rest, even at the bottom of a potential well, and undergoes quantum fluctuations.
- The stationary states (states of definite energy) have nonzero values also in regions beyond classical turning points. When in the ground state, a quantum oscillator is most likely to be found around the position of the minimum of the potential well, which is the least-likely

position for a classical oscillator.

- For high quantum numbers, the motion of a quantum oscillator becomes more similar to the motion of a classical oscillator, in accordance with Bohr's correspondence principle.

7.6 The Quantum Tunneling of Particles through Potential Barriers

- A quantum particle that is incident on a potential barrier of a finite width and height may cross the barrier and appear on its other side. This phenomenon is called 'quantum tunneling.' It does not have a classical analog.
- To find the probability of quantum tunneling, we assume the energy of an incident particle and solve the stationary Schrödinger equation to find wave functions inside and outside the barrier. The tunneling probability is a ratio of squared amplitudes of the wave past the barrier to the incident wave.
- The tunneling probability depends on the energy of the incident particle relative to the height of the barrier and on the width of the barrier. It is strongly affected by the width of the barrier in a nonlinear, exponential way so that a small change in the barrier width causes a disproportionately large change in the transmission probability.
- Quantum-tunneling phenomena govern radioactive nuclear decays. They are utilized in many modern technologies such as STM and nano-electronics. STM allows us to see individual atoms on metal surfaces. Electron-tunneling devices have revolutionized electronics and allow us to build fast electronic devices of miniature sizes.

Conceptual Questions

7.1 Wave Functions

1. What is the physical unit of a wave function, $\Psi(x, t)$? What is the physical unit of the square of this wave function?
2. Can the magnitude of a wave function ($\Psi^*(x, t) \Psi(x, t)$) be a negative number? Explain.
3. What kind of physical quantity does a wave function of an electron represent?
4. What is the physical meaning of a wave function of a particle?
5. What is the meaning of the expression "expectation value?" Explain.

7.2 The Heisenberg Uncertainty Principle

6. If the formalism of quantum mechanics is 'more exact' than that of classical mechanics, why don't we use quantum mechanics to describe the motion of a leaping frog? Explain.
7. Can the de Broglie wavelength of a particle be known precisely? Can the position of a particle be known precisely?
8. Can we measure the energy of a free localized particle with complete precision?
9. Can we measure both the position and momentum of a particle with complete precision?

7.3 The Schrödinger Equation

- What is the difference between a wave function $\psi(x, y, z)$ and a wave function $\Psi(x, y, z, t)$ for the same particle?
- If a quantum particle is in a stationary state, does it mean that it does not move?
- Explain the difference between time-dependent and -independent Schrödinger's equations.
- Suppose a wave function is discontinuous at some point. Can this function represent a quantum state of some physical particle? Why? Why not?

7.4 The Quantum Particle in a Box

- Using the quantum particle in a box model, describe how the possible energies of the particle are related to the size of the box.
- Is it possible that when we measure the energy of a quantum particle in a box, the measurement may return a smaller value than the ground state energy? What is the highest value of the energy that we can measure for this particle?
- For a quantum particle in a box, the first excited state (Ψ_2) has zero value at the midpoint position in the box, so that the probability density of finding a particle at this point is exactly zero. Explain what is wrong with the following reasoning: "If the probability of finding a quantum particle at the midpoint is zero, the particle is never at this point, right? How does it come then that the particle can cross this point on its way from the left side to the right side of the box?"

Problems

7.1 Wave Functions

- Compute $|\Psi(x, t)|^2$ for the function $\Psi(x, t) = \psi(x) \sin \omega t$, where ω is a real constant.
- Given the complex-valued function $f(x, y) = (x - iy)/(x + iy)$, calculate $|f(x, y)|^2$.
- Which one of the following functions, and why, qualifies to be a wave function of a particle that can move along the entire real axis? (a)

7.5 The Quantum Harmonic Oscillator

- Is it possible to measure energy of $0.75\hbar\omega$ for a quantum harmonic oscillator? Why? Why not? Explain.
- Explain the connection between Planck's hypothesis of energy quanta and the energies of the quantum harmonic oscillator.
- If a classical harmonic oscillator can be at rest, why can the quantum harmonic oscillator never be at rest? Does this violate Bohr's correspondence principle?
- Use an example of a quantum particle in a box or a quantum oscillator to explain the physical meaning of Bohr's correspondence principle.
- Can we simultaneously measure position and energy of a quantum oscillator? Why? Why not?

7.6 The Quantum Tunneling of Particles through Potential Barriers

- When an electron and a proton of the same kinetic energy encounter a potential barrier of the same height and width, which one of them will tunnel through the barrier more easily? Why?
- What decreases the tunneling probability most: doubling the barrier width or halving the kinetic energy of the incident particle?
- Explain the difference between a box-potential and a potential of a quantum dot.
- Can a quantum particle 'escape' from an infinite potential well like that in a box? Why? Why not?
- A tunnel diode and a resonant-tunneling diode both utilize the same physics principle of quantum tunneling. In what important way are they different?

$$\psi(x) = Ae^{-x^2};$$

$$(b) \psi(x) = Ae^{-x}; (c) \psi(x) = A \tan x;$$

$$(d) \psi(x) = A(\sin x)/x; (e) \psi(x) = Ae^{-|x|}.$$

- A particle with mass m moving along the x -axis and its quantum state is represented by the following wave function:

$$\Psi(x, t) = \begin{cases} 0, & x < 0, \\ Axe^{-\alpha x} e^{-iEt/\hbar}, & x \geq 0, \end{cases}$$

where $\alpha = 2.0 \times 10^{10} \text{ m}^{-1}$. (a) Find the normalization constant. (b) Find the probability that the particle can be found on the interval $0 \leq x \leq L$. (c) Find the expectation value of

position. (d) Find the expectation value of kinetic energy.

31. A wave function of a particle with mass m is given by

$$\psi(x) = \begin{cases} A \cos \alpha x, & -\frac{\pi}{2\alpha} \leq x \leq +\frac{\pi}{2\alpha}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha = 1.00 \times 10^{10}/\text{m}$. (a) Find the normalization constant. (b) Find the probability that the particle can be found on the interval $0 \leq x \leq 0.5 \times 10^{-10} \text{ m}$. (c) Find the particle's average position. (d) Find its average momentum. (e) Find its average kinetic energy $-0.5 \times 10^{-10} \text{ m} \leq x \leq +0.5 \times 10^{-10} \text{ m}$.

7.2 The Heisenberg Uncertainty Principle

32. A velocity measurement of an α -particle has been performed with a precision of 0.02 mm/s. What is the minimum uncertainty in its position?
33. A gas of helium atoms at 273 K is in a cubical container with 25.0 cm on a side. (a) What is the minimum uncertainty in momentum components of helium atoms? (b) What is the minimum uncertainty in velocity components? (c) Find the ratio of the uncertainties in (b) to the mean speed of an atom in each direction.
34. If the uncertainty in the y -component of a proton's position is 2.0 pm, find the minimum uncertainty in the simultaneous measurement of the proton's y -component of velocity. What is the minimum uncertainty in the simultaneous measurement of the proton's x -component of velocity?
35. Some unstable elementary particle has a rest energy of 80.41 GeV and an uncertainty in rest energy of 2.06 GeV. Estimate the lifetime of this particle.
36. An atom in a metastable state has a lifetime of 5.2 ms. Find the minimum uncertainty in the measurement of energy of the excited state.
37. Measurements indicate that an atom remains in an excited state for an average time of 50.0 ns before making a transition to the ground state with the simultaneous emission of a 2.1-eV photon. (a) Estimate the uncertainty in the frequency of the photon. (b) What fraction of the photon's average frequency is this?
38. Suppose an electron is confined to a region of length 0.1 nm (of the order of the size of a hydrogen atom). (a) What is the minimum uncertainty of its momentum? (b) What would the uncertainty in momentum be if the confined

length region doubled to 0.2 nm?

7.3 The Schrödinger Equation

39. Combine Equation 7.17 and Equation 7.18 to show $k^2 = \frac{\omega^2}{c^2}$.
40. Show that $\Psi(x, t) = Ae^{i(kx - \omega t)}$ is a valid solution to Schrödinger's time-dependent equation.
41. Show that $\Psi(x, t) = A \sin(kx - \omega t)$ and $\Psi(x, t) = A \cos(kx - \omega t)$ do not obey Schrödinger's time-dependent equation.
42. Show that when $\Psi_1(x, t)$ and $\Psi_2(x, t)$ are solutions to the time-dependent Schrödinger equation and A, B are numbers, then a function $\Psi(x, t)$ that is a superposition of these functions is also a solution:
 $\Psi(x, t) = A\Psi_1(x, t) + B\Psi_2(x, t)$.
43. A particle with mass m is described by the following wave function:
 $\psi(x) = A \cos kx + B \sin kx$, where A, B , and k are constants. Assuming that the particle is free, show that this function is the solution of the stationary Schrödinger equation for this particle and find the energy that the particle has in this state.
44. Find the expectation value of the kinetic energy for the particle in the state,
 $\Psi(x, t) = Ae^{i(kx - \omega t)}$. What conclusion can you draw from your solution?
45. Find the expectation value of the square of the momentum squared for the particle in the state,
 $\Psi(x, t) = Ae^{i(kx - \omega t)}$. What conclusion can you draw from your solution?
46. A free proton has a wave function given by
 $\Psi(x, t) = Ae^{i(5.02 \times 10^{11}x - 8.00 \times 10^{15}t)}$.
 The coefficient of x is inverse meters (m^{-1}) and the coefficient on t is inverse seconds (s^{-1}). Find its momentum and energy.

7.4 The Quantum Particle in a Box

47. Assume that an electron in an atom can be treated as if it were confined to a box of width 2.0 Å. What is the ground state energy of the electron? Compare your result to the ground state kinetic energy of the hydrogen atom in the Bohr's model of the hydrogen atom.
48. Assume that a proton in a nucleus can be treated as if it were confined to a one-dimensional box of width 10.0 fm. (a) What are the energies of the proton when it is in the states corresponding to $n = 1$, $n = 2$, and $n = 3$?

- (b) What are the energies of the photons emitted when the proton makes the transitions from the first and second excited states to the ground state?
49. An electron confined to a box has the ground state energy of 2.5 eV. What is the width of the box?
 50. What is the ground state energy (in eV) of a proton confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?
 51. What is the ground state energy (in eV) of an α -particle confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?
 52. To excite an electron in a one-dimensional box from its first excited state to its third excited state requires 20.0 eV. What is the width of the box?
 53. An electron confined to a box of width 0.15 nm by infinite potential energy barriers emits a photon when it makes a transition from the first excited state to the ground state. Find the wavelength of the emitted photon.
 54. If the energy of the first excited state of the electron in the box is 25.0 eV, what is the width of the box?
 55. Suppose an electron confined to a box emits photons. The longest wavelength that is registered is 500.0 nm. What is the width of the box?
 56. Hydrogen H_2 molecules are kept at 300.0 K in a cubical container with a side length of 20.0 cm. Assume that you can treat the molecules as though they were moving in a one-dimensional box. (a) Find the ground state energy of the hydrogen molecule in the container. (b) Assume that the molecule has a thermal energy given by $k_B T/2$ and find the corresponding quantum number n of the quantum state that would correspond to this thermal energy.
 57. An electron is confined to a box of width 0.25 nm. (a) Draw an energy-level diagram representing the first five states of the electron. (b) Calculate the wavelengths of the emitted photons when the electron makes transitions between the fourth and the second excited states, between the second excited state and the ground state, and between the third and the second excited states.
 58. An electron in a box is in the ground state with energy 2.0 eV. (a) Find the width of the box. (b) How much energy is needed to excite the

electron to its first excited state? (c) If the electron makes a transition from an excited state to the ground state with the simultaneous emission of 30.0-eV photon, find the quantum number of the excited state?

7.5 The Quantum Harmonic Oscillator

59. Show that the two lowest energy states of the simple harmonic oscillator, $\psi_0(x)$ and $\psi_1(x)$ from [Equation 7.57](#), satisfy [Equation 7.55](#).
60. If the ground state energy of a simple harmonic oscillator is 1.25 eV, what is the frequency of its motion?
61. When a quantum harmonic oscillator makes a transition from the $(n + 1)$ state to the n state and emits a 450-nm photon, what is its frequency?
62. Vibrations of the hydrogen molecule H_2 can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 \times 10^3 \text{ N/m}$ and mass $m = 1.67 \times 10^{-27} \text{ kg}$. (a) What is the vibrational frequency of this molecule? (b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states?
63. A particle with mass 0.030 kg oscillates back-and-forth on a spring with frequency 4.0 Hz. At the equilibrium position, it has a speed of 0.60 m/s. If the particle is in a state of definite energy, find its energy quantum number.
64. Find the expectation value $\langle x^2 \rangle$ of the square of the position for a quantum harmonic oscillator in the ground state. Note:

$$\int_{-\infty}^{+\infty} dx x^2 e^{-ax^2} = \sqrt{\pi}(2a^{-3/2})^{-1}.$$
65. Determine the expectation value of the potential energy for a quantum harmonic oscillator in the ground state. Use this to calculate the expectation value of the kinetic energy.
66. Verify that $\psi_1(x)$ given by [Equation 7.57](#) is a solution of Schrödinger's equation for the quantum harmonic oscillator.
67. Estimate the ground state energy of the quantum harmonic oscillator by Heisenberg's uncertainty principle. Start by assuming that the product of the uncertainties Δx and Δp is at its minimum. Write Δp in terms of Δx and assume that for the ground state $x \approx \Delta x$ and $p \approx \Delta p$, then write the ground state energy in terms of x . Finally, find the value of x that

minimizes the energy and find the minimum of the energy.

68. A mass of 0.250 kg oscillates on a spring with the force constant 110 N/m. Calculate the ground energy level and the separation between the adjacent energy levels. Express the results in joules and in electron-volts. Are quantum effects important?

7.6 The Quantum Tunneling of Particles through Potential Barriers

69. Show that the wave function in (a) Equation 7.68 satisfies Equation 7.61, and (b) Equation 7.69 satisfies Equation 7.63.
70. A 6.0-eV electron impacts on a barrier with height 11.0 eV. Find the probability of the electron to tunnel through the barrier if the barrier width is (a) 0.80 nm and (b) 0.40 nm.
71. A 5.0-eV electron impacts on a barrier of width 0.60 nm. Find the probability of the electron to tunnel through the barrier if the barrier height is (a) 7.0 eV; (b) 9.0 eV; and (c) 13.0 eV.
72. A 12.0-eV electron encounters a barrier of height 15.0 eV. If the probability of the electron tunneling through the barrier is 2.5 %, find its width.
73. A quantum particle with initial kinetic energy

32.0 eV encounters a square barrier with height 41.0 eV and width 0.25 nm. Find probability that the particle tunnels through this barrier if the particle is (a) an electron and, (b) a proton.

74. A simple model of a radioactive nuclear decay assumes that α -particles are trapped inside a well of nuclear potential that walls are the barriers of a finite width 2.0 fm and height 30.0 MeV. Find the tunneling probability across the potential barrier of the wall for α -particles having kinetic energy (a) 29.0 MeV and (b) 20.0 MeV. The mass of the α -particle is $m = 6.64 \times 10^{-27}$ kg.
75. A muon, a quantum particle with a mass approximately 200 times that of an electron, is incident on a potential barrier of height 10.0 eV. The kinetic energy of the impacting muon is 5.5 eV and only about 0.10% of the squared amplitude of its incoming wave function filters through the barrier. What is the barrier's width?
76. A grain of sand with mass 1.0 mg and kinetic energy 1.0 J is incident on a potential energy barrier with height 1.000001 J and width 2500 nm. How many grains of sand have to fall on this barrier before, on the average, one passes through?

Additional Problems

77. Show that if the uncertainty in the position of a particle is on the order of its de Broglie's wavelength, then the uncertainty in its momentum is on the order of the value of its momentum.
78. The mass of a ρ -meson is measured to be $770 \text{ MeV}/c^2$ with an uncertainty of $100 \text{ MeV}/c^2$. Estimate the lifetime of this meson.
79. A particle of mass m is confined to a box of width L . If the particle is in the first excited state, what are the probabilities of finding the particle in a region of width $0.020 L$ around the given point x : (a) $x = 0.25L$; (b) $x = 0.40L$; (c) $x = 0.75L$; and (d) $x = 0.90L$.
80. A particle in a box $[0;L]$ is in the third excited state. What are its most probable positions?
81. A 0.20-kg billiard ball bounces back and forth without losing its energy between the cushions of a 1.5 m long table. (a) If the ball is in its ground state, how many years does it need to get from one cushion to the other? You may compare this time interval to the age of the universe. (b) How much energy is required to make the ball go from its ground state to its first excited state? Compare it with the kinetic energy of the ball moving at 2.0 m/s.
82. Find the expectation value of the position squared when the particle in the box is in its third excited state and the length of the box is L .
83. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Show that the function $\psi(x) = A \sin kx$ is the solution to the stationary Schrödinger equation for the particle in a box only if $k = \sqrt{2mE}/\hbar$. Explain why this is an acceptable wave function only if k is an integer multiple of π/L .

- 84.** Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Explain why the function $\psi(x) = A \cos kx$ is not a solution to the stationary Schrödinger equation for the particle in a box.
- 85.** Atoms in a crystal lattice vibrate in simple harmonic motion. Assuming a lattice atom has a mass of 9.4×10^{-26} kg, what is the force constant of the lattice if a lattice atom makes a transition from the ground state to first excited state when it absorbs a $525\text{-}\mu\text{m}$ photon?
- 86.** A diatomic molecule behaves like a quantum harmonic oscillator with the force constant 12.0 N/m and mass 5.60×10^{-26} kg. (a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state? (b) Find the ground state energy of vibrations for this diatomic molecule.
- 87.** An electron with kinetic energy 2.0 MeV encounters a potential energy barrier of height 16.0 MeV and width 2.00 nm. What is the probability that the electron emerges on the other side of the barrier?
- 88.** A beam of mono-energetic protons with energy 2.0 MeV falls on a potential energy barrier of height 20.0 MeV and of width 1.5 fm. What percentage of the beam is transmitted through the barrier?

Challenge Problems

- 89.** An electron in a long, organic molecule used in a dye laser behaves approximately like a quantum particle in a box with width 4.18 nm. Find the emitted photon when the electron makes a transition from the first excited state to the ground state and from the second excited state to the first excited state.
- 90.** In STM, an elevation of the tip above the surface being scanned can be determined with a great precision, because the tunneling-electron current between surface atoms and the atoms of the tip is extremely sensitive to the variation of the separation gap between them from point to point along the surface. Assuming that the tunneling-electron current is in direct proportion to the tunneling probability and that the tunneling probability is to a good approximation expressed by the exponential function $e^{-2\beta L}$ with $\beta = 10.0/\text{nm}$, determine the ratio of the tunneling current when the tip is 0.500 nm above the surface to the current when the tip is 0.515 nm above the surface.
- 91.** If STM is to detect surface features with local heights of about 0.00200 nm, what percent change in tunneling-electron current must the STM electronics be able to detect? Assume that the tunneling-electron current has characteristics given in the preceding problem.
- 92.** Use Heisenberg's uncertainty principle to estimate the ground state energy of a particle oscillating on a spring with angular frequency, $\omega = \sqrt{k/m}$, where k is the spring constant and m is the mass.
- 93.** Suppose an infinite square well extends from $-L/2$ to $+L/2$. Solve the time-independent Schrödinger's equation to find the allowed energies and stationary states of a particle with mass m that is confined to this well. Then show that these solutions can be obtained by making the coordinate transformation $x' = x - L/2$ for the solutions obtained for the well extending between 0 and L .
- 94.** A particle of mass m confined to a box of width L is in its first excited state $\psi_2(x)$. (a) Find its average position (which is the expectation value of the position). (b) Where is the particle most likely to be found?

CHAPTER 10

Nuclear Physics

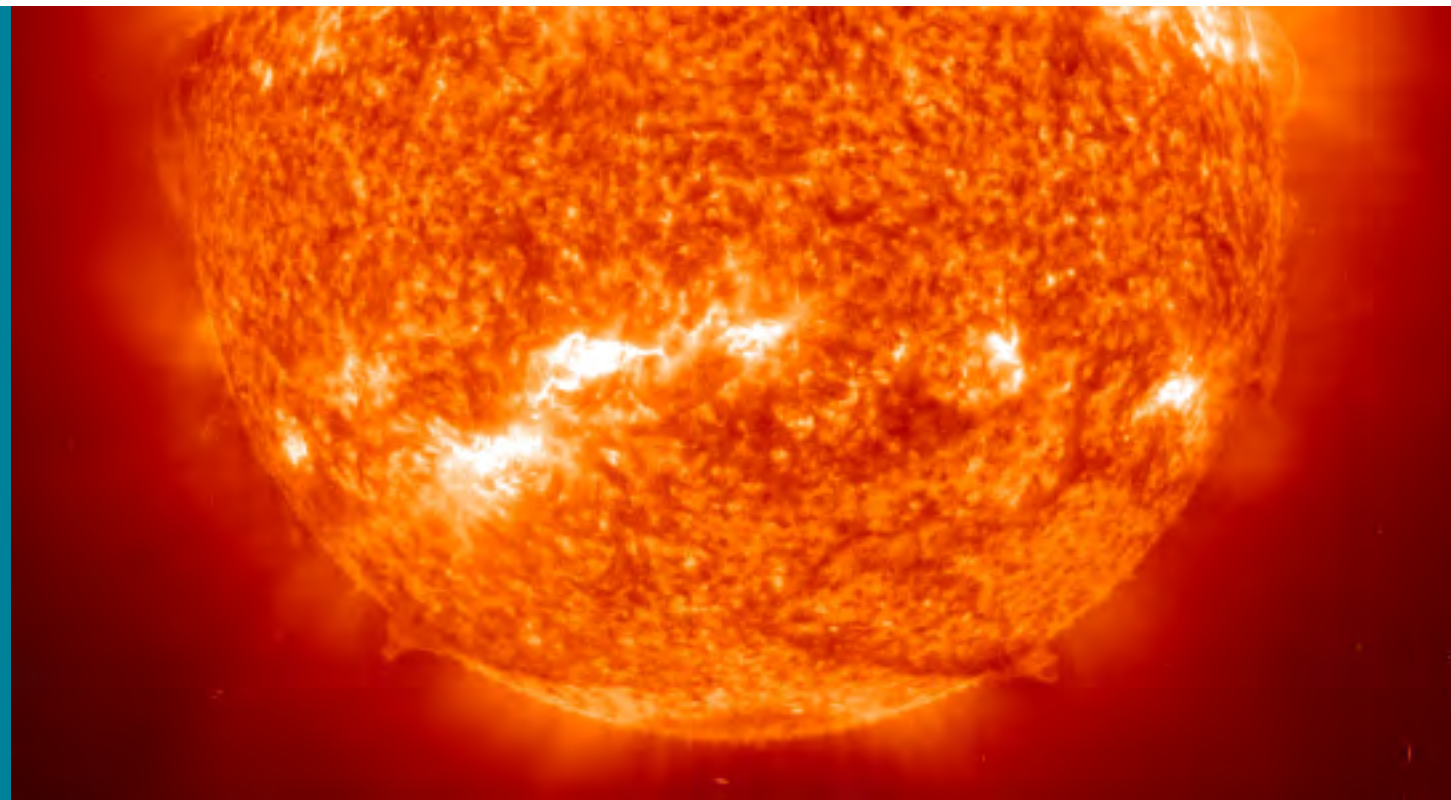


Figure 10.1 The Sun is powered by nuclear fusion in its core. The core converts approximately 10^{38} protons/second into helium at a temperature of 14 million K. This process releases energy in the form of photons, neutrinos, and other particles. (credit: modification of work by EIT SOHO Consortium, ESA, NASA)

Chapter Outline

[10.1 Properties of Nuclei](#)

[10.2 Nuclear Binding Energy](#)

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[10.4 Nuclear Reactions](#)

[10.5 Fission](#)

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[10.7 Medical Applications and Biological Effects of Nuclear Radiation](#)

INTRODUCTION In this chapter, we study the composition and properties of the atomic nucleus. The nucleus lies at the center of an atom, and consists of protons and neutrons. A deep understanding of the nucleus leads to numerous valuable technologies, including devices to date ancient rocks, map the galactic arms of the Milky

Way, and generate electrical power.

The Sun is the main source of energy in the solar system. The Sun is 109 Earth diameters across, and accounts for more than 99% of the total mass of the solar system. The Sun shines by fusing hydrogen nuclei—protons—deep inside its interior. Once this fuel is spent, the Sun will burn helium and, later, other nuclei. Nuclear fusion in the Sun is discussed toward the end of this chapter. In the meantime, we will investigate nuclear properties that govern all nuclear processes, including fusion.

10.1 Properties of Nuclei

Learning Objectives

By the end of this section, you will be able to:

- Describe the composition and size of an atomic nucleus
- Use a nuclear symbol to express the composition of an atomic nucleus
- Explain why the number of neutrons is greater than protons in heavy nuclei
- Calculate the atomic mass of an element given its isotopes

The **atomic nucleus** is composed of **protons** and **neutrons** (Figure 10.2). Protons and neutrons have approximately the same mass, but protons carry one unit of positive charge ($+e$), and neutrons carry no charge. These particles are packed together into an extremely small space at the center of an atom. According to scattering experiments, the nucleus is spherical or ellipsoidal in shape, and about 1/100,000th the size of a hydrogen atom. If an atom were the size of a major league baseball stadium, the nucleus would be roughly the size of the baseball. Protons and neutrons within the nucleus are called **nucleons**.

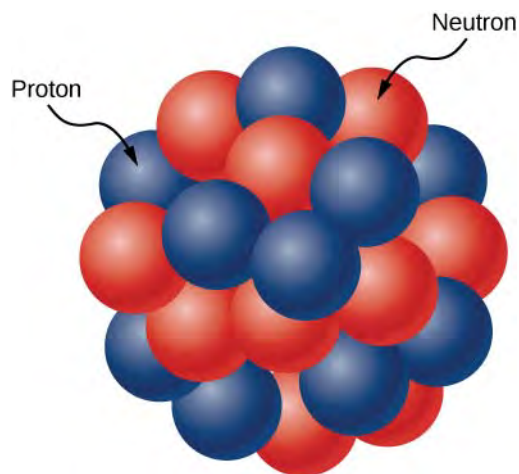


Figure 10.2 The atomic nucleus is composed of protons and neutrons. Protons are shown in blue, and neutrons are shown in red.

Counts of Nucleons

The number of protons in the nucleus is given by the **atomic number**, Z . The number of neutrons in the nucleus is the **neutron number**, N . The total number of nucleons is the **mass number**, A . These numbers are related by

$$A = Z + N. \quad 10.1$$

A nucleus is represented symbolically by

$${}^A_Z X, \quad 10.2$$

where X represents the chemical element, A is the mass number, and Z is the atomic number. For example, ${}^{12}_6\text{C}$ represents the carbon nucleus with six protons and six neutrons (or 12 nucleons).

A graph of the number N of neutrons versus the number Z of protons for a range of stable nuclei (**nuclides**) is shown in Figure 10.3. For a given value of Z , multiple values of N (blue points) are possible. For small values of

Z , the number of neutrons equals the number of protons ($N = P$), and the data fall on the red line. For large values of Z , the number of neutrons is greater than the number of protons ($N > P$), and the data points fall above the red line. The number of neutrons is generally greater than the number of protons for $Z > 15$.

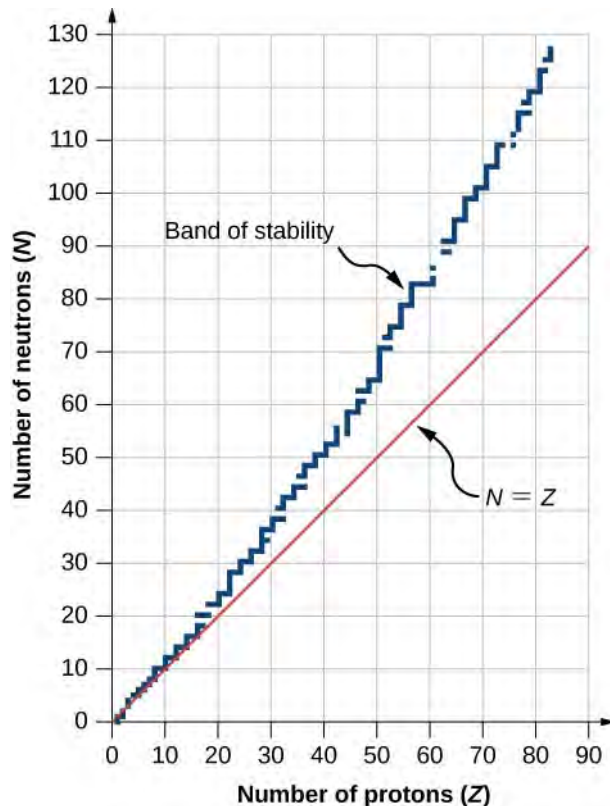


Figure 10.3 This graph plots the number of neutrons N against the number of protons Z for stable atomic nuclei. Larger nuclei, have more neutrons than protons.

A chart based on this graph that provides more detailed information about each nucleus is given in [Figure 10.4](#). This chart is called a **chart of the nuclides**. Each cell or tile represents a separate nucleus. The nuclei are arranged in order of ascending Z (along the horizontal direction) and ascending N (along the vertical direction).

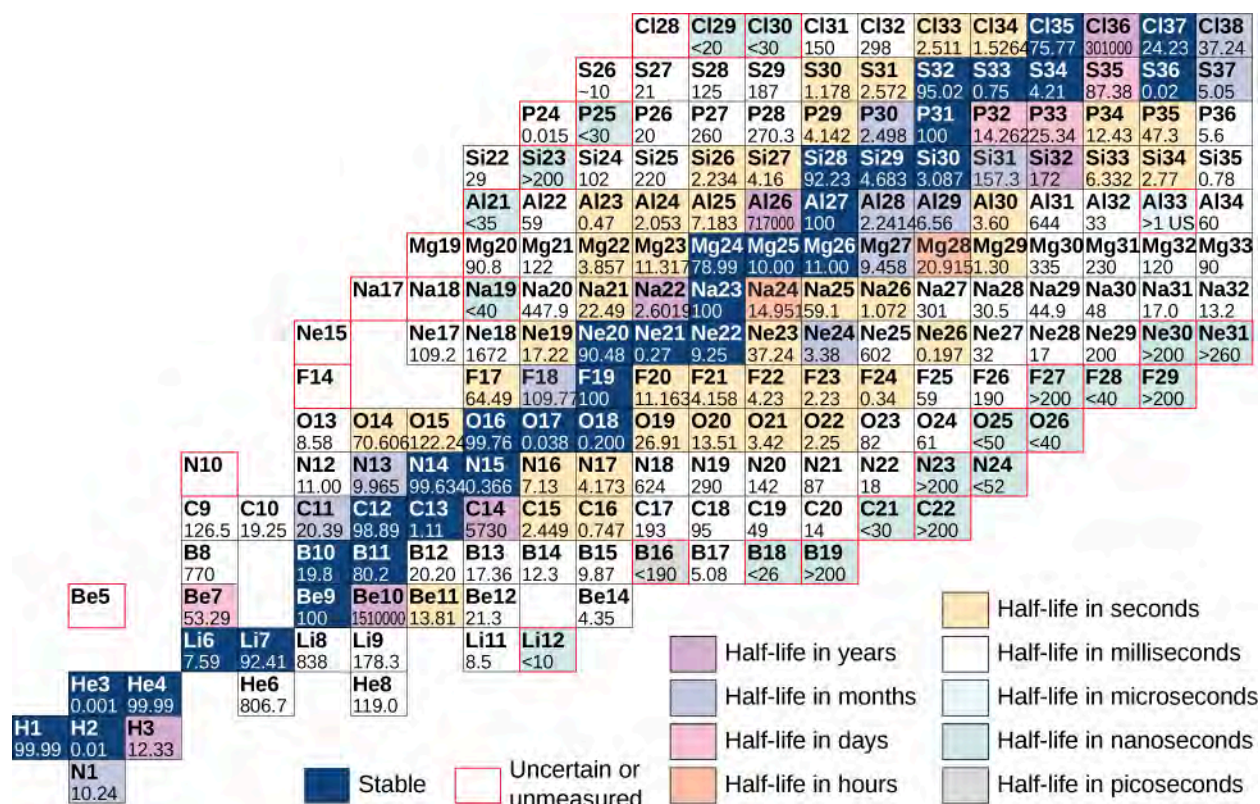


Figure 10.4 Partial chart of the nuclides. For stable nuclei (dark blue backgrounds), cell values represent the percentage of nuclei found on Earth with the same atomic number (percent abundance). For the unstable nuclei, the number represents the half-life.

Atoms that contain nuclei with the same number of protons (Z) and different numbers of neutrons (N) are called **isotopes**. For example, hydrogen has three isotopes: normal hydrogen (1 proton, no neutrons), deuterium (one proton and one neutron), and tritium (one proton and two neutrons). Isotopes of a given atom share the same chemical properties, since these properties are determined by interactions between the outer electrons of the atom, and not the nucleons. For example, water that contains deuterium rather than hydrogen (“heavy water”) looks and tastes like normal water. The following table shows a list of common isotopes.

Element	Symbol	Mass Number	Mass (Atomic Mass Units)	Percent Abundance*	Half-life**
Hydrogen	H	1	1.0078	99.99	stable
	^2H or D	2	2.0141	0.01	stable
	^3H	3	3.0160	–	12.32 y
Carbon	^{12}C	12	12.0000	98.91	stable
	^{13}C	13	13.0034	1.1	stable
	^{14}C	14	14.0032	–	5730 y
Nitrogen	^{14}N	14	14.0031	99.6	stable
	^{15}N	15	15.0001	0.4	stable

Element	Symbol	Mass Number	Mass (Atomic Mass Units)	Percent Abundance*	Half-life**
	^{16}N	16	16.0061	–	7.13 s
Oxygen	^{16}O	16	15.9949	99.76	stable
	^{17}O	17	16.9991	0.04	stable
	^{18}O	18	17.9992	0.20	stable
	^{19}O	19	19.0035	–	26.46 s

Table 10.1 Common Isotopes *No entry if less than 0.001 (trace amount).

**Stable if half-life > 10 seconds.

Why do neutrons outnumber protons in heavier nuclei ([Figure 10.5](#))? The answer to this question requires an understanding of forces inside the nucleus. Two types of forces exist: (1) the long-range electrostatic (Coulomb) force that makes the positively charged protons repel one another; and (2) the short-range **strong nuclear force** that makes all nucleons in the nucleus attract one another. You may also have heard of a “weak” nuclear force. This force is responsible for some nuclear decays, but as the name implies, it does not play a role in stabilizing the nucleus against the strong Coulomb repulsion it experiences. We discuss strong nuclear force in more detail in the next chapter when we cover particle physics. Nuclear stability occurs when the attractive forces between nucleons compensate for the repulsive, long-range electrostatic forces between all protons in the nucleus. For heavy nuclei ($Z > 15$), excess neutrons are necessary to keep the electrostatic interactions from breaking the nucleus apart, as shown in [Figure 10.3](#).

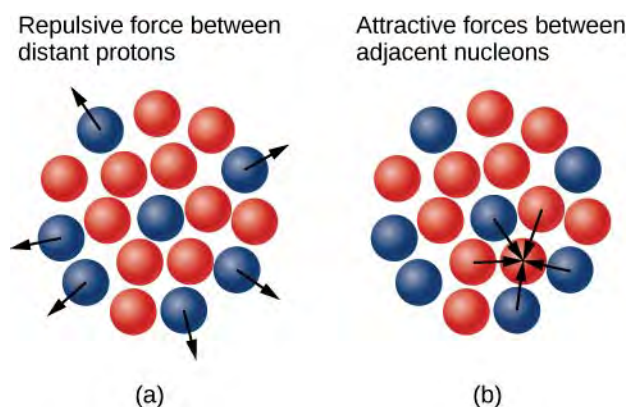


Figure 10.5 (a) The electrostatic force is repulsive and has long range. The arrows represent outward forces on protons (in blue) at the nuclear surface by a proton (also in blue) at the center. (b) The strong nuclear force acts between neighboring nucleons. The arrows represent attractive forces exerted by a neutron (in red) on its nearest neighbors.

Because of the existence of stable isotopes, we must take special care when quoting the mass of an element. For example, Copper (Cu) has two stable isotopes:

$^{63}_{29}\text{Cu}$ (62.929595 g/mol) with an abundance of 69.09%

$^{65}_{29}\text{Cu}$ (64.927786 g/mol) with an abundance of 30.91%

Given these two “versions” of Cu, what is the mass of this element? The **atomic mass** of an element is defined as the weighted average of the masses of its isotopes. Thus, the atomic mass of Cu is $m_{\text{Cu}} = (62.929595)(0.6909) + (64.927786)(0.3091) = 63.55$ g/mol. The mass of an individual nucleus is often expressed in **atomic mass units** (u), where $u = 1.66054 \times 10^{-27}$ kg. (An atomic mass unit is defined as 1/12th the mass of a ^{12}C nucleus.) In atomic mass units, the mass of a helium nucleus ($A = 4$) is approximately

4 u. A helium nucleus is also called an alpha (α) particle.

Nuclear Size

The simplest model of the nucleus is a densely packed sphere of nucleons. The volume V of the nucleus is therefore proportional to the number of nucleons A , expressed by

$$V = \frac{4}{3} \pi r^3 = kA,$$

where r is the **radius of a nucleus** and k is a constant with units of volume. Solving for r , we have

$$r = r_0 A^{1/3} \quad 10.3$$

where r_0 is a constant. For hydrogen ($A = 1$), r_0 corresponds to the radius of a single proton. Scattering experiments support this general relationship for a wide range of nuclei, and they imply that neutrons have approximately the same radius as protons. The experimentally measured value for r_0 is approximately 1.2 femtometer (recall that 1 fm = 10^{-15} m).



EXAMPLE 10.1

The Iron Nucleus

Find the radius (r) and approximate density (ρ) of a Fe-56 nucleus. Assume the mass of the Fe-56 nucleus is approximately 56 u.

Strategy

(a) Finding the radius of ^{56}Fe is a straightforward application of $r = r_0 A^{1/3}$, given $A = 56$. (b) To find the approximate density of this nucleus, assume the nucleus is spherical. Calculate its volume using the radius found in part (a), and then find its density from $\rho = m/V$.

Solution

- a. The radius of a nucleus is given by

$$r = r_0 A^{1/3}.$$

Substituting the values for r_0 and A yields

$$\begin{aligned} r &= (1.2 \text{ fm})(56)^{1/3} = (1.2 \text{ fm})(3.83) \\ &= 4.6 \text{ fm}. \end{aligned}$$

- b. Density is defined to be $\rho = m/V$, which for a sphere of radius r is

$$\rho = \frac{m}{V} = \frac{m}{(4/3)\pi r^3}.$$

Substituting known values gives

$$\rho = \frac{56 \text{ u}}{(1.33)(3.14)(4.6 \text{ fm})^3} = 0.138 \text{ u/fm}^3.$$

Converting to units of kg/m^3 , we find

$$\rho = (0.138 \text{ u/fm}^3)(1.66 \times 10^{-27} \text{ kg/u}) \left(\frac{1 \text{ fm}}{10^{-15} \text{ m}} \right) = 2.3 \times 10^{17} \text{ kg/m}^3.$$

Significance

- The radius of the Fe-56 nucleus is found to be approximately 5 fm, so its diameter is about 10 fm, or 10^{-14} m. In previous discussions of Rutherford's scattering experiments, a light nucleus was estimated to be 10^{-15} m in diameter. Therefore, the result shown for a mid-sized nucleus is reasonable.
- The density found here may seem incredible. However, it is consistent with earlier comments about the nucleus containing nearly all of the mass of the atom in a tiny region of space. One cubic meter of nuclear

matter has the same mass as a cube of water 61 km on each side.

✓ CHECK YOUR UNDERSTANDING 10.1

Nucleus X is two times larger than nucleus Y. What is the ratio of their atomic masses?

10.2 Nuclear Binding Energy

Learning Objectives

By the end of this section, you will be able to:

- Calculate the mass defect and binding energy for a wide range of nuclei
- Use a graph of binding energy per nucleon (BEN) versus mass number (A) graph to assess the relative stability of a nucleus
- Compare the binding energy of a nucleon in a nucleus to the ionization energy of an electron in an atom

The forces that bind nucleons together in an atomic nucleus are much greater than those that bind an electron to an atom through electrostatic attraction. This is evident by the relative sizes of the atomic nucleus and the atom (10^{-15} and 10^{-10} m, respectively). The energy required to pry a nucleon from the nucleus is therefore much larger than that required to remove (or ionize) an electron in an atom. In general, all nuclear changes involve large amounts of energy per particle undergoing the reaction. This has numerous practical applications.

Mass Defect

According to nuclear particle experiments, the total mass of a nucleus (m_{nuc}) is *less* than the sum of the masses of its constituent nucleons (protons and neutrons). The mass difference, or **mass defect**, is given by

$$\Delta m = Zm_p + (A - Z)m_n - m_{\text{nuc}} \quad 10.4$$

where Zm_p is the total mass of the protons, $(A - Z)m_n$ is the total mass of the neutrons, and m_{nuc} is the mass of the nucleus. According to Einstein's special theory of relativity, mass is a measure of the total energy of a system ($E = mc^2$). Thus, the total energy of a nucleus is less than the sum of the energies of its constituent nucleons. The formation of a nucleus from a system of isolated protons and neutrons is therefore an exothermic reaction—meaning that it releases energy. The energy emitted, or radiated, in this process is $(\Delta m)c^2$.

Now imagine this process occurs in reverse. Instead of forming a nucleus, energy is put into the system to break apart the nucleus (Figure 10.6). The amount of energy required is called the total **binding energy (BE)**, E_b .

Binding Energy

The binding energy is equal to the amount of energy released in forming the nucleus, and is therefore given by

$$E_b = (\Delta m)c^2. \quad 10.5$$

Experimental results indicate that the binding energy for a nucleus with mass number $A > 8$ is roughly proportional to the total number of nucleons in the nucleus, A . The binding energy of a magnesium nucleus (^{24}Mg), for example, is approximately two times greater than for the carbon nucleus (^{12}C).

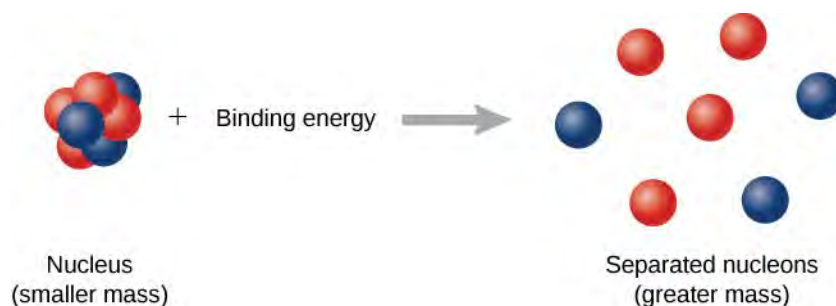


Figure 10.6 The binding energy is the energy required to break a nucleus into its constituent protons and neutrons. A system of separated nucleons has a greater mass than a system of bound nucleons.



EXAMPLE 10.2

Mass Defect and Binding Energy of the Deuteron

Calculate the mass defect and the binding energy of the deuteron. The mass of the deuteron is $m_D = 3.34359 \times 10^{-27} \text{ kg}$ or $1875.61 \text{ MeV}/c^2$.

Solution

From [Equation 10.4](#), the mass defect for the deuteron is

$$\begin{aligned}\Delta m &= m_p + m_n - m_D \\ &= 938.28 \text{ MeV}/c^2 + 939.57 \text{ MeV}/c^2 - 1875.61 \text{ MeV}/c^2 \\ &= 2.24 \text{ MeV}/c^2.\end{aligned}$$

The binding energy of the deuteron is then

$$E_b = (\Delta m) c^2 = (2.24 \text{ MeV}/c^2) (c^2) = 2.24 \text{ MeV}.$$

Over two million electron volts are needed to break apart a deuteron into a proton and a neutron. This very large value indicates the great strength of the nuclear force. By comparison, the greatest amount of energy required to liberate an electron bound to a hydrogen atom by an attractive Coulomb force (an electromagnetic force) is about 10 eV.

Graph of Binding Energy per Nucleon

In nuclear physics, one of the most important experimental quantities is the **binding energy per nucleon (BEN)**, which is defined by

$$BEN = \frac{E_b}{A} \quad 10.6$$

This quantity is the average energy required to remove an individual nucleon from a nucleus—analogueous to the ionization energy of an electron in an atom. If the BEN is relatively large, the nucleus is relatively stable. BEN values are estimated from nuclear scattering experiments.

A graph of binding energy per nucleon versus mass number A is given in [Figure 10.7](#). This graph is considered by many physicists to be one of the most important graphs in physics. Two notes are in order. First, typical BEN values range from 6–10 MeV, with an average value of about 8 MeV. In other words, it takes several million electron volts to pry a nucleon from a typical nucleus, as compared to just 13.6 eV to ionize an electron in the ground state of hydrogen. This is why nuclear force is referred to as the “strong” nuclear force.

Second, the graph rises at low A , peaks very near iron (Fe , $A = 56$), and then tapers off at high A . The peak value suggests that the iron nucleus is the most stable nucleus in nature (it is also why nuclear fusion in the cores of stars ends with Fe). The reason the graph rises and tapers off has to do with competing forces in the

nucleus. At low values of A , attractive nuclear forces between nucleons dominate over repulsive electrostatic forces between protons. But at high values of A , repulsive electrostatic forces between forces begin to dominate, and these forces tend to break apart the nucleus rather than hold it together.

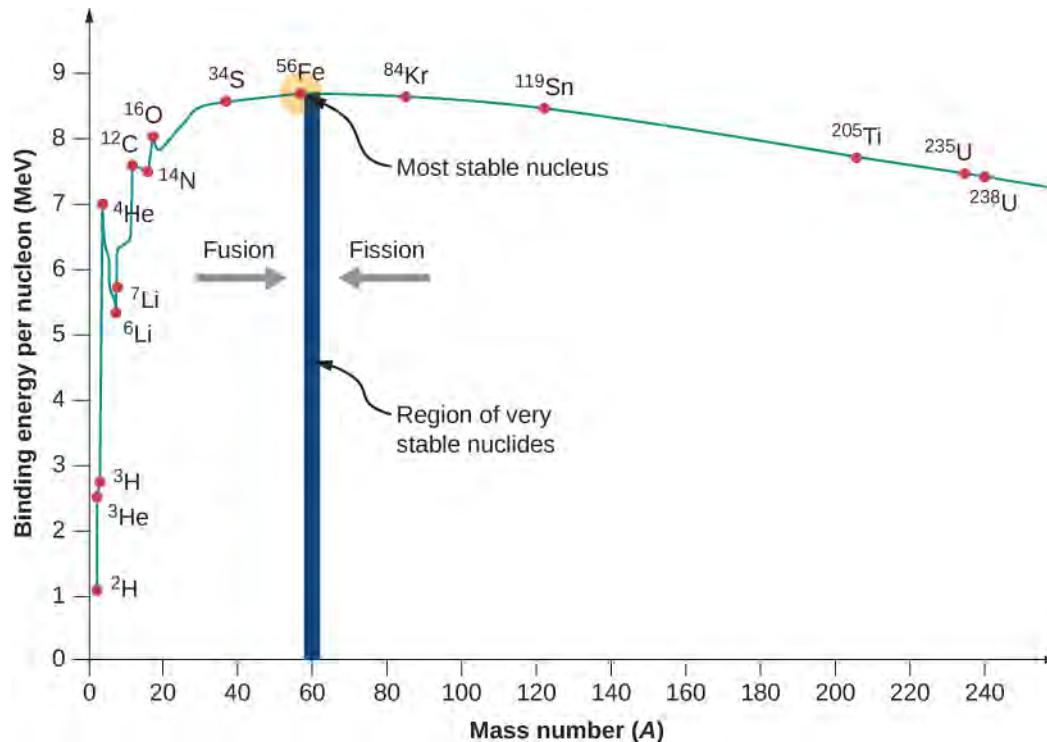


Figure 10.7 In this graph of binding energy per nucleon for stable nuclei, the BEN is greatest for nuclei with a mass near ^{56}Fe . Therefore, fusion of nuclei with mass numbers much less than that of Fe, and fission of nuclei with mass numbers greater than that of Fe, are exothermic processes.

As we will see, the BEN-versus- A graph implies that nuclei divided or combined release an enormous amount of energy. This is the basis for a wide range of phenomena, from the production of electricity at a nuclear power plant to sunlight.



EXAMPLE 10.3

Tightly Bound Alpha Nuclides

Calculate the binding energy per nucleon of an ^4He (α particle).

Strategy

Determine the total binding energy (BE) using the equation $\text{BE} = (\Delta m)c^2$, where Δm is the mass defect. The binding energy per nucleon (BEN) is BE divided by A .

Solution

For ^4He , we have $Z = N = 2$. The total binding energy is

$$\text{BE} = \{ [2m_p + 2m_n] - m(^4\text{He}) \} c^2.$$

These masses are $m(^4\text{He}) = 4.002602 \text{ u}$, $m_p = 1.007825 \text{ u}$, and $m_n = 1.008665 \text{ u}$. Thus we have,

$$\text{BE} = (0.030378 \text{ u}) c^2.$$

Noting that $1 \text{ u} = 931.5 \text{ MeV}/c^2$, we find

$$\begin{aligned} \text{BE} &= (0.030378) (931.5 \text{ MeV}/c^2) c^2 \\ &= 28.3 \text{ MeV}. \end{aligned}$$

Since $A = 4$, the total binding energy per nucleon is

$$\text{BEN} = 7.07 \text{ MeV/nucleon}.$$

Significance

Notice that the binding energy per nucleon for ${}^4\text{He}$ is much greater than for the hydrogen isotopes (only $\approx 3 \text{ MeV/nucleon}$). Therefore, helium nuclei cannot break down hydrogen isotopes without energy being put into the system.

✓ CHECK YOUR UNDERSTANDING 10.2

If the binding energy per nucleon is large, does this make it harder or easier to strip off a nucleon from a nucleus?

10.3 Radioactive Decay

Learning Objectives

By the end of this section, you will be able to:

- Describe the decay of a radioactive substance in terms of its decay constant and half-life
- Use the radioactive decay law to estimate the age of a substance
- Explain the natural processes that allow the dating of living tissue using ${}^{14}\text{C}$

In 1896, Antoine Becquerel discovered that a uranium-rich rock emits invisible rays that can darken a photographic plate in an enclosed container. Scientists offer three arguments for the nuclear origin of these rays. First, the effects of the radiation do not vary with chemical state; that is, whether the emitting material is in the form of an element or compound. Second, the radiation does not vary with changes in temperature or pressure—both factors that in sufficient degree can affect electrons in an atom. Third, the very large energy of the invisible rays (up to hundreds of eV) is not consistent with atomic electron transitions (only a few eV). Today, this radiation is explained by the conversion of mass into energy deep within the nucleus of an atom. The spontaneous emission of radiation from nuclei is called nuclear **radioactivity** ([Figure 10.8](#)).



Figure 10.8 The international ionizing radiation symbol is universally recognized as the warning symbol for nuclear radiation.

Radioactive Decay Law

When an individual nucleus transforms into another with the emission of radiation, the nucleus is said to **decay**. Radioactive decay occurs for all nuclei with $Z > 82$, and also for some unstable isotopes with $Z < 83$. The decay rate is proportional to the number of original (undecayed) nuclei N in a substance. The number of nuclei lost to decay, $-dN$ in time interval dt , is written

$$-\frac{dN}{dt} = \lambda N \quad 10.7$$

where λ is called the **decay constant**. (The minus sign indicates the number of original nuclei decreases over time.) In other words, the more nuclei available to decay, the more that do decay (in time dt). This equation can be rewritten as

$$\frac{dN}{N} = -\lambda dt. \quad 10.8$$

Integrating both sides of the equation, and defining N_0 to be the number of nuclei at $t = 0$, we obtain

$$\int_{N_0}^N \frac{dN'}{N'} = - \int_0^t \lambda dt'. \quad 10.9$$

This gives us

$$\ln \frac{N}{N_0} = -\lambda t. \quad 10.10$$

Taking the left and right sides of the equation as a power of e , we have the **radioactive decay law**.

Radioactive Decay Law

The total number N of radioactive nuclei remaining after time t is

$$N = N_0 e^{-\lambda t} \quad 10.11$$

where λ is the decay constant for the particular nucleus.

The total number of nuclei drops very rapidly at first, and then more slowly (Figure 10.9).

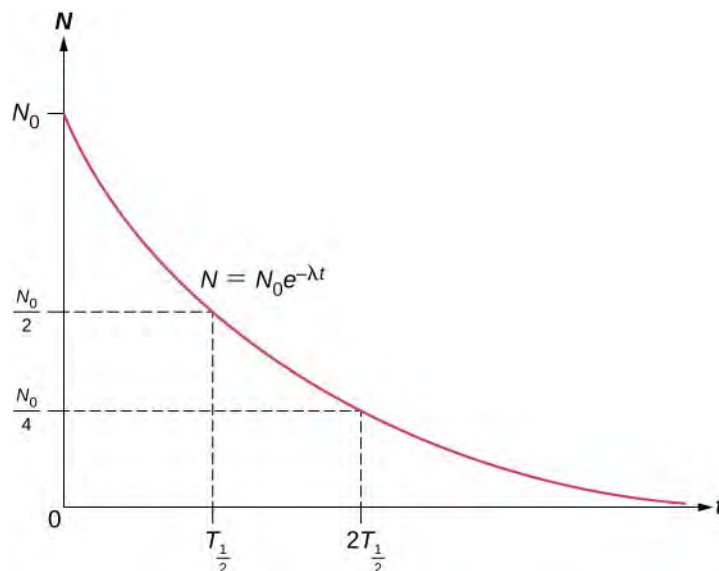


Figure 10.9 A plot of the radioactive decay law demonstrates that the number of nuclei remaining in a decay sample drops dramatically during the first moments of decay.

The **half-life** ($T_{1/2}$) of a radioactive substance is defined as the time for half of the original nuclei to decay (or the time at which half of the original nuclei remain). The half-lives of unstable isotopes are shown in the chart of nuclides in Figure 10.4. The number of radioactive nuclei remaining after an integer (n) number of half-lives is therefore

$$N = \frac{N_0}{2^n} \quad 10.12$$

If the decay constant (λ) is large, the half-life is small, and vice versa. To determine the relationship between

these quantities, note that when $t = T_{1/2}$, then $N = N_0/2$. Thus, [Equation 10.10](#) can be rewritten as

$$\frac{N_0}{2} = N_0 e^{-\lambda T_{1/2}}. \quad 10.13$$

Dividing both sides by N_0 and taking the natural logarithm yields

$$\ln \frac{1}{2} = \ln e^{-\lambda T_{1/2}} \quad 10.14$$

which reduces to

$$\lambda = \frac{0.693}{T_{1/2}}. \quad 10.15$$

Thus, if we know the half-life $T_{1/2}$ of a radioactive substance, we can find its decay constant. The **lifetime** $\bar{T}$ of a radioactive substance is defined as the average amount of time that a nucleus exists before decaying. The lifetime of a substance is just the reciprocal of the decay constant, written as

$$\bar{T} = \frac{1}{\lambda}. \quad 10.16$$

The **activity** A is defined as the magnitude of the decay rate, or

$$A = -\frac{dN}{dt} = \lambda N = \lambda N_0 e^{-\lambda t}. \quad 10.17$$

The infinitesimal change dN in the time interval dt is negative because the number of parent (undecayed) particles is decreasing, so the activity (A) is positive. Defining the initial activity as $A_0 = \lambda N_0$, we have

$$A = A_0 e^{-\lambda t}. \quad 10.18$$

Thus, the activity A of a radioactive substance decreases exponentially with time ([Figure 10.10](#)).

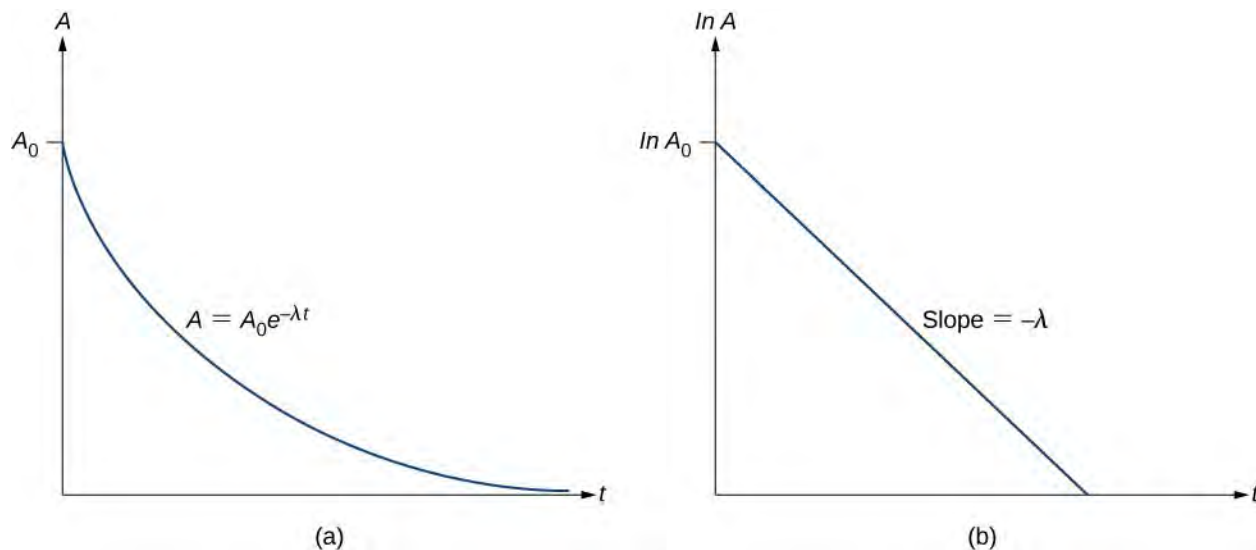


Figure 10.10 (a) A plot of the activity as a function of time (b) If we measure the activity at different times, we can plot $\ln A$ versus t , and obtain a straight line.



EXAMPLE 10.4

Decay Constant and Activity of Strontium-90

The half-life of strontium-90, $^{90}_{38}\text{Sr}$, is 28.8 y. Find (a) its decay constant and (b) the initial activity of 1.00 g of the material.

Strategy

We can find the decay constant directly from [Equation 10.15](#). To determine the activity, we first need to find the number of nuclei present.

Solution

- a. The decay constant is found to be

$$\lambda = \frac{0.693}{T_{1/2}} = \left(\frac{0.693}{T_{1/2}} \right) \left(\frac{1 \text{ yr}}{3.16 \times 10^7 \text{ s}} \right) = 7.61 \times 10^{-10} \text{ s}^{-1}.$$

- b. The atomic mass of $^{90}_{38}\text{Sr}$ is 89.91 g. Using Avogadro's number $N_A = 6.022 \times 10^{23}$ atoms/mol, we find the initial number of nuclei in 1.00 g of the material:

$$N_0 = \frac{1.00 \text{ g}}{89.91 \text{ g}} (N_A) = 6.70 \times 10^{21} \text{ nuclei}.$$

From this, we find that the activity A_0 at $t = 0$ for 1.00 g of strontium-90 is

$$\begin{aligned} A_0 &= \lambda N_0 \\ &= (7.61 \times 10^{-10} \text{ s}^{-1})(6.70 \times 10^{21} \text{ nuclei}) \\ &= 5.10 \times 10^{12} \text{ decays/s.} \end{aligned}$$

Expressing λ in terms of the half-life of the substance, we get

$$A = A_0 e^{-(0.693/T_{1/2})T_{1/2}} = A_0 e^{-0.693} = A_0/2. \quad 10.19$$

Therefore, the activity is halved after one half-life. We can determine the decay constant λ by measuring the activity as a function of time. Taking the natural logarithm of the left and right sides of [Equation 10.17](#), we get

$$\ln A = -\lambda t + \ln A_0. \quad 10.20$$

This equation follows the linear form $y = mx + b$. If we plot $\ln A$ versus t , we expect a straight line with slope $-\lambda$ and y -intercept $\ln A_0$ ([Figure 10.10\(b\)](#)). Activity A is expressed in units of **becquerels** (Bq), where one 1 Bq = 1 decay per second. This quantity can also be expressed in decays per minute or decays per year. One of the most common units for activity is the **curie (Ci)**, defined to be the activity of 1 g of ^{226}Ra . The relationship between the Bq and Ci is

$$1 \text{ Ci} = 3.70 \times 10^{10} \text{ Bq}.$$



EXAMPLE 10.5

What is ^{14}C Activity in Living Tissue?

Approximately 20% of the human body by mass is carbon. Calculate the activity due to ^{14}C in 1.00 kg of carbon found in a living organism. Express the activity in units of Bq and Ci.

Strategy

The activity of ^{14}C is determined using the equation $A_0 = \lambda N_0$, where λ is the decay constant and N_0 is the number of radioactive nuclei. The number of ^{14}C nuclei in a 1.00-kg sample is determined in two steps. First, we determine the number of ^{12}C nuclei using the concept of a mole. Second, we multiply this value by

1.3×10^{-12} (the known abundance of ^{14}C in a carbon sample from a living organism) to determine the number of ^{14}C nuclei in a living organism. The decay constant is determined from the known half-life of ^{14}C (available from [Figure 10.4](#)).

Solution

One mole of carbon has a mass of 12.0 g, since it is nearly pure ^{12}C . Thus, the number of carbon nuclei in a kilogram is

$$N(^{12}\text{C}) = \frac{6.02 \times 10^{23} \text{ mol}^{-1}}{12.0 \text{ g/mol}} \times (1000 \text{ g}) = 5.02 \times 10^{25}.$$

The number of ^{14}C nuclei in 1 kg of carbon is therefore

$$N(^{14}\text{C}) = (5.02 \times 10^{25}) (1.3 \times 10^{-12}) = 6.52 \times 10^{13}.$$

Now we can find the activity A by using the equation $A = \frac{0.693 N}{t_{1/2}}$. Entering known values gives us

$$A = \frac{0.693 (6.52 \times 10^{13})}{5730 \text{ y}} = 7.89 \times 10^9 \text{ y}^{-1}$$

or 7.89×10^9 decays per year. To convert this to the unit Bq, we simply convert years to seconds. Thus,

$$A = (7.89 \times 10^9 \text{ y}^{-1}) \frac{1.00 \text{ y}}{3.16 \times 10^7 \text{ s}} = 250 \text{ Bq},$$

or 250 decays per second. To express A in curies, we use the definition of a curie,

$$A = \frac{250 \text{ Bq}}{3.7 \times 10^{10} \text{ Bq/Ci}} = 6.76 \times 10^{-9} \text{ Ci}.$$

Thus,

$$A = 6.76 \text{ nCi}.$$

Significance

Approximately 20% of the human body by weight is carbon. Hundreds of ^{14}C decays take place in the human body every second. Carbon-14 and other naturally occurring radioactive substances in the body compose a person's background exposure to nuclear radiation. As we will see later in this chapter, this activity level is well below the maximum recommended dosages.

Radioactive Dating

Radioactive dating is a technique that uses naturally occurring radioactivity to determine the age of a material, such as a rock or an ancient artifact. The basic approach is to estimate the original number of nuclei in a material and the present number of nuclei in the material (after decay), and then use the known value of the decay constant λ and [Equation 10.10](#) to calculate the total time of the decay, t .

An important method of radioactive dating is **carbon-14 dating**. Carbon-14 nuclei are produced when high-energy solar radiation strikes ^{14}N nuclei in the upper atmosphere and subsequently decay with a half-life of 5730 years. Radioactive carbon has the same chemistry as stable carbon, so it combines with the ecosphere and eventually becomes part of every living organism. Carbon-14 has an abundance of 1.3 parts per trillion of normal carbon. Therefore, if you know the number of carbon nuclei in an object, you multiply that number by 1.3×10^{-12} to find the number of ^{14}C nuclei in that object. When an organism dies, carbon exchange with the environment ceases, and ^{14}C is not replenished as it decays.

By comparing the abundance of ^{14}C in an artifact, such as mummy wrappings, with the normal abundance in living tissue, it is possible to determine the mummy's age (or the time since the person's death). Carbon-14 dating can be used for biological tissues as old as 50,000 years, but is generally most accurate for younger samples, since the abundance of ^{14}C nuclei in them is greater. Very old biological materials contain no ^{14}C at

all. The validity of carbon dating can be checked by other means, such as by historical knowledge or by tree-ring counting.



EXAMPLE 10.6

An Ancient Burial Cave

In an ancient burial cave, your team of archaeologists discovers ancient wood furniture. Only 80% of the original ^{14}C remains in the wood. How old is the furniture?

Strategy

The problem statement implies that $N/N_0 = 0.80$. Therefore, the equation $N = N_0 e^{-\lambda t}$ can be used to find the product, λt . We know the half-life of ^{14}C is 5730 y, so we also know the decay constant, and therefore the total decay time t .

Solution

Solving the equation $N = N_0 e^{-\lambda t}$ for N/N_0 gives us

$$\frac{N}{N_0} = e^{-\lambda t}.$$

Thus,

$$0.80 = e^{-\lambda t}.$$

Taking the natural logarithm of both sides of the equation yields

$$\ln 0.80 = -\lambda t,$$

so that

$$-0.223 = -\lambda t.$$

Rearranging the equation to isolate t gives us

$$t = \frac{0.223}{\lambda},$$

where

$$\lambda = \frac{0.693}{t_{1/2}} = \frac{0.693}{5730 \text{ y}}.$$

Combining this information yields

$$t = \frac{0.223}{\left(\frac{0.693}{5730 \text{ y}}\right)} = 1844 \text{ y}.$$

Significance

The furniture is almost 2000 years old—an impressive discovery. The typical uncertainty on carbon-14 dating is about 5%, so the furniture is anywhere between 1750 and 1950 years old. This date range must be confirmed by other evidence, such as historical records.



CHECK YOUR UNDERSTANDING 10.3

A radioactive nuclide has a high decay rate. What does this mean for its half-life and activity?



INTERACTIVE

Visit the [Radioactive Dating Game \(https://openstax.org/l/21raddatgame\)](https://openstax.org/l/21raddatgame) to learn about the types of

radiometric dating and try your hand at dating some ancient objects.

10.4 Nuclear Reactions

Learning Objectives

By the end of this section, you will be able to:

- Describe and compare three types of nuclear radiation
- Use nuclear symbols to describe changes that occur during nuclear reactions
- Describe processes involved in the decay series of heavy elements

Early experiments revealed three types of nuclear “rays” or radiation: **alpha (α) rays**, **beta (β) rays**, and **gamma (γ) rays**. These three types of radiation are differentiated by their ability to penetrate matter. Alpha radiation is barely able to pass through a thin sheet of paper. Beta radiation can penetrate aluminum to a depth of about 3 mm, and gamma radiation can penetrate lead to a depth of 2 or more centimeters ([Figure 10.11](#)).

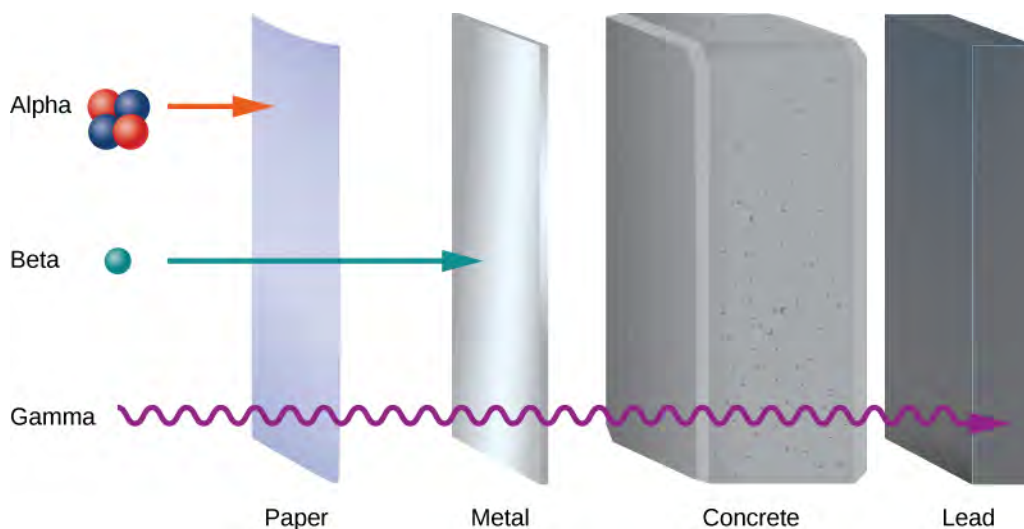


Figure 10.11 A comparison of the penetration depths of alpha (α), beta (β), and gamma (γ) radiation through various materials.

The electrical properties of these three types of radiation are investigated by passing them through a uniform magnetic field, as shown in [Figure 10.12](#). According to the magnetic force equation $\vec{F} = q\vec{v} \times \vec{B}$, positively charged particles are deflected upward, negatively charged particles are deflected downward, and particles with no charge pass through the magnetic field undeflected. Eventually, α rays were identified with helium nuclei (${}^4\text{He}$), β rays with electrons and **positrons** (positively charged electrons or **antielectrons**), and γ rays with high-energy photons. We discuss alpha, beta, and gamma radiation in detail in the remainder of this section.

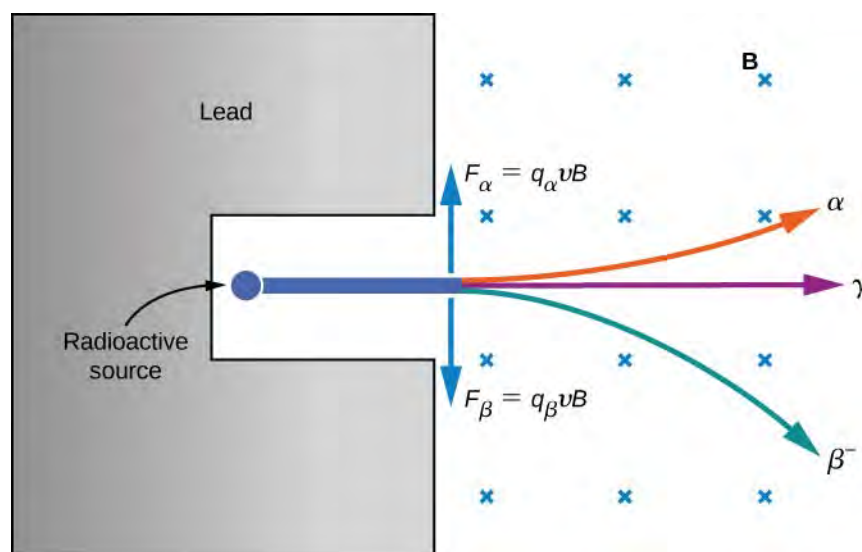
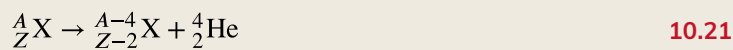


Figure 10.12 The effect of a magnetic field on alpha (α), beta (β), and gamma (γ) radiation. This figure is a schematic only. The relative paths of the particles depend on their masses and initial kinetic energies.

Alpha Decay

Heavy unstable nuclei emit α radiation. In α -particle decay (or **alpha decay**), the nucleus loses two protons and two neutrons, so the atomic number decreases by two, whereas its mass number decreases by four. Before the decay, the nucleus is called the **parent nucleus**. The nucleus or nuclei produced in the decay are referred to as the **daughter nucleus** or daughter nuclei. We represent an α decay symbolically by

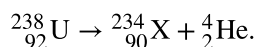


where ${}^A_Z\text{X}$ is the parent nucleus, ${}^{A-4}_{Z-2}\text{X}$ is the daughter nucleus, and ${}^4_2\text{He}$ is the α particle. In α decay, a nucleus of atomic number Z decays into a nucleus of atomic number $Z - 2$ and atomic mass $A - 4$. Interestingly, the dream of the ancient alchemists to turn other metals into gold is scientifically feasible through the alpha-decay process. The efforts of the alchemists failed because they relied on chemical interactions rather than nuclear interactions.

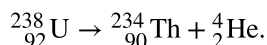
INTERACTIVE

Watch alpha particles escape from a polonium nucleus, causing radioactive alpha decay. See how random decay times relate to the half-life. To try a simulation of alpha decay, visit [alpha particles \(https://openstax.org/l/21alphapartid\)](https://openstax.org/l/21alphapartid)

An example of alpha decay is uranium-238:



The atomic number has dropped from 92 to 90. The chemical element with $Z = 90$ is thorium. Hence, Uranium-238 has decayed to Thorium-234 by the emission of an α particle, written



Subsequently, ${}^{234}_{90}\text{Th}$ decays by β emission with a half-life of 24 days. The energy released in this alpha decay takes the form of kinetic energies of the thorium and helium nuclei, although the kinetic energy of thorium is smaller than helium due to its heavier mass and smaller velocity.



EXAMPLE 10.7

Plutonium Alpha Decay

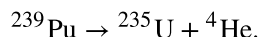
Find the energy emitted in the α decay of ^{239}Pu .

Strategy

The energy emitted in the α decay of ^{239}Pu can be found using the equation $E = (\Delta m) c^2$. We must first find Δm , the difference in mass between the parent nucleus and the products of the decay.

Solution

The decay equation is



Thus, the pertinent masses are those of ^{239}Pu , ^{235}U , and the α particle or ^4He , all of which are known. The initial mass was $m(^{239}\text{Pu}) = 239.052157 \text{ u}$. The final mass is the sum

$$\begin{aligned} m(^{235}\text{U}) + m(^4\text{He}) &= 235.043924 \text{ u} + 4.002602 \text{ u} \\ &= 239.046526 \text{ u}. \end{aligned}$$

Thus,

$$\begin{aligned} \Delta m &= m(^{239}\text{Pu}) - [m(^{235}\text{U}) + m(^4\text{He})] \\ &= 239.052157 \text{ u} - 239.046526 \text{ u} \\ &= 0.005631 \text{ u}. \end{aligned}$$

Now we can find E by entering Δm into the equation:

$$E = (\Delta m) c^2 = (0.005631 \text{ u}) c^2.$$

We know $1 \text{ u} = 931.5 \text{ MeV}/c^2$, so we have

$$\begin{aligned} E &= (0.005631) (931.5 \text{ MeV}/c^2) (c^2) \\ &= 5.25 \text{ MeV}. \end{aligned}$$

Significance

The energy released in this α decay is in the MeV range, many times greater than chemical reaction energies. Most of this energy becomes kinetic energy of the α particle (or ^4He nucleus), which moves away at high speed. The energy carried away by the recoil of the ^{235}U nucleus is much smaller due to its relatively large mass. The ^{235}U nucleus can be left in an excited state to later emit photons (γ rays).

Beta Decay

In most β particle decays (or **beta decay**), either an electron (β^-) or positron (β^+) is emitted by a nucleus. A positron has the same mass as the electron, but its charge is $+e$. For this reason, a positron is sometimes called an antielectron. How does β decay occur? A possible explanation is the electron (positron) is confined to the nucleus prior to the decay and somehow escapes. To obtain a rough estimate of the escape energy, consider a simplified model of an electron trapped in a box (or in the terminology of quantum mechanics, a one-dimensional square well) that has the width of a typical nucleus (10^{-14} m). According to the Heisenberg uncertainty principle in [Quantum Mechanics](#), the uncertainty of the momentum of the electron is:

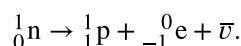
$$\Delta p > \frac{h}{\Delta x} = \frac{6.6 \times 10^{-34} \text{ m}^2 \cdot \text{kg/s}}{10^{-14} \text{ m}} = 6.6 \times 10^{-20} \text{ kg} \cdot \text{m/s}.$$

Taking this momentum value (an underestimate) to be the “true value,” the kinetic energy of the electron on escape is approximately

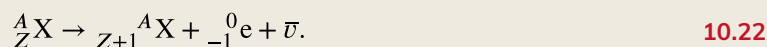
$$\frac{(\Delta p)^2}{2m_e} = \frac{(6.6 \times 10^{-20} \text{ kg} \cdot \text{m/s})^2}{2(9.1 \times 10^{-31} \text{ kg})} = 2.0 \times 10^{-9} \text{ J} = 12,400 \text{ MeV}.$$

Experimentally, the electrons emitted in β^- decay are found to have kinetic energies of the order of only a few MeV. We therefore conclude that the electron is somehow produced in the decay rather than escaping the nucleus. Particle production (annihilation) is described by theories that combine quantum mechanics and relativity, a subject of a more advanced course in physics.

Nuclear beta decay involves the conversion of one nucleon into another. For example, a neutron can decay to a proton by the emission of an electron (β^-) and a nearly massless particle called an **antineutrino** ($\bar{\nu}$):



The notation ${}_{-1}^0\text{e}$ is used to designate the electron. Its mass number is 0 because it is not a nucleon, and its atomic number is -1 to signify that it has a charge of $-e$. The proton is represented by ${}_1^1\text{p}$ because its mass number and atomic number are 1. When this occurs within an atomic nucleus, we have the following equation for beta decay:

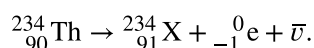


As discussed in another chapter, this process occurs due to the weak nuclear force.

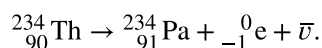
INTERACTIVE

Watch [beta decay \(https://openstax.org/l/21betadecayvid\)](https://openstax.org/l/21betadecayvid) occur for a collection of nuclei or for an individual nucleus.

As an example, the isotope ${}_{90}^{234}\text{Th}$ is unstable and decays by β^- emission with a half-life of 24 days. Its decay can be represented as

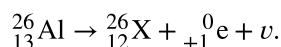


Since the chemical element with atomic number 91 is protactinium (Pa), we can write the β^- decay of thorium as

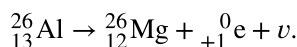


The reverse process is also possible: A proton can decay to a neutron by the emission of a positron (β^+) and a nearly massless particle called a **neutrino** (ν). This reaction is written as ${}_1^1\text{p} \rightarrow {}_0^1\text{n} + {}_{+1}^0\text{e} + \nu$.

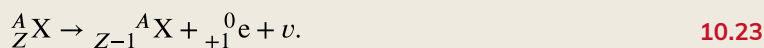
The positron ${}_{+1}^0\text{e}$ is emitted with the neutrino ν , and the neutron remains in the nucleus. (Like β^- decay, the positron does not precede the decay but is produced in the decay.) For an isolated proton, this process is impossible because the neutron is heavier than the proton. However, this process is possible within the nucleus because the proton can receive energy from other nucleons for the transition. As an example, the isotope of aluminum ${}_{13}^{26}\text{Al}$ decays by β^+ emission with a half-life of 7.40×10^5 y. The decay is written as



The atomic number 12 corresponds to magnesium. Hence,



As a nuclear reaction, positron emission can be written as



The neutrino was not detected in the early experiments on β decay. However, the laws of energy and momentum seemed to require such a particle. Later, neutrinos were detected through their interactions with nuclei.



EXAMPLE 10.8

Bismuth Alpha and Beta Decay

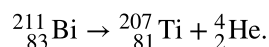
The ${}_{83}^{211}\text{Bi}$ nucleus undergoes both α and β^- decay. For each case, what is the daughter nucleus?

Strategy

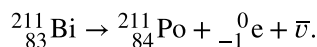
We can use the processes described by [Equation 10.21](#) and [Equation 10.22](#), as well as the Periodic Table, to identify the resulting elements.

Solution

The atomic number and the mass number for the α particle are 2 and 4, respectively. Thus, when a bismuth-211 nucleus emits an α particle, the daughter nucleus has an atomic number of 81 and a mass number of 207. The element with an atomic number of 81 is thallium, so the decay is given by



In β^- decay, the atomic number increases by 1, while the mass number stays the same. The element with an atomic number of 84 is polonium, so the decay is given by

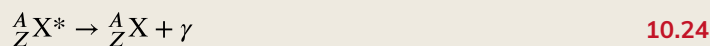


CHECK YOUR UNDERSTANDING 10.4

In radioactive beta decay, does the atomic mass number, A , increase or decrease?

Gamma Decay

A nucleus in an excited state can decay to a lower-level state by the emission of a “gamma-ray” photon, and this is known as **gamma decay**. This is analogous to de-excitation of an atomic electron. Gamma decay is represented symbolically by

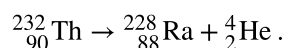


where the asterisk (*) on the nucleus indicates an excited state. In γ decay, neither the atomic number nor the mass number changes, so the type of nucleus does not change.

Radioactive Decay Series

Nuclei with $Z > 82$ are unstable and decay naturally. Many of these nuclei have very short lifetimes, so they are not found in nature. Notable exceptions include ${}_{90}^{232}\text{Th}$ (or Th-232) with a half-life of 1.39×10^{10} years, and ${}_{92}^{238}\text{U}$ (or U-238) with a half-life of 7.04×10^8 years. When a heavy nucleus decays to a lighter one, the lighter daughter nucleus can become the parent nucleus for the next decay, and so on. This process can produce a long series of nuclear decays called a **decay series**. The series ends with a stable nucleus.

To illustrate the concept of a decay series, consider the decay of Th-232 series ([Figure 10.13](#)). The neutron number, N , is plotted on the vertical y -axis, and the atomic number, Z , is plotted on the horizontal x -axis, so Th-232 is found at the coordinates $(N, Z) = (142, 90)$. Th-232 decays by α emission with a half-life of 1.39×10^{10} years. Alpha decay decreases the atomic number by 2 and the mass number by 4, so we have



The neutron number for Radium-228 is 140, so it is found in the diagram at the coordinates $(N, Z) = (140, 88)$. Radium-228 is also unstable and decays by β^- emission with a half-life of 5.76 years to Actinium-228. The atomic number increases by 1, the mass number remains the same, and the neutron number decreases by 1. Notice that in the graph, α emission appears as a line sloping downward to the left, with both N and Z decreasing by 2. Beta emission, on the other hand, appears as a line sloping downward to the right with N decreasing by 1, and Z increasing by 1. After several additional alpha and beta decays, the series ends with the stable nucleus Pb-208.

The relative frequency of different types of radioactive decays (alpha, beta, and gamma) depends on many factors, including the strength of the forces involved and the number of ways a given reaction can occur without violating the conservation of energy and momentum. How often a radioactive decay occurs often depends on a sensitive balance of the strong and electromagnetic forces. These forces are discussed in [Particle Physics and Cosmology](#).

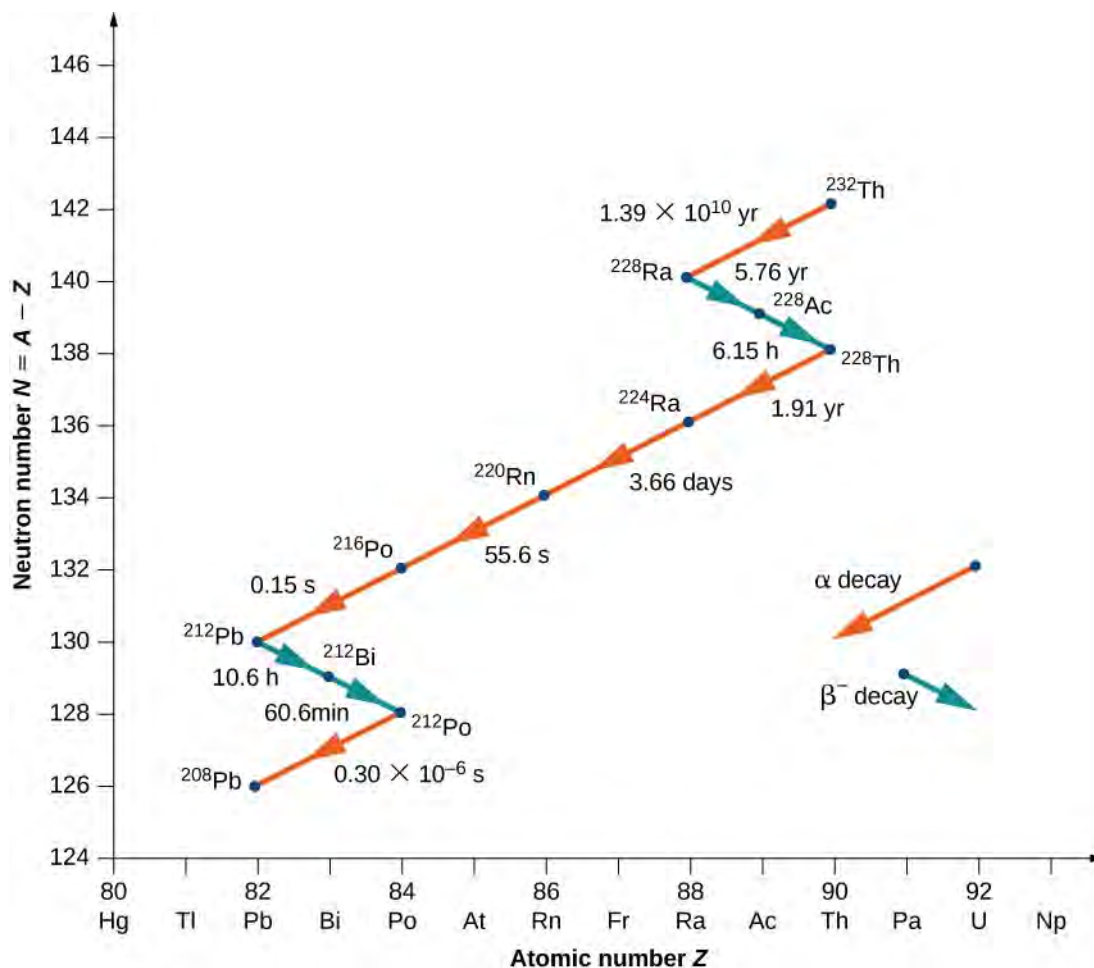


Figure 10.13 In the thorium ${}_{90}^{232}\text{Th}$ decay series, alpha (α) decays reduce the atomic number, as indicated by the red arrows. Beta (β^-) decays increase the atomic number, as indicated by the blue arrows. The series ends at the stable nucleus Pb-208.

As another example, consider the U-238 decay series shown in [Figure 10.14](#). After numerous alpha and beta decays, the series ends with the stable nucleus Pb-206. An example of a decay whose parent nucleus no longer exists naturally is shown in [Figure 10.15](#). It starts with Neptunium-237 and ends in the stable nucleus Bismuth-209. Neptunium is called a **transuranic element** because it lies beyond uranium in the periodic table. Uranium has the highest atomic number ($Z = 92$) of any element found in nature. Elements with $Z > 92$ can be produced only in the laboratory. They most probably also existed in nature at the time of the formation of Earth, but because of their relatively short lifetimes, they have completely decayed. There is

nothing fundamentally different between naturally occurring and artificial elements.

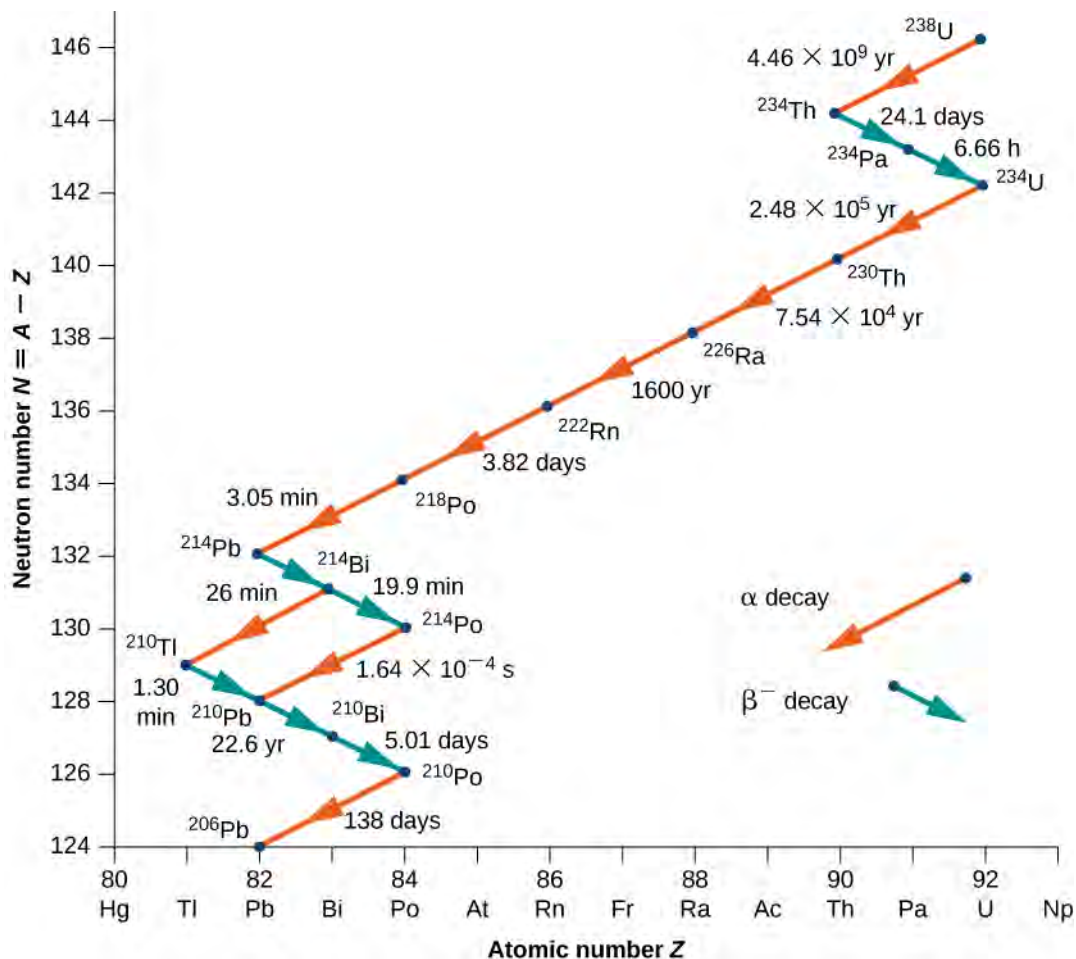


Figure 10.14 In the Uranium-238 decay series, alpha (α) decays reduce the atomic number, as indicated by the red arrows. Beta (β^-) decays increase the atomic number, as indicated by the blue arrows. The series ends at the stable nucleus Pb-206.

Notice that for Bi (21), the decay may proceed through either alpha or beta decay.

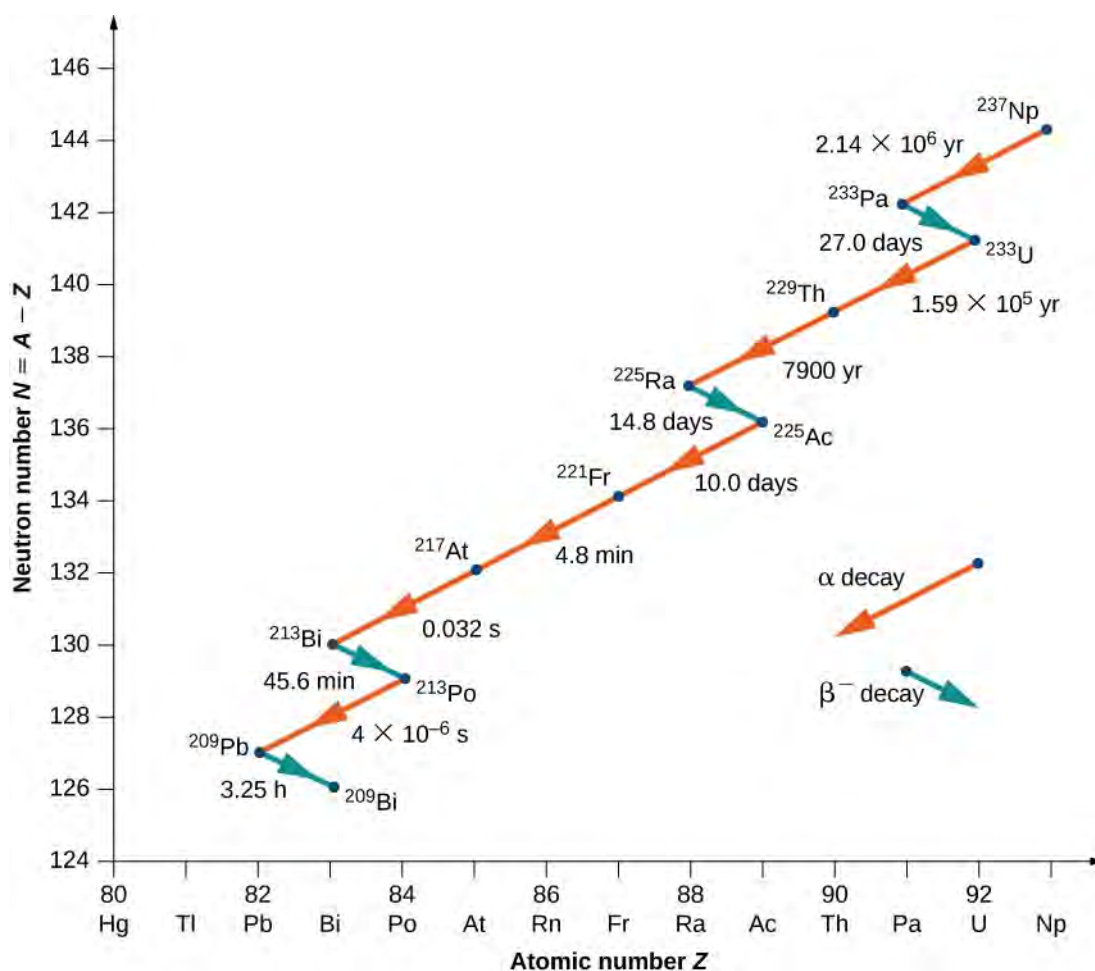


Figure 10.15 In the Neptunium-237 decay series, alpha (α) decays reduce the atomic number, as indicated by the red arrows. Beta (β^-) decays increase the atomic number, as indicated by the blue arrows. The series ends at the stable nucleus Bi-209.

Radioactivity in the Earth

According to geologists, if there were no heat source, Earth should have cooled to its present temperature in no more than 1 billion years. Yet, Earth is more than 4 billion years old. Why is Earth cooling so slowly? The answer is nuclear radioactivity, that is, high-energy particles produced in radioactive decays heat Earth from the inside (Figure 10.16).

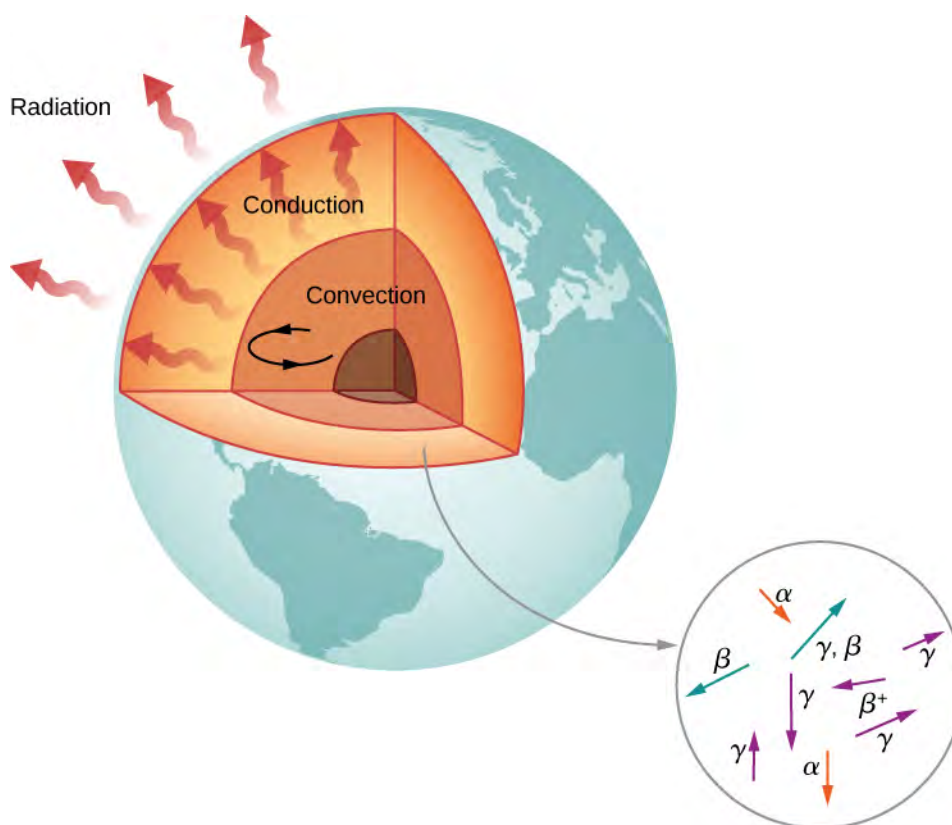


Figure 10.16 Earth is heated by nuclear reactions (alpha, beta, and gamma decays). Without these reactions, Earth's core and mantle would be much cooler than it is now.

Candidate nuclei for this heating model are ^{238}U and ^{40}K , which possess half-lives similar to or longer than the age of Earth. The energy produced by these decays (per second per cubic meter) is small, but the energy cannot escape easily, so Earth's core is very hot. Thermal energy in Earth's core is transferred to Earth's surface and away from it through the processes of convection, conduction, and radiation.

10.5 Fission

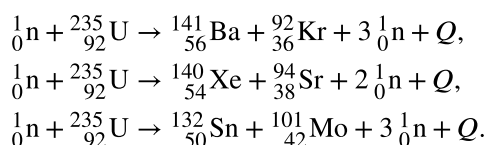
Learning Objectives

By the end of this section, you will be able to:

- Describe the process of nuclear fission in terms of its product and reactants
- Calculate the energies of particles produced by a fission reaction
- Explain the fission concept in the context of fission bombs and nuclear reactions

In 1934, Enrico Fermi bombarded chemical elements with neutrons in order to create isotopes of other elements. He assumed that bombarding uranium with neutrons would make it unstable and produce a new element. Unfortunately, Fermi could not determine the products of the reaction. Several years later, Otto Hahn and Fritz Strassman reproduced these experiments and discovered that the products of these reactions were smaller nuclei. From this, they concluded that the uranium nucleus had split into two smaller nuclei.

The splitting of a nucleus is called **fission**. Interestingly, U-235 fission does not always produce the same fragments. Example fission reactions include:



In each case, the sum of the masses of the product nuclei are less than the masses of the reactants, so the fission of uranium is an exothermic process ($Q > 0$). This is the idea behind the use of fission reactors as

sources of energy (Figure 10.17). The energy carried away by the reaction takes the form of particles with kinetic energy. The percent yield of fragments from a U-235 fission is given in Figure 10.18.



Figure 10.17 The Phillipsburg Nuclear Power Plant in Germany uses a fission reactor to generate electricity.

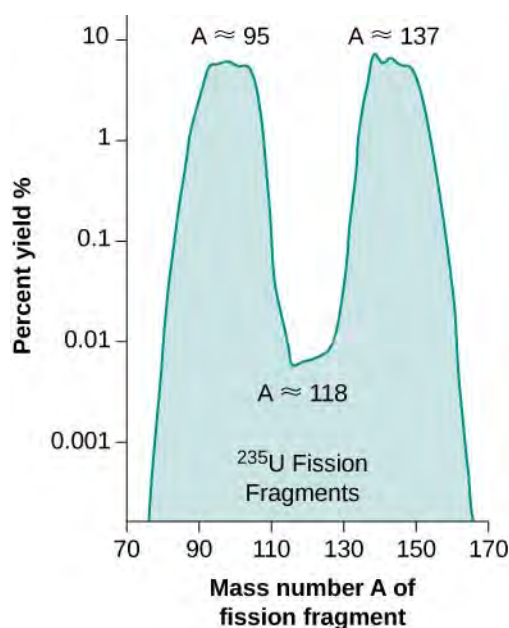


Figure 10.18 In this graph of fission fragments from U-235, the peaks in the graph indicate nuclei that are produced in the greatest abundance by the fission process.

Energy changes in a nuclear fission reaction can be understood in terms of the binding energy per nucleon curve (Figure 10.7). The BEN value for uranium ($A = 236$) is slightly lower than its daughter nuclei, which lie closer to the iron (Fe) peak. This means that nucleons in the nuclear fragments are more tightly bound than

those in the U-235 nucleus. Therefore, a fission reaction results in a drop in the average energy of a nucleon. This energy is carried away by high-energy neutrons.

Niels Bohr and John Wheeler developed the **liquid drop model** to understand the fission process. According to this model, firing a neutron at a nucleus is analogous to disturbing a droplet of water ([Figure 10.19](#)). The analogy works because short-range forces between nucleons in a nucleus are similar to the attractive forces between water molecules in a water droplet. In particular, forces between nucleons at the surface of the nucleus result in a surface tension similar to that of a water droplet. A neutron fired into a uranium nucleus can set the nucleus into vibration. If this vibration is violent enough, the nucleus divides into smaller nuclei and also emits two or three individual neutrons.

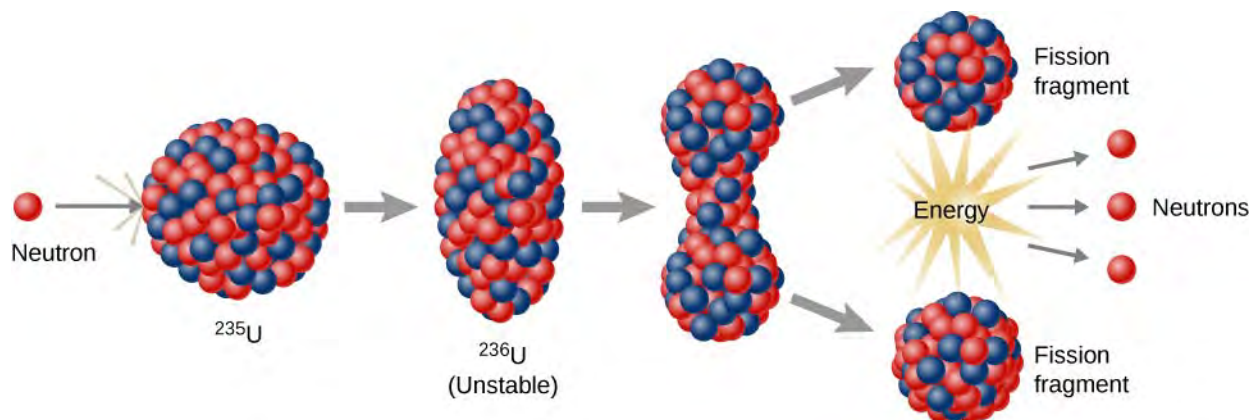


Figure 10.19 In the liquid drop model of nuclear fission, the uranium nucleus is split into two lighter nuclei by a high-energy neutron.

U-235 fission can produce a chain reaction. In a compound consisting of many U-235 nuclei, neutrons in the decay of one U-235 nucleus can initiate the fission of additional U-235 nuclei ([Figure 10.20](#)). This chain reaction can proceed in a controlled manner, as in a nuclear reactor at a power plant, or proceed uncontrollably, as in an explosion.

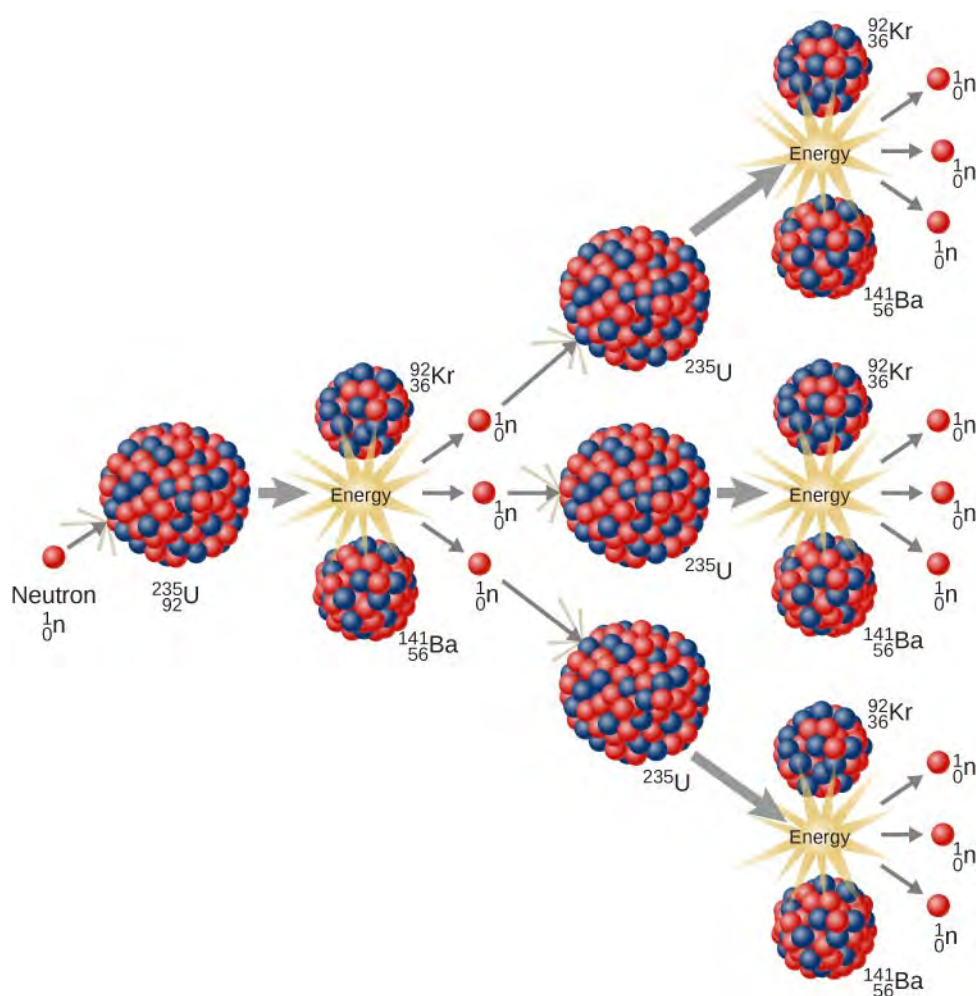


Figure 10.20 In a U-235 fission chain reaction, the fission of the uranium nucleus produces high-energy neutrons that go on to split more nuclei. The energy released in this process can be used to produce electricity.

INTERACTIVE

View a simulation on [nuclear fission \(https://openstax.org/l/21nuclrfissvid\)](https://openstax.org/l/21nuclrfissvid) to start a chain reaction, or introduce nonradioactive isotopes to prevent one. Control energy production in a nuclear reactor.

The Atomic Bomb

The possibility of a chain reaction in uranium, with its extremely large energy release, led nuclear scientists to conceive of making a bomb—an atomic bomb. (These discoveries were taking place in the years just prior to the Second World War and many of the European physicists involved in these discoveries came from countries that were being overrun.) Natural uranium contains 99.3% U-238 and only 0.7% U-235, and does not produce a chain reaction. To produce a controlled, sustainable chain reaction, the percentage of U-235 must be increased to about 50%. In addition, the uranium sample must be massive enough so a typical neutron is more likely to induce fission than it is to escape. The minimum mass needed for the chain reaction to occur is called the **critical mass**. When the critical mass reaches a point at which the chain reaction becomes self-sustaining, this is a condition known as **criticality**. The original design required two pieces of U-235 below the critical mass. When one piece in the form of a bullet is fired into the second piece, the critical mass is exceeded and a chain reaction is produced.

An important obstacle to the U-235 bomb is the production of a critical mass of fissionable material. Therefore, scientists developed a plutonium-239 bomb because Pu-239 is more fissionable than U-235 and thus requires a smaller critical mass. The bomb was made in the form of a sphere with pieces of plutonium, each below the

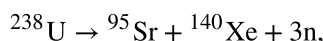
critical mass, at the edge of the sphere. A series of chemical explosions fired the plutonium pieces toward the center of the sphere simultaneously. When all these pieces of plutonium came together, the combination exceeded the critical mass and produced a chain reaction. Both the U-235 and Pu-239 bombs were used in World War II. Whether to develop and use atomic weapons remain two of the most important questions faced by human civilization.



EXAMPLE 10.9

Calculating Energy Released by Fission

Calculate the energy released in the following rare spontaneous fission reaction:



The atomic masses are $m(^{238}\text{U}) = 238.050784 \text{ u}$, $m(^{95}\text{Sr}) = 94.919388 \text{ u}$, $m(^{140}\text{Xe}) = 139.921610 \text{ u}$, and $m(\text{n}) = 1.008665 \text{ u}$.

Strategy

As always, the energy released is equal to the mass destroyed times c^2 , so we must find the difference in mass between the parent ^{238}U and the fission products.

Solution

The products have a total mass of

$$\begin{aligned} m_{\text{products}} &= 94.919388 \text{ u} + 139.921610 \text{ u} + 3(1.008665 \text{ u}) \\ &= 237.866993 \text{ u}. \end{aligned}$$

The mass lost is the mass of $^{238}\text{U} - m_{\text{products}}$ or

$$\Delta m = 238.050784 \text{ u} - 237.866993 \text{ u} = 0.183791 \text{ u}.$$

Therefore, the energy released is

$$E = (\Delta m) c^2 = (0.183791 \text{ u}) \frac{931.5 \text{ MeV}/c^2}{\text{u}} c^2 = 171.2 \text{ MeV}.$$

Significance

Several important things arise in this example. The energy release is large but less than it would be if the nucleus split into two equal parts, since energy is carried away by neutrons. However, this fission reaction produces neutrons and does not split the nucleus into two equal parts. Fission of a given nuclide, such as ^{238}U , does not always produce the same products. Fission is a statistical process in which an entire range of products are produced with various probabilities. Most fission produces neutrons, although the number varies. This is an extremely important aspect of fission, because *neutrons can induce more fission*, enabling self-sustaining chain reactions.

Fission Nuclear Reactors

The first nuclear reactor was built by Enrico Fermi on a squash court on the campus of the University of Chicago on December 2, 1942. The reactor itself contained U-238 enriched with 3.6% U-235. Neutrons produced by the chain reaction move too fast to initiate fission reactions. One way to slow them down is to enclose the entire reactor in a water bath under high pressure. The neutrons collide with the water molecules and are slowed enough to be used in the fission process. The slowed neutrons split more U-235 nuclei and a chain reaction occurs. The rate at which the chain reaction proceeds is controlled by a series of “control” rods made of cadmium inserted into the reactor. Cadmium is capable of absorbing a large number of neutrons without becoming unstable.

A nuclear reactor design, called a pressurized water reactor, can also be used to generate electricity ([Figure 10.21](#)). A pressurized water reactor (on the left in the figure) is designed to control the fission of large amounts

of ^{235}U . The energy released in this process is absorbed by water flowing through pipes in the system (the “primary loop”) and steam is produced. Cadmium control rods adjust the neutron flux (the rate of flow of neutrons passing through the system) and therefore control the reaction. In case the reactor overheats and the water boils away, the chain reaction terminates, because water is used to thermalize the neutrons. (This safety feature can be overwhelmed in extreme circumstances.) The hot, high-pressure water then passes through a pipe to a second tank of water at normal pressure in the steam generator. The steam produced at one end of the steam generator fills a chamber that contains a turbine. This steam is at a very high pressure. Meanwhile, a steam condenser connected to the other side of the turbine chamber maintains steam at low pressure. The pressure differences force steam through the chamber, which turns the turbine. The turbine, in turn, powers an electric generator.

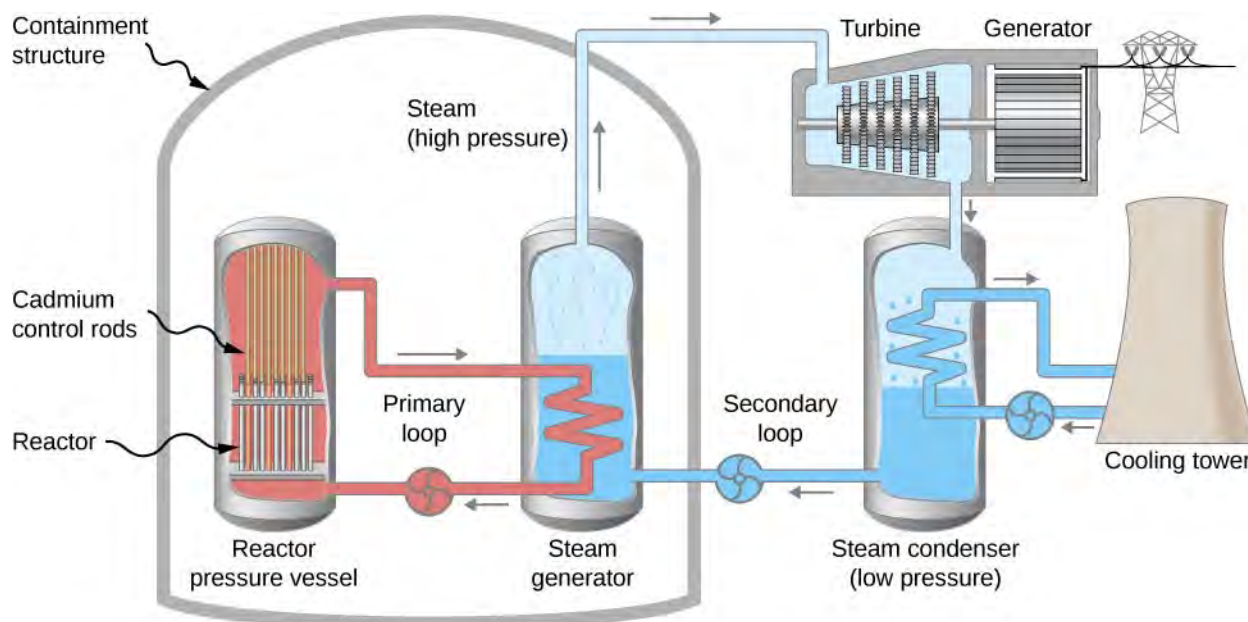
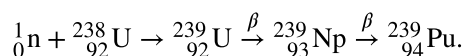


Figure 10.21 A nuclear reactor uses the energy produced in the fission of U-235 to produce electricity. Energy from a nuclear fission reaction produces hot, high-pressure steam that turns a turbine. As the turbine turns, electricity is produced.

The major drawback to a fission reactor is nuclear waste. U-235 fission produces nuclei with long half-lives such as ^{236}U that must be stored. These products cannot be dumped into oceans or left in any place where they will contaminate the environment, such as through the soil, air, or water. Many scientists believe that the best place to store nuclear waste is the bottom of old salt mines or inside of stable mountains.

Many people are fearful that a nuclear reactor may explode like an atomic bomb. However, a nuclear reactor does not contain enough U-235 to do this. Also, a nuclear reactor is designed so that failure of any mechanism of the reactor causes the cadmium control rods to fall fully into the reactor, stopping the fission process. As evidenced by the Fukushima and Chernobyl disasters, such systems can fail. Systems and procedures to avoid such disasters is an important priority for advocates of nuclear energy.

If all electrical power were produced by nuclear fission of U-235, Earth’s known reserves of uranium would be depleted in less than a century. However, Earth’s supply of fissionable material can be expanded considerably using a **breeder reactor**. A breeder reactor operates for the first time using the fission of U-235 as just described for the pressurized water reactor. But in addition to producing energy, some of the fast neutrons originating from the fission of U-235 are absorbed by U-238, resulting in the production of Pu-239 via the set of reactions



The Pu-239 is itself highly fissionable and can therefore be used as a nuclear fuel in place of U-235. Since 99.3% of naturally occurring uranium is the U-238 isotope, the use of breeder reactors should increase our supply of nuclear fuel by roughly a factor of 100. Breeder reactors are now in operation in Great Britain,

France, and Russia. Breeder reactors also have drawbacks. First, breeder reactors produce plutonium, which can, if leaked into the environment, produce serious public health problems. Second, plutonium can be used to build bombs, thus increasing significantly the risk of nuclear proliferation.



EXAMPLE 10.10

Calculating Energy of Fissionable Fuel

Calculate the amount of energy produced by the fission of 1.00 kg of ^{235}U given that the average fission reaction of ^{235}U produces 200 MeV.

Strategy

The total energy produced is the number of ^{235}U atoms times the given energy per ^{235}U fission. We should therefore find the number of ^{235}U atoms in 1.00 kg.

Solution

The number of ^{235}U atoms in 1.00 kg is Avogadro's number times the number of moles. One mole of ^{235}U has a mass of 235.04 g; thus, there are $(1000 \text{ g}) / (235.04 \text{ g/mol}) = 4.25 \text{ mol}$. The number of ^{235}U atoms is therefore

$$(4.25 \text{ mol}) (6.02 \times 10^{23} \text{ } ^{235}\text{U/mol}) = 2.56 \times 10^{24} \text{ } ^{235}\text{U}.$$

Thus, the total energy released is

$$E = (2.56 \times 10^{24} \text{ } ^{235}\text{U}) \left(\frac{200 \text{ MeV}}{^{235}\text{U}} \right) \left(\frac{1.60 \times 10^{-13} \text{ J}}{\text{MeV}} \right) = 8.21 \times 10^{13} \text{ J}.$$

Significance

This is another impressively large amount of energy, equivalent to about 14,000 barrels of crude oil or 600,000 gallons of gasoline. However, it is only one-fourth the energy produced by the fusion of a kilogram mixture of deuterium and tritium. Even though each fission reaction yields about 10 times the energy of a fusion reaction, the energy per kilogram of fission fuel is less, because there are far fewer moles per kilogram of the heavy nuclides. Fission fuel is also much scarcer than fusion fuel, and less than 1% of uranium (the ^{235}U) is readily usable.



CHECK YOUR UNDERSTANDING 10.5

Which has a larger energy yield per fission reaction, a large or small sample of pure ^{235}U ?

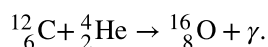
10.6 Nuclear Fusion

Learning Objectives

By the end of this section, you will be able to:

- Describe the process of nuclear fusion in terms of its product and reactants
- Calculate the energies of particles produced by a fusion reaction
- Explain the fission concept in the context of fusion bombs, the production of energy by the Sun, and nucleosynthesis

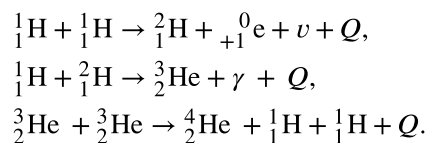
The process of combining lighter nuclei to make heavier nuclei is called **nuclear fusion**. As with fission reactions, fusion reactions are exothermic—they release energy. Suppose that we fuse a carbon and helium nuclei to produce oxygen:



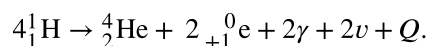
The energy changes in this reaction can be understood using a graph of binding energy per nucleon ([Figure 10.7](#)). Comparing the binding energy per nucleon for oxygen, carbon, and helium, the oxygen nucleus is much

more tightly bound than the carbon and helium nuclei, indicating that the reaction produces a drop in the energy of the system. This energy is released in the form of gamma radiation. Fusion reactions are said to be exothermic when the amount of energy released (known as the Q value) in each reaction is greater than zero ($Q > 0$).

An important example of nuclear fusion in nature is the production of energy in the Sun. In 1938, Hans Bethe proposed that the Sun produces energy when hydrogen nuclei (${}^1\text{H}$) fuse into stable helium nuclei (${}^4\text{He}$) in the Sun's core (Figure 10.22). This process, called the **proton-proton chain**, is summarized by three reactions:



Thus, a stable helium nucleus is formed from the fusion of the nuclei of the hydrogen atom. These three reactions can be summarized by



The net Q value is about 26 MeV. The release of this energy produces an outward thermal gas pressure that prevents the Sun from gravitational collapse. Astrophysicists find that hydrogen fusion supplies the energy stars require to maintain energy balance over most of a star's life span.

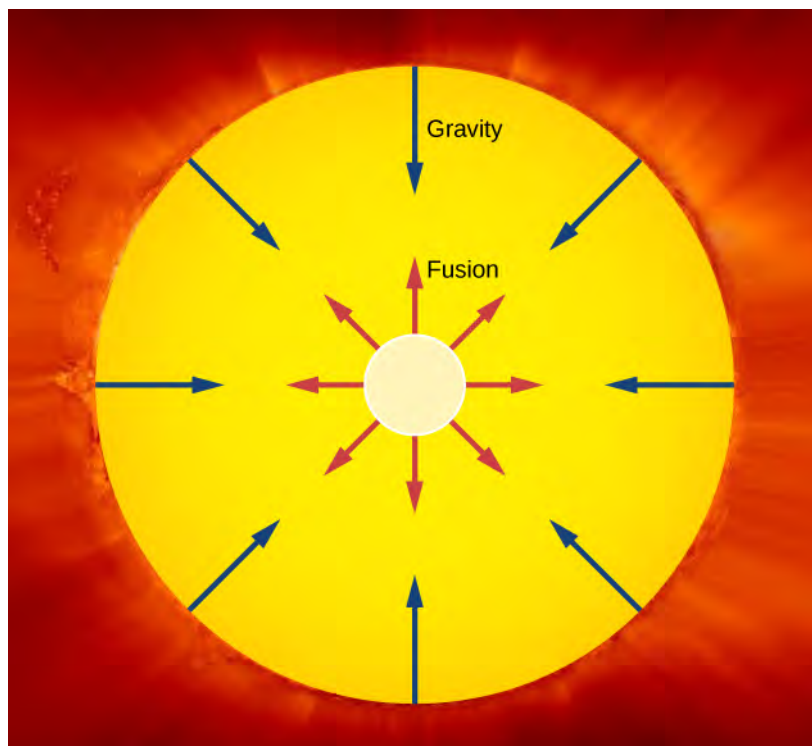
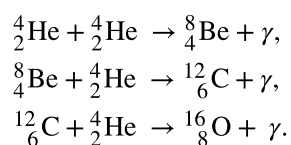


Figure 10.22 The Sun produces energy by fusing hydrogen into helium at the Sun's core. The red arrows show outward pressure due to thermal gas, which tends to make the Sun expand. The blue arrows show inward pressure due to gravity, which tends to make the Sun contract. These two influences balance each other.

Nucleosynthesis

Scientists now believe that many heavy elements found on Earth and throughout the universe were originally synthesized by fusion within the hot cores of the stars. This process is known as **nucleosynthesis**. For example, in lighter stars, hydrogen combines to form helium through the proton-proton chain. Once the hydrogen fuel is exhausted, the star enters the next stage of its life and fuses helium. An example of a nuclear reaction chain that can occur is:



Carbon and oxygen nuclei produced in such processes eventually reach the star's surface by convection. Near the end of its lifetime, the star loses its outer layers into space, thus enriching the interstellar medium with the nuclei of heavier elements ([Figure 10.23](#)).

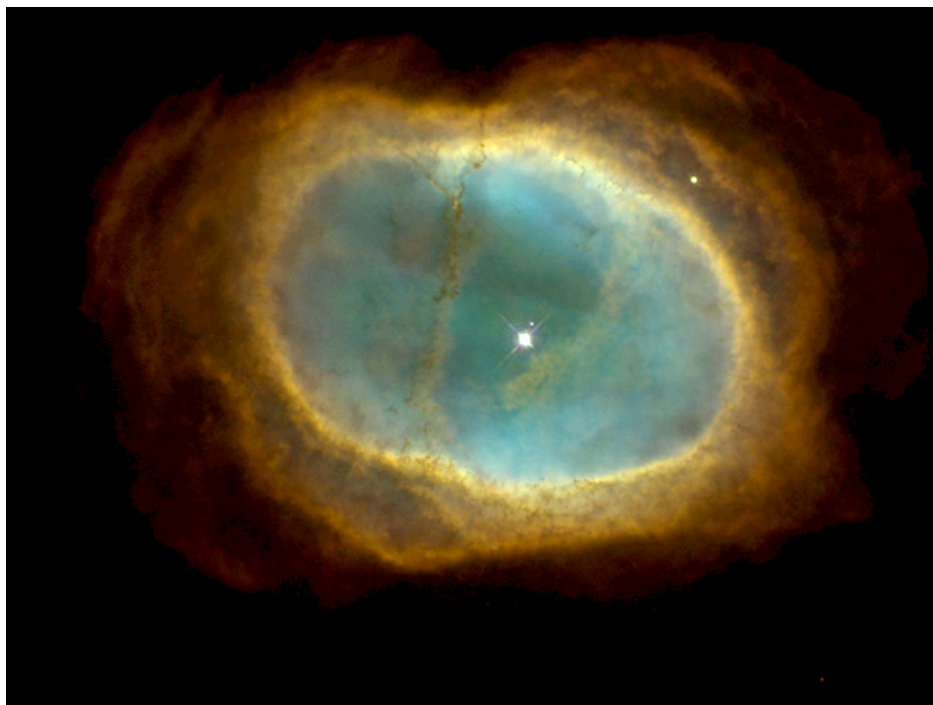
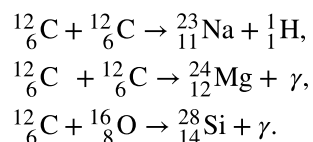


Figure 10.23 A planetary nebula is produced at the end of the life of a star. The greenish color of this planetary nebula comes from oxygen ions. (credit: Hubble Heritage Team (STScI/AURA/NASA/ESA))

Stars similar in mass to the Sun do not become hot enough to fuse nuclei as heavy (or heavier) than oxygen nuclei. However, in massive stars whose cores become much hotter ($T > 6 \times 10^8 \text{ K}$), even more complex nuclei are produced. Some representative reactions are



Nucleosynthesis continues until the core is primarily iron-nickel metal. Now, iron has the peculiar property that any fusion or fission reaction involving the iron nucleus is endothermic, meaning that energy is absorbed rather than produced. Hence, nuclear energy cannot be generated in an iron-rich core. Lacking an outward pressure from fusion reactions, the star begins to contract due to gravity. This process heats the core to a temperature on the order of $5 \times 10^9 \text{ K}$. Expanding shock waves generated within the star due to the collapse cause the star to quickly explode. The luminosity of the star can increase temporarily to nearly that of an entire galaxy. During this event, the flood of energetic neutrons reacts with iron and the other nuclei to produce elements heavier than iron. These elements, along with much of the star, are ejected into space by the explosion. Supernovae and the formation of planetary nebulas together play a major role in the dispersal of chemical elements into space.

Eventually, much of the material lost by stars is pulled together through the gravitational force, and it condenses into a new generation of stars and accompanying planets. Recent images from the Hubble Space Telescope provide a glimpse of this magnificent process taking place in the constellation Serpens ([Figure](#)

10.24). The new generation of stars begins the nucleosynthesis process anew, with a higher percentage of heavier elements. Thus, stars are “factories” for the chemical elements, and many of the atoms in our bodies were once a part of stars.



Figure 10.24 This image taken by NASA’s Spitzer Space Telescope and the Two Micron All Sky Survey (2MASS), shows the Serpens Cloud Core, a star-forming region in the constellation Serpens (the “Serpent”). Located about 750 light-years away, this cluster of stars is formed from cooling dust and gases. Infrared light has been used to reveal the youngest stars in orange and yellow. (credit: NASA/JPL-Caltech/2MASS)



EXAMPLE 10.11

Energy of the Sun

The power output of the Sun is approximately 3.8×10^{26} J/s. Most of this energy is produced in the Sun’s core by the proton-proton chain. This energy is transmitted outward by the processes of convection and radiation. (a) How many of these fusion reactions per second must occur to supply the power radiated by the Sun? (b) What is the rate at which the mass of the Sun decreases? (c) In about five billion years, the central core of the Sun will be depleted of hydrogen. By what percentage will the mass of the Sun have decreased from its present value when the core is depleted of hydrogen?

Strategy

The total energy output per second is given in the problem statement. If we know the energy released in each fusion reaction, we can determine the rate of the fusion reactions. If the mass loss per fusion reaction is known, the mass loss rate is known. Multiplying this rate by five billion years gives the total mass lost by the Sun. This value is divided by the original mass of the Sun to determine the percentage of the Sun’s mass that has been lost when the hydrogen fuel is depleted.

Solution

- a. The decrease in mass for the fusion reaction is

$$\begin{aligned}\Delta m &= 4m({}_1^1\text{H}) - m({}_2^4\text{He}) - 2m({}_{+1}^0\text{e}) \\ &= 4(1.007825 \text{ u}) - 4.002603 \text{ u} - 2(0.000549 \text{ u}) \\ &= 0.0276 \text{ u}.\end{aligned}$$

The energy released per fusion reaction is

$$Q = (0.0276 \text{ u})(931.49 \text{ MeV/u}) = 25.7 \text{ MeV}.$$

Thus, to supply 3.8×10^{26} J/s = 2.38×10^{39} MeV/s, there must be

$$\frac{2.38 \times 10^{39} \text{ MeV/s}}{25.7 \text{ MeV/reaction}} = 9.26 \times 10^{37} \text{ reaction/s}.$$

- b. The Sun’s mass decreases by $0.0276 \text{ u} = 4.58 \times 10^{-29} \text{ kg}$ per fusion reaction, so the rate at which its mass decreases is

$$(9.26 \times 10^{37} \text{ reaction/s}) (4.58 \times 10^{-29} \text{ kg/reaction}) = 4.24 \times 10^9 \text{ kg/s}.$$

c. In $5 \times 10^9 \text{ y} = 1.6 \times 10^{17} \text{ s}$, the Sun's mass will therefore decrease by

$$\Delta M = (4.24 \times 10^9 \text{ kg/s}) (1.6 \times 10^{17} \text{ s}) = 6.8 \times 10^{26} \text{ kg}.$$

The current mass of the Sun is about $2.0 \times 10^{30} \text{ kg}$, so the percentage decrease in its mass when its hydrogen fuel is depleted will be

$$\left(\frac{6.8 \times 10^{26} \text{ kg}}{2.0 \times 10^{30} \text{ kg}} \right) \times 100\% = 0.034\%.$$

Significance

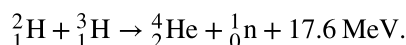
After five billion years, the Sun is very nearly the same mass as it is now. Hydrogen burning does very little to change the mass of the Sun. This calculation assumes that only the proton-proton decay change is responsible for the power output of the Sun.

CHECK YOUR UNDERSTANDING 10.6

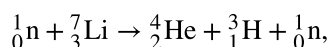
Where does the energy from the Sun originate?

The Hydrogen Bomb

In 1942, Robert Oppenheimer suggested that the extremely high temperature of an atomic bomb could be used to trigger a fusion reaction between deuterium and tritium, thus producing a fusion (or hydrogen) bomb. The reaction between deuterium and tritium, both isotopes of hydrogen, is given by



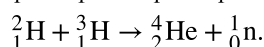
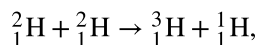
Deuterium is relatively abundant in ocean water but tritium is scarce. However, tritium can be generated in a nuclear reactor through a reaction involving lithium. The neutrons from the reactor cause the reaction



to produce the desired tritium. The first hydrogen bomb was detonated in 1952 on the remote island of Eniwetok in the Marshall Islands. A hydrogen bomb has never been used in war. Modern hydrogen bombs are approximately 1000 times more powerful than the fission bombs dropped on Hiroshima and Nagasaki in World War II.

The Fusion Reactor

The fusion chain believed to be the most practical for use in a **nuclear fusion reactor** is the following two-step process:



This chain, like the proton-proton chain, produces energy without any radioactive by-product. However, there is a very difficult problem that must be overcome before fusion can be used to produce significant amounts of energy: Extremely high temperatures ($\sim 10^7 \text{ K}$) are needed to drive the fusion process. To meet this challenge, test fusion reactors are being developed to withstand temperatures 20 times greater than the Sun's core temperature. An example is the Joint European Torus (JET) shown in [Figure 10.25](#). A great deal of work still has to be done on fusion reactor technology, but many scientists predict that fusion energy will power the world's cities by the end of the twentieth century.



Figure 10.25 The Joint European Torus (JET) tokamak fusion detector uses magnetic fields to fuse deuterium and tritium nuclei (credit: EUROfusion).

10.7 Medical Applications and Biological Effects of Nuclear Radiation

Learning Objectives

By the end of this section, you will be able to:

- Describe two medical uses of nuclear technology
- Explain the origin of biological effects due to nuclear radiation
- List common sources of radiation and their effects
- Estimate exposure for nuclear radiation using common dosage units

Nuclear physics is an integral part of our everyday lives ([Figure 10.26](#)). Radioactive compounds are used to identify cancer, study ancient artifacts, and power our cities. Nuclear fusion also powers the Sun, the primary source of energy on Earth. The focus of this chapter is nuclear radiation. In this section, we ask such questions as: How is nuclear radiation used to benefit society? What are its health risks? How much nuclear radiation is the average person exposed to in a lifetime?



Figure 10.26 Dr. Tori Randall, a curator at the San Diego Museum of Man, uses nuclear radiation to study a 500-year-old Peruvian child mummy. The origin of this radiation is the transformation of one nucleus to another. (credit: Samantha A. Lewis, U.S. Navy)

Medical Applications

Medical use of nuclear radiation is quite common in today's hospitals and clinics. One of the most important uses of nuclear radiation is the location and study of diseased tissue. This application requires a special drug called a **radiopharmaceutical**. A radiopharmaceutical contains an unstable radioactive isotope. When the drug enters the body, it tends to concentrate in inflamed regions of the body. (Recall that the interaction of the drug with the body does not depend on whether a given nucleus is replaced by one of its isotopes, since this interaction is determined by chemical interactions.) Radiation detectors used outside the body use nuclear radiation from the radioisotopes to locate the diseased tissue. Radiopharmaceuticals are called **radioactive tags** because they allow doctors to track the movement of drugs in the body. Radioactive tags are for many purposes, including the identification of cancer cells in the bones, brain tumors, and Alzheimer's disease ([Figure 10.27](#)). Radioactive tags are also used to monitor the function of body organs, such as blood flow, heart muscle activity, and iodine uptake in the thyroid gland.

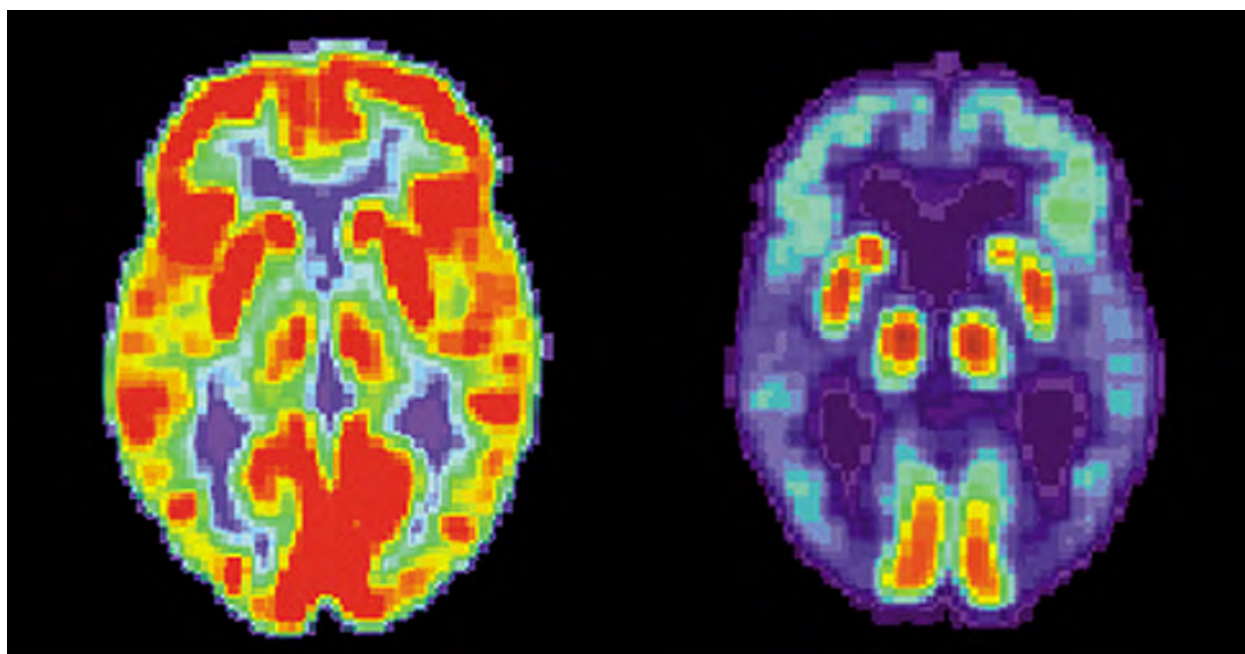


Figure 10.27 These brain images are produced using a radiopharmaceutical. The colors indicate relative metabolic or biochemical activity (red indicates high activity and blue indicates low activity). The figure on the left shows the normal brain of an individual and the figure on

the right shows the brain of someone diagnosed with Alzheimer's disease. The brain image of the normal brain indicates much greater metabolic activity (a larger fraction of red and orange areas). (credit: modification of works by National Institutes of Health)

[Table 10.2](#) lists some medical diagnostic uses of radiopharmaceuticals, including isotopes and typical activity (A) levels. One common diagnostic test uses iodine to image the thyroid, since iodine is concentrated in that organ. Another common nuclear diagnostic is the thallium scan for the cardiovascular system, which reveals blockages in the coronary arteries and examines heart activity. The salt TlCl can be used because it acts like NaCl and follows the blood. Note that [Table 10.2](#) lists many diagnostic uses for $^{99\text{m}}\text{Tc}$, where “m” stands for a metastable state of the technetium nucleus. This isotope is used in many compounds to image the skeleton, heart, lungs, and kidneys. About 80% of all radiopharmaceuticals employ $^{99\text{m}}\text{Tc}$ because it produces a single, easily identified, 0.142-MeV γ ray and has a short 6.0-h half-life, which reduces radiation exposure.

Procedure, Isotope		Activity (mCi), where 1 mCi = 3.7×10^7 Bq	Procedure, Isotope		Activity (mCi), where 1 mCi = 3.7×10^7 Bq
<i>Brain scan</i>			<i>Thyroid scan</i>		
$^{99\text{m}}\text{Tc}$	7.5		^{131}I	0.05	
^{15}O (PET)	50		^{123}I	0.07	
<i>Lung scan</i>			<i>Liver scan</i>		
^{133}Xe	7.5		^{198}Au (colloid)	0.1	
$^{99\text{m}}\text{Tc}$	2		$^{99\text{m}}\text{Tc}$ (colloid)	2	
<i>Cardiovascular blood pool</i>			<i>Bone scan</i>		
^{131}I	0.2		^{85}Sr	0.1	
$^{99\text{m}}\text{Tc}$	2		$^{99\text{m}}\text{Tc}$	10	
<i>Cardiovascular arterial flow</i>			<i>Kidney scan</i>		
^{201}Tl	3		^{197}Hg	0.1	
^{24}Na	7.5		$^{99\text{m}}\text{Tc}$	1.5	

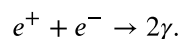
Table 10.2 Diagnostic Uses of Radiopharmaceuticals

The first radiation detectors produced two-dimensional images, like a photo taken from a camera. However, a circular array of detectors that can be rotated can be used to produce three-dimensional images. This technique is similar to that used in X-ray computed tomography (CT) scans. One application of this technique is called **single-photon-emission CT (SPECT)** ([Figure 10.28](#)). The spatial resolution of this technique is about 1 cm.



Figure 10.28 The SPECT machine uses radiopharmaceutical compounds to produce an image of the human body. The machine takes advantage of the physics of nuclear beta decays and electron-positron collisions. (credit: "Woldo"/Wikimedia Commons)

Improved image resolution is achieved by a technique known as **positron emission tomography (PET)**. This technique uses radioisotopes that decay by β^+ radiation. When a positron encounters an electron, these particles annihilate to produce two gamma-ray photons. This reaction is represented by



These γ -ray photons have identical 0.511-MeV energies and move directly away from one another ([Figure 10.29](#)). This easily identified decay signature can be used to identify the location of the radioactive isotope. Examples of β^+ -emitting isotopes used in PET include ^{11}C , ^{13}N , ^{15}O , and ^{18}F . The nuclei have the advantage of being able to function as tags for natural body compounds. Its resolution of 0.5 cm is better than that of SPECT.

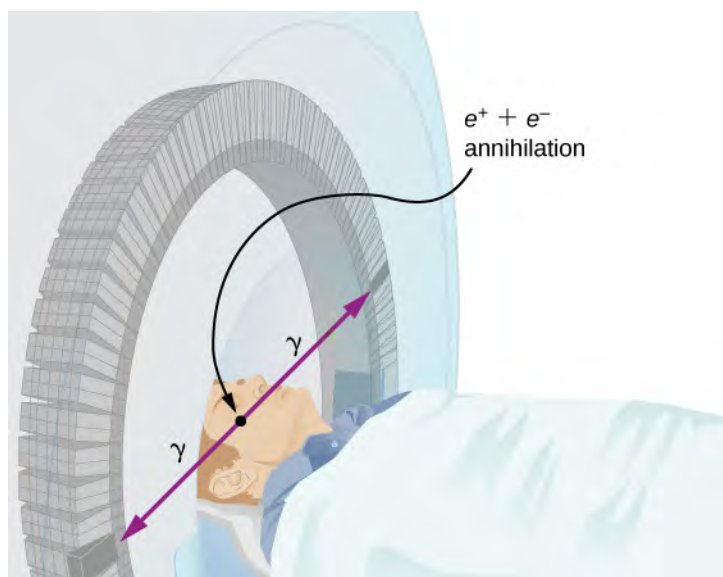


Figure 10.29 A PET system takes advantage of the two identical γ -ray photons produced by positron-electron annihilation. These γ rays are emitted in opposite directions, so that the line along which each pair is emitted is determined.

PET scans are especially useful to examine the brain's anatomy and function. For example, PET scans can be used to monitor the brain's use of oxygen and water, identify regions of decreased metabolism (linked to Alzheimer's disease), and locate different parts of the brain responsible for sight, speech, and fine motor activity

INTERACTIVE

Is it a tumor? View an [animation \(https://openstax.org/l/21simmagresimg\)](https://openstax.org/l/21simmagresimg) of simplified magnetic resonance imaging (MRI) to see if you can tell. Your head is full of tiny radio transmitters (the nuclear spins of the hydrogen nuclei of your water molecules). In an MRI unit, these little radios can be made to broadcast their positions, giving a detailed picture of the inside of your head.

Biological Effects

Nuclear radiation can have both positive and negative effects on biological systems. However, it can also be used to treat and even cure cancer. How do we understand these effects? To answer this question, consider molecules within cells, particularly DNA molecules.

Cells have long, double-helical DNA molecules containing chemical codes that govern the function and processes of the cell. Nuclear radiation can alter the structural features of the DNA chain, leading to changes in the genetic code. In human cells, we can have as many as a million individual instances of damage to DNA per cell per day. DNA contains codes that check whether the DNA is damaged and can repair itself. This repair ability of DNA is vital for maintaining the integrity of the genetic code and for the normal functioning of the entire organism. It should be constantly active and needs to respond rapidly. The rate of DNA repair depends on various factors such as the type and age of the cell. If nuclear radiation damages the ability of the cell to repair DNA, the cell can

1. Retreat to an irreversible state of dormancy (known as senescence);
2. Commit suicide (known as programmed cell death); or
3. Progress into unregulated cell division, possibly leading to tumors and cancers.

Nuclear radiation can harm the human body in many other ways as well. For example, high doses of nuclear radiation can cause burns and even hair loss.

Biological effects of nuclear radiation are expressed by many different physical quantities and in many different units. A common unit to express the biological effects of nuclear radiation is the **rad** or **radiation dose unit**. One rad is equal to 1/100 of a joule of nuclear energy deposited per kilogram of tissue, written:

$$1 \text{ rad} = 0.01 \text{ J/kg}.$$

For example, if a 50.0-kg person is exposed to nuclear radiation over her entire body and she absorbs 1.00 J, then her whole-body radiation dose is

$$(1.00 \text{ J}) / (50.0 \text{ kg}) = 0.0200 \text{ J/kg} = 2.00 \text{ rad}.$$

Nuclear radiation damages cells by ionizing atoms in the cells as they pass through the cells ([Figure 10.30](#)). The effects of ionizing radiation depend on the dose in rads, but also on the type of radiation (alpha, beta, gamma, or X-ray) and the type of tissue. For example, if the range of the radiation is small, as it is for α rays, then the ionization and the damage created is more concentrated and harder for the organism to repair. To account for such effects, we define the **relative biological effectiveness** (RBE). Sample RBE values for several types of ionizing nuclear radiation are given in [Table 10.3](#).

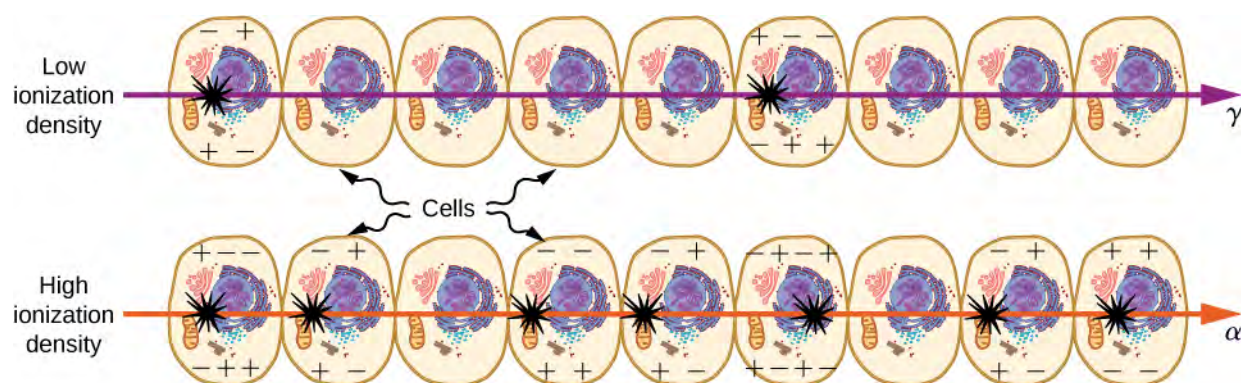


Figure 10.30 The image shows ionization created in cells by α and γ radiation. Because of its shorter range, the ionization and damage created by α rays is more concentrated and harder for the organism to repair. Thus, the RBE for α rays is greater than the RBE for γ rays, even though they create the same amount of ionization at the same energy.

Type and Energy of Radiation	RBE ^[1]
X-rays	1
γ rays	1
β rays greater than 32 keV	1
β rays less than 32 keV	1.7
Neutrons, thermal to slow (<20 keV)	2–5
Neutrons, fast (1–10 MeV)	10 (body), 32 (eyes)
Protons (1–10 MeV)	10 (body), 32 (eyes)
α rays from radioactive decay	10–20
Heavy ions from accelerators	10–20

Table 10.3 Relative Biological Effectiveness ^[1] Values approximate. Difficult to determine.

A dose unit more closely related to effects in biological tissue is called the **roentgen equivalent man (rem)** and is defined to be the dose (in rads) multiplied by the relative biological effectiveness (RBE). Thus, if a person had a whole-body dose of 2.00 rad of γ radiation, the dose in rem would be $(2.00 \text{ rad})(1) = 2.00 \text{ rem}$ for the whole body. If the person had a whole-body dose of 2.00 rad of α radiation, then the dose in rem would be $(2.00 \text{ rad})(20) = 40.0 \text{ rem}$ for the whole body. The α rays would have 20 times the effect on the person than the γ rays for the same deposited energy. The SI equivalent of the rem, and the more standard term, is the **sievert (Sv)** is

$$1 \text{ Sv} = 100 \text{ rem.}$$

The RBEs given in [Table 10.3](#) are approximate but reflect an understanding of nuclear radiation and its interaction with living tissue. For example, neutrons are known to cause more damage than γ rays, although both are neutral and have large ranges, due to secondary radiation. Any dose less than 100 mSv (10 rem) is called a **low dose**, 0.1 Sv to 1 Sv (10 to 100 rem) is called a **moderate dose**, and anything greater than 1 Sv (100 rem) is called a **high dose**. It is difficult to determine if a person has been exposed to less than 10 mSv.

Biological effects of different levels of nuclear radiation on the human body are given in [Table 10.4](#). The first

clue that a person has been exposed to radiation is a change in blood count, which is not surprising since blood cells are the most rapidly reproducing cells in the body. At higher doses, nausea and hair loss are observed, which may be due to interference with cell reproduction. Cells in the lining of the digestive system also rapidly reproduce, and their destruction causes nausea. When the growth of hair cells slows, the hair follicles become thin and break off. High doses cause significant cell death in all systems, but the lowest doses that cause fatalities do so by weakening the immune system through the loss of white blood cells.

Dose in Sv ^[1]	Effect
0–0.10	No observable effect.
0.1–1	Slight to moderate decrease in white blood cell counts.
0.5	Temporary sterility; 0.35 for women, 0.50 for men.
1–2	Significant reduction in blood cell counts, brief nausea and vomiting. Rarely fatal.
2–5	Nausea, vomiting, hair loss, severe blood damage, hemorrhage, fatalities.
4.5	Lethal to 50% of the population within 32 days after exposure if not treated.
5–20	Worst effects due to malfunction of small intestine and blood systems. Limited survival.
>20	Fatal within hours due to collapse of central nervous system.

Table 10.4 Immediate Effects of Radiation (Adults, Whole-Body, Single Exposure) ^[1] Multiply by 100 to obtain dose in rem.

Sources of Radiation

Human are also exposed to many sources of nuclear radiation. A summary of average radiation doses for different sources by country is given in [Table 10.5](#). Earth emits radiation due to the isotopes of uranium, thorium, and potassium. Radiation levels from these sources depend on location and can vary by a factor of 10. Fertilizers contain isotopes of potassium and uranium, which we digest in the food we eat. Fertilizers have more than 3000 Bq/kg radioactivity, compared to just 66 Bq/kg for Carbon-14.

Source	Dose (mSv/y) ^[1]			
	Australia	Germany	US	World
Natural radiation – external				
Cosmic rays	0.30	0.28	0.30	0.39
Soil, building materials	0.40	0.40	0.30	0.48
Radon gas	0.90	1.1	2.0	1.2
Natural radiation – internal				
⁴⁰ K, ¹⁴ C, ²²⁶ Ra	0.24	0.28	0.40	0.29

Source	Dose (mSv/y) ^[1]			
Artificial radiation				
Medical and dental	0.80	0.90	0.53	0.40
TOTAL	2.6	3.0	3.5	2.8

Table 10.5 Background Radiation Sources and Average Doses ^[1] Multiply by 100 to obtain does in mrem/y.

Medical visits are also a source of nuclear radiation. A sample of common nuclear radiation doses is given in [Table 10.6](#). These doses are generally low and can be lowered further with improved techniques and more sensitive detectors. With the possible exception of routine dental X-rays, medical use of nuclear radiation is used only when the risk-benefit is favorable. Chest X-rays give the lowest doses—about 0.1 mSv to the tissue affected, with less than 5% scattering into tissues that are not directly imaged. Other X-ray procedures range upward to about 10 mSv in a CT scan, and about 5 mSv (0.5 rem) per dental X-ray, again both only affecting the tissue imaged. Medical images with radiopharmaceuticals give doses ranging from 1 to 5 mSv, usually localized.

Procedure	Effective Dose (mSv)
Chest	0.02
Dental	0.01
Skull	0.07
Leg	0.02
Mammogram	0.40
Barium enema	7.0
Upper GI	3.0
CT head	2.0
CT abdomen	10.0

Table 10.6 Typical Doses Received During Diagnostic X-Ray Exams



EXAMPLE 10.12

What Mass of ^{137}Cs Escaped Chernobyl?

The Chernobyl accident in Ukraine (formerly in the Soviet Union) exposed the surrounding population to a large amount of radiation through the decay of ^{137}Cs . The initial radioactivity level was approximately $A = 6.0 \text{ MCi}$. Calculate the total mass of ^{137}Cs involved in this accident.

Strategy

The total number of nuclei, N , can be determined from the known half-life and activity of ^{137}Cs (30.2 y). The mass can be calculated from N using the concept of a mole.

Solution

Solving the equation $A = \frac{0.693 N}{t_{1/2}}$ for N gives

$$N = \frac{A t_{1/2}}{0.693}.$$

Entering the given values yields

$$N = \frac{(6.0 \text{ MCi}) (30.2 \text{ y})}{0.693}.$$

To convert from curies to becquerels and years to seconds, we write

$$N = \frac{(6.0 \times 10^6 \text{ Ci}) (3.7 \times 10^{10} \text{ Bq/Ci}) (30.2 \text{ y}) (3.16 \times 10^7 \text{ s/y})}{0.693} = 3.1 \times 10^{26}.$$

One mole of a nuclide ${}^A\text{X}$ has a mass of A grams, so that one mole of ${}^{137}\text{Cs}$ has a mass of 137 g. A mole has 6.02×10^{23} nuclei. Thus the mass of ${}^{137}\text{Cs}$ released was

$$m = \left(\frac{137 \text{ g}}{6.02 \times 10^{23}} \right) (3.1 \times 10^{26}) = 70 \times 10^3 \text{ g} = 70 \text{ kg}.$$

Significance

The mass of ${}^{137}\text{Cs}$ involved in the Chernobyl accident is a small material compared to the typical amount of fuel used in a nuclear reactor. However, approximately 250 people were admitted to local hospitals immediately after the accident, and diagnosed as suffering acute radiation syndrome. They received external radiation dosages between 1 and 16 Sv. Referring to biological effects in [Table 10.4](#), these dosages are extremely hazardous. The eventual death toll is estimated to be around 4000 people, primarily due to radiation-induced cancer.

 CHECK YOUR UNDERSTANDING 10.7

Radiation propagates in all directions from its source, much as electromagnetic radiation from a light bulb. Is *activity* concept more analogous to power, intensity, or brightness?

CHAPTER REVIEW

Key Terms

activity magnitude of the decay rate for radioactive nuclides

alpha (α) rays one of the types of rays emitted from the nucleus of an atom as alpha particles

alpha decay radioactive nuclear decay associated with the emission of an alpha particle

antielectrons another term for positrons

antineutrino antiparticle of an electron's neutrino in β^- decay

atomic mass total mass of the protons, neutrons, and electrons in a single atom

atomic mass unit unit used to express the mass of an individual nucleus, where
 $1\text{u} = 1.66054 \times 10^{-27}\text{ kg}$

atomic nucleus tightly packed group of nucleons at the center of an atom

atomic number number of protons in a nucleus

becquerel (Bq) SI unit for the decay rate of a radioactive material, equal to 1 decay/second

beta (β) rays one of the types of rays emitted from the nucleus of an atom as beta particles

beta decay radioactive nuclear decay associated with the emission of a beta particle

binding energy (BE) energy needed to break a nucleus into its constituent protons and neutrons

binding energy per nucleon (BEN) energy need to remove a nucleon from a nucleus

breeder reactor reactor that is designed to make plutonium

carbon-14 dating method to determine the age of formerly living tissue using the ratio $^{14}\text{C}/^{12}\text{C}$

chart of the nuclides graph comprising stable and unstable nuclei

critical mass minimum mass required of a given nuclide in order for self-sustained fission to occur

criticality condition in which a chain reaction easily becomes self-sustaining

curie (Ci) unit of decay rate, or the activity of 1 g of ^{226}Ra , equal to $3.70 \times 10^{10}\text{ Bq}$

daughter nucleus nucleus produced by the decay of a parent nucleus

decay process by which an individual atomic nucleus of an unstable atom loses mass and energy by emitting ionizing particles

decay constant quantity that is inversely proportional to the half-life and that is used in equation for number of nuclei as a function of time

decay series series of nuclear decays ending in a stable nucleus

fission splitting of a nucleus

gamma (γ) rays one of the types of rays emitted from the nucleus of an atom as gamma particles

gamma decay radioactive nuclear decay associated with the emission of gamma radiation

half-life time for half of the original nuclei to decay (or half of the original nuclei remain)

high dose dose of radiation greater than 1 Sv (100 rem)

isotopes nuclei having the same number of protons but different numbers of neutrons

lifetime average time that a nucleus exists before decaying

liquid drop model model of nucleus (only to understand some of its features) in which nucleons in a nucleus act like atoms in a drop

low dose dose of radiation less than 100 mSv (10 rem)

mass defect difference between the mass of a nucleus and the total mass of its constituent nucleons

mass number number of nucleons in a nucleus

moderate dose dose of radiation from 0.1 Sv to 1 Sv (10 to 100 rem)

neutrino subatomic elementary particle which has no net electric charge

neutron number number of neutrons in a nucleus

nuclear fusion process of combining lighter nuclei to make heavier nuclei

nuclear fusion reactor nuclear reactor that uses the fusion chain to produce energy

nucleons protons and neutrons found inside the nucleus of an atom

nucleosynthesis process of fusion by which all elements on Earth are believed to have been created

nuclide nucleus

parent nucleus original nucleus before decay

positron electron with positive charge

positron emission tomography (PET) tomography technique that uses β^+ emitters and detects the two annihilation γ rays, aiding in source localization

proton-proton chain combined reactions that fuse hydrogen nuclei to produce He nuclei

radiation dose unit (rad) ionizing energy deposited per kilogram of tissue

radioactive dating application of radioactive decay in which the age of a material is determined by the amount of radioactivity of a particular type

that occurs

radioactive decay law describes the exponential decrease of parent nuclei in a radioactive sample

radioactive tags special drugs (radiopharmaceuticals) that allow doctors to track movement of other drugs in the body

radioactivity spontaneous emission of radiation from nuclei

radiopharmaceutical compound used for medical imaging

radius of a nucleus radius of a nucleus is defined as $r = r_0 A^{1/3}$

relative biological effectiveness (RBE) number

that expresses the relative amount of damage that a fixed amount of ionizing radiation of a given type can inflict on biological tissues

roentgen equivalent man (rem) dose unit more closely related to effects in biological tissue

sievert (Sv) SI equivalent of the rem

single-photon-emission computed tomography (SPECT) tomography performed with γ -emitting radiopharmaceuticals

strong nuclear force force that binds nucleons together in the nucleus

transuranic element element that lies beyond uranium in the periodic table

Key Equations

Atomic mass number	$A = Z + N$
Standard format for expressing an isotope	${}_Z^A\text{X}$
Nuclear radius, where r_0 is the radius of a single proton	$r = r_0 A^{1/3}$
Mass defect	$\Delta m = Zm_p + (A - Z)m_n - m_{\text{nuc}}$
Binding energy	$E = (\Delta m)c^2$
Binding energy per nucleon	$BE N = \frac{E_b}{A}$
Radioactive decay rate	$-\frac{dN}{dt} = \lambda N$
Radioactive decay law	$N = N_0 e^{-\lambda t}$
Decay constant	$\lambda = \frac{0.693}{T_{1/2}}$
Lifetime of a substance	$\bar{T} = \frac{1}{\lambda}$
Activity of a radioactive substance	$A = A_0 e^{-\lambda t}$
Activity of a radioactive substance (linear form)	$\ln A = -\lambda t + \ln A_0$
Alpha decay	${}_Z^A\text{X} \rightarrow {}_{Z-2}^{A-4}\text{X} + {}_2^4\text{He}$
Beta decay	${}_Z^A\text{X} \rightarrow {}_{Z+1}^A\text{X} + {}_{-1}^0\text{e} + \bar{\nu}$
Positron emission	${}_Z^A\text{X} \rightarrow {}_{Z-1}^A\text{X} + {}_{+1}^0\text{e} + \nu$
Gamma decay	${}_Z^A\text{X}^* \rightarrow {}_Z^A\text{X} + \gamma$

Summary

10.1 Properties of Nuclei

- The atomic nucleus is composed of protons and neutrons.
- The number of protons in the nucleus is given by the atomic number, Z . The number of neutrons in the nucleus is the neutron number, N . The number of nucleons is mass number, A .
- Atomic nuclei with the same atomic number, Z , but different neutron numbers, N , are isotopes of the same element.
- The atomic mass of an element is the weighted average of the masses of its isotopes.

10.2 Nuclear Binding Energy

- The mass defect of a nucleus is the difference between the total mass of a nucleus and the sum of the masses of all its constituent nucleons.
- The binding energy (BE) of a nucleus is equal to the amount of energy released in forming the nucleus, or the mass defect multiplied by the speed of light squared.
- A graph of binding energy per nucleon (BEN) versus atomic number A implies that nuclei divided or combined release an enormous amount of energy.
- The binding energy of a nucleon in a nucleus is analogous to the ionization energy of an electron in an atom.

10.3 Radioactive Decay

- In the decay of a radioactive substance, if the decay constant (λ) is large, the half-life is small, and vice versa.
- The radioactive decay law, $N = N_0 e^{-\lambda t}$, uses the properties of radioactive substances to estimate the age of a substance.
- Radioactive carbon has the same chemistry as stable carbon, so it mixes into the ecosphere and eventually becomes part of every living organism. By comparing the abundance of ^{14}C in an artifact with the normal abundance in living tissue, it is possible to determine the artifact's age.

10.4 Nuclear Reactions

- The three types of nuclear radiation are alpha (α) rays, beta (β) rays, and gamma (γ) rays.
- We represent α decay symbolically by ${}_Z^AX \rightarrow {}_{Z-2}^{A-4}X + {}_2^4\text{He}$. There are two types of β decay: either an electron (β^-) or a positron (β^+) is emitted by a nucleus. γ decay is represented

symbolically by ${}_Z^AX^* \rightarrow {}_Z^AX + \gamma$.

- When a heavy nucleus decays to a lighter one, the lighter daughter nucleus can become the parent nucleus for the next decay, and so on, producing a decay series.

10.5 Fission

- Nuclear fission is a process in which the sum of the masses of the product nuclei are less than the masses of the reactants.
- Energy changes in a nuclear fission reaction can be understood in terms of the binding energy per nucleon curve.
- The production of new or different isotopes by nuclear transformation is called breeding, and reactors designed for this purpose are called breeder reactors.

10.6 Nuclear Fusion

- Nuclear fusion is a reaction in which two nuclei are combined to form a larger nucleus; energy is released when light nuclei are fused to form medium-mass nuclei.
- The amount of energy released by a fusion reaction is known as the Q value.
- Nuclear fusion explains the reaction between deuterium and tritium that produces a fusion (or hydrogen) bomb; fusion also explains the production of energy in the Sun, the process of nucleosynthesis, and the creation of the heavy elements.

10.7 Medical Applications and Biological Effects of Nuclear Radiation

- Nuclear technology is used in medicine to locate and study diseased tissue using special drugs called radiopharmaceuticals. Radioactive tags are used to identify cancer cells in the bones, brain tumors, and Alzheimer's disease, and to monitor the function of body organs, such as blood flow, heart muscle activity, and iodine uptake in the thyroid gland.
- The biological effects of ionizing radiation are due to two effects it has on cells: interference with cell reproduction and destruction of cell function.
- Common sources of radiation include that emitted by Earth due to the isotopes of uranium, thorium, and potassium; natural radiation from cosmic rays, soils, and building materials, and artificial sources from medical and dental

diagnostic tests.

- Biological effects of nuclear radiation are expressed by many different physical quantities

Conceptual Questions

10.1 Properties of Nuclei

1. Define and make clear distinctions between the terms neutron, nucleon, nucleus, and nuclide.
2. What are isotopes? Why do isotopes of the same atom share the same chemical properties?

10.2 Nuclear Binding Energy

3. Explain why a bound system should have less mass than its components. Why is this not observed traditionally, say, for a building made of bricks?
4. Why is the number of neutrons greater than the number of protons in stable nuclei that have an A greater than about 40? Why is this effect more pronounced for the heaviest nuclei?
5. To obtain the most precise value of the binding energy per nucleon, it is important to take into account forces between nucleons at the surface of the nucleus. Will surface effects increase or decrease estimates of BEN?

10.3 Radioactive Decay

6. How is the initial activity rate of a radioactive substance related to its half-life?
7. For the carbon dating described in this chapter, what important assumption is made about the time variation in the intensity of cosmic rays?

10.4 Nuclear Reactions

8. What is the key difference and the key similarity between beta (β^-) decay and alpha decay?
9. What is the difference between γ rays and characteristic X-rays and visible light?

Problems

10.1 Properties of Nuclei

21. Find the atomic numbers, mass numbers, and neutron numbers for (a) $^{58}_{29}\text{Cu}$, (b) $^{24}_{11}\text{Na}$, (c) $^{210}_{84}\text{Po}$, (d) $^{45}_{20}\text{Ca}$, and (e) $^{206}_{82}\text{Pb}$.
22. Silver has two stable isotopes. The nucleus, $^{107}_{47}\text{Ag}$, has atomic mass 106.905095 g/mol with an abundance of 51.83%; whereas $^{109}_{47}\text{Ag}$ has atomic mass 108.904754 g/mol with an abundance of 48.17%. Find the atomic mass of

and in many different units, including the rad or radiation dose unit.

10. What characteristics of radioactivity show it to be nuclear in origin and not atomic?
11. Consider [Figure 10.12](#). If the magnetic field is replaced by an electric field pointed in toward the page, in which directions will the α^- , β^+ , and γ rays bend?
12. Why is Earth's core molten?

10.5 Fission

13. Should an atomic bomb really be called *nuclear* bomb?
14. Why does a chain reaction occur during a fission reaction?
15. In what way is an atomic nucleus like a liquid drop?

10.6 Nuclear Fusion

16. Explain the difference between nuclear fission and nuclear fusion.
17. Why does the fusion of light nuclei into heavier nuclei release energy?

10.7 Medical Applications and Biological Effects of Nuclear Radiation

18. Why is a PET scan more accurate than a SPECT scan?
19. Isotopes that emit α radiation are relatively safe outside the body and exceptionally hazardous inside. Explain why.
20. Ionizing radiation can impair the ability of a cell to repair DNA. What are the three ways the cell can respond?

the element silver.

23. The mass (M) and the radius (r) of a nucleus can be expressed in terms of the mass number, A . (a) Show that the density of a nucleus is independent of A . (b) Calculate the density of a gold (Au) nucleus. Compare your answer to that for iron (Fe).
24. A particle has a mass equal to 10 u. If this mass is converted completely into energy, how much energy is released? Express your answer in

mega-electron volts (MeV). (Recall that $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$.)

25. Find the length of a side of a cube having a mass of 1.0 kg and the density of nuclear matter.
26. The detail that you can observe using a probe is limited by its wavelength. Calculate the energy of a particle that has a wavelength of $1 \times 10^{-16} \text{ m}$, small enough to detect details about one-tenth the size of a nucleon.

10.2 Nuclear Binding Energy

27. How much energy would be released if six hydrogen atoms and six neutrons were combined to form ${}^{12}_6\text{C}$?
28. Find the mass defect and the binding energy for the helium-4 nucleus.
29. ${}^{56}\text{Fe}$ is among the most tightly bound of all nuclides. It makes up more than 90% of natural iron. Note that ${}^{56}\text{Fe}$ has even numbers of protons and neutrons. Calculate the binding energy per nucleon for ${}^{56}\text{Fe}$ and compare it with the approximate value obtained from the graph in [Figure 10.7](#).
30. ${}^{209}\text{Bi}$ is the heaviest stable nuclide, and its BEN is low compared with medium-mass nuclides. Calculate BEN for this nucleus and compare it with the approximate value obtained from the graph in [Figure 10.7](#).
31. (a) Calculate BEN for ${}^{235}\text{U}$, the rarer of the two most common uranium isotopes; (b) Calculate BEN for ${}^{238}\text{U}$. (Most of uranium is ${}^{238}\text{U}$.)
32. The fact that BEN peaks at roughly $A = 60$ implies that the *range* of the strong nuclear force is about the diameter of this nucleus.
(a) Calculate the diameter of $A = 60$ nucleus.
(b) Compare BEN for ${}^{58}\text{Ni}$ and ${}^{90}\text{Sr}$. The first is one of the most tightly bound nuclides, whereas the second is larger and less tightly bound.

10.3 Radioactive Decay

33. A sample of radioactive material is obtained from a very old rock. A plot $\ln A$ versus t yields a slope value of -10^{-9} s^{-1} (see [Figure 10.10\(b\)](#)). What is the half-life of this material?
34. Show that: $\bar{T} = \frac{1}{\lambda}$.
35. The half-life of strontium-91, ${}^{91}_{38}\text{Sr}$ is 9.70 h. Find (a) its decay constant and (b) for an initial 1.00-g sample, the activity after 15 hours.
36. A sample of pure carbon-14 ($T_{1/2} = 5730 \text{ y}$) has an activity of $1.0 \mu \text{ Ci}$. What is the mass of the sample?

37. A radioactive sample initially contains $2.40 \times 10^{-2} \text{ mol}$ of a radioactive material whose half-life is 6.00 h. How many moles of the radioactive material remain after 6.00 h? After 12.0 h? After 36.0 h?
38. An old campfire is uncovered during an archaeological dig. Its charcoal is found to contain less than 1/1000 the normal amount of ${}^{14}\text{C}$. Estimate the minimum age of the charcoal, noting that $2^{10} = 1024$.
39. Calculate the activity R , in curies of 1.00 g of ${}^{226}\text{Ra}$. (b) Explain why your answer is not exactly 1.00 Ci, given that the curie was originally supposed to be exactly the activity of a gram of radium.
40. Natural uranium consists of ${}^{235}\text{U}$ (percent abundance = 0.7200%, $\lambda = 3.12 \times 10^{-17} \text{ /s}$) and ${}^{238}\text{U}$ (percent abundance = 99.27%, $\lambda = 4.92 \times 10^{-18} \text{ /s}$). What were the values for percent abundance of ${}^{235}\text{U}$ and ${}^{238}\text{U}$ when Earth formed 4.5×10^9 years ago?
41. World War II aircraft had instruments with glowing radium-painted dials. The activity of one such instrument was $1.0 \times 10^5 \text{ Bq}$ when new. (a) What mass of ${}^{226}\text{Ra}$ was present? (b) After some years, the phosphors on the dials deteriorated chemically, but the radium did not escape. What is the activity of this instrument 57.0 years after it was made?
42. The ${}^{210}\text{Po}$ source used in a physics laboratory is labeled as having an activity of $1.0 \mu \text{ Ci}$ on the date it was prepared. A student measures the radioactivity of this source with a Geiger counter and observes 1500 counts per minute. She notices that the source was prepared 120 days before her lab. What fraction of the decays is she observing with her apparatus?
43. Armor-piercing shells with depleted uranium cores are fired by aircraft at tanks. (The high density of the uranium makes them effective.) The uranium is called depleted because it has had its ${}^{235}\text{U}$ removed for reactor use and is nearly pure ${}^{238}\text{U}$. Depleted uranium has been erroneously called nonradioactive. To demonstrate that this is wrong: (a) Calculate the activity of 60.0 g of pure ${}^{238}\text{U}$. (b) Calculate the activity of 60.0 g of natural uranium, neglecting the ${}^{234}\text{U}$ and all daughter nuclides.

10.4 Nuclear Reactions

44. ${}^{249}\text{Cf}$ undergoes alpha decay. (a) Write the

reaction equation. (b) Find the energy released in the decay.

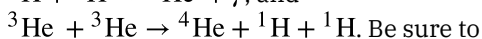
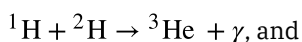
45. (a) Calculate the energy released in the α decay of ^{238}U . (b) What fraction of the mass of a single ^{238}U is destroyed in the decay? The mass of ^{234}Th is 234.043593 u. (c) Although the fractional mass loss is large for a single nucleus, it is difficult to observe for an entire macroscopic sample of uranium. Why is this?
46. The β^- particles emitted in the decay of ^3H (tritium) interact with matter to create light in a glow-in-the-dark exit sign. At the time of manufacture, such a sign contains 15.0 Ci of ^3H . (a) What is the mass of the tritium? (b) What is its activity 5.00 y after manufacture?
47. (a) Write the complete β^- decay equation for ^{90}Sr , a major waste product of nuclear reactors. (b) Find the energy released in the decay.
48. Write a nuclear β^- decay reaction that produces the ^{90}Y nucleus. (*Hint:* The parent nuclide is a major waste product of reactors and has chemistry similar to calcium, so that it is concentrated in bones if ingested.)
49. Write the complete decay equation in the complete ${}_Z^AX_N$ notation for the beta (β^-) decay of ^3H (tritium), a manufactured isotope of hydrogen used in some digital watch displays, and manufactured primarily for use in hydrogen bombs.
50. If a 1.50-cm-thick piece of lead can absorb 90.0% of the rays from a radioactive source, how many centimeters of lead are needed to absorb all but 0.100% of the rays?
51. An electron can interact with a nucleus through the beta-decay process:
 ${}_Z^AX + e^- \rightarrow Y + \nu_e$.
 (a) Write the complete reaction equation for electron capture by ^7Be .
 (b) Calculate the energy released.
52. (a) Write the complete reaction equation for electron capture by ^{15}O .
 (b) Calculate the energy released.
53. A rare decay mode has been observed in which ^{222}Ra emits a ^{14}C nucleus. (a) The decay equation is $^{222}\text{Ra} \rightarrow {}^AX + {}^{14}\text{C}$. Identify the nuclide ${}^AX$. (b) Find the energy emitted in the decay. The mass of ^{222}Ra is 222.015353 u.

10.5 Fission

54. A large power reactor that has been in operation for some months is turned off, but residual activity in the core still produces 150 MW of power. If the average energy per decay of the fission products is 1.00 MeV, what is the core activity?
55. (a) Calculate the energy released in this rare neutron-induced fission
 $n + {}^{238}\text{U} \rightarrow {}^{96}\text{Sr} + {}^{140}\text{Xe} + 3n$, given
 $m({}^{96}\text{Sr}) = 95.921750$ u and
 $m({}^{140}\text{Xe}) = 139.92164$.
 (b) This result is about 6 MeV greater than the result for spontaneous fission. Why?
 (c) Confirm that the total number of nucleons and total charge are conserved in this reaction.
56. (a) Calculate the energy released in the neutron-induced fission reaction
 $n + {}^{235}\text{U} \rightarrow {}^{92}\text{Kr} + {}^{142}\text{Ba} + 2n$, given
 $m({}^{92}\text{Kr}) = 91.926269$ u and
 $m({}^{142}\text{Ba}) = 141.916361$ u. (b) Confirm that the total number of nucleons and total charge are conserved in this reaction.
57. The electrical power output of a large nuclear reactor facility is 900 MW. It has a 35.0% efficiency in converting nuclear power to electrical power.
 (a) What is the thermal nuclear power output in megawatts?
 (b) How many ^{235}U nuclei fission each second, assuming the average fission produces 200 MeV?
 (c) What mass of ^{235}U is fissioned in 1 year of full-power operation?
58. Find the total energy released if 1.00 kg of $^{235}_{92}\text{U}$ were to undergo fission.

10.6 Nuclear Fusion

59. Verify that the total number of nucleons, and total charge are conserved for each of the following fusion reactions in the proton-proton chain.
 (i) ${}^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu_e$,
 (ii) ${}^1\text{H} + {}^2\text{H} \rightarrow {}^3\text{He} + \gamma$, and (iii)
 ${}^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^1\text{H} + {}^1\text{H}$.
 (List the value of each of the conserved quantities before and after each of the reactions.)
60. Calculate the energy output in each of the fusion reactions in the proton-proton chain, and verify the values determined in the preceding problem.
61. Show that the total energy released in the proton-proton chain is 26.7 MeV, considering the overall effect in ${}^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu_e$,



include the annihilation energy.

62. Two fusion reactions mentioned in the text are $n + {}^3\text{He} \rightarrow {}^4\text{He} + \gamma$ and $n + {}^1\text{H} \rightarrow {}^2\text{H} + \gamma$. Both reactions release energy, but the second also creates more fuel. Confirm that the energies produced in the reactions are 20.58 and 2.22 MeV, respectively. Comment on which product nuclide is most tightly bound, ${}^4\text{He}$ or ${}^2\text{H}$.
63. The power output of the Sun is $4 \times 10^{26} \text{ W}$. (a) If 90% of this energy is supplied by the proton-proton chain, how many protons are consumed per second? (b) How many neutrinos per second should there be per square meter at the surface of Earth from this process?
64. Another set of reactions that fuses hydrogen into helium in the Sun and especially in hotter stars is called the CNO cycle:
- $${}^{12}\text{C} + {}^1\text{H} \rightarrow {}^{13}\text{N} + \gamma$$
- $${}^{13}\text{N} \rightarrow {}^{13}\text{C} + e^+ + \nu_e$$
- $${}^{13}\text{C} + {}^1\text{H} \rightarrow {}^{14}\text{N} + \gamma$$
- $${}^{14}\text{N} + {}^1\text{H} \rightarrow {}^{15}\text{O} + \gamma$$
- $${}^{15}\text{O} \rightarrow {}^{15}\text{N} + e^+ + \nu_e$$
- $${}^{15}\text{N} + {}^1\text{H} \rightarrow {}^{12}\text{C} + {}^4\text{He}$$
- This process is a “cycle” because ${}^{12}\text{C}$ appears at the beginning and end of these reactions. Write down the overall effect of this cycle (as done for the proton-proton chain in $2e^- + 4{}^1\text{H} \rightarrow {}^4\text{He} + 2\nu_e + 6\gamma$). Assume that the positrons annihilate electrons to form more γ rays.
65. (a) Calculate the energy released by the fusion of a 1.00-kg mixture of deuterium and tritium, which produces helium. There are equal numbers of deuterium and tritium nuclei in the mixture.
(b) If this process takes place continuously over a period of a year, what is the average power output?

Additional Problems

75. The wiki-phony site states that the atomic mass of chlorine is 40 g/mol. Check this result. *Hint:* The two, most common stable isotopes of chlorine are: ${}^{35}_{17}\text{Cl}$ and ${}^{37}_{17}\text{Cl}$. (The abundance of Cl-35 is 75.8%, and the abundance of Cl-37 is 24.2%.)

10.7 Medical Applications and Biological Effects of Nuclear Radiation

66. What is the dose in mSv for: (a) a 0.1-Gy X-ray? (b) 2.5 mGy of neutron exposure to the eye? (c) 1.5m Gy of α exposure?
67. Find the radiation dose in Gy for: (a) A 10-mSv fluoroscopic X-ray series. (b) 50 mSv of skin exposure by an α emitter. (c) 160 mSv of β^- and γ rays from the ${}^{40}\text{K}$ in your body.
68. Find the mass of ${}^{239}\text{Pu}$ that has an activity of 1.00 μCi .
69. In the 1980s, the term picowave was used to describe food irradiation in order to overcome public resistance by playing on the well-known safety of microwave radiation. Find the energy in MeV of a photon having a wavelength of a picometer.
70. What is the dose in Sv in a cancer treatment that exposes the patient to 200 Gy of γ rays?
71. One half the γ rays from ${}^{99\text{m}}\text{Tc}$ are absorbed by a 0.170-mm-thick lead shielding. Half of the γ rays that pass through the first layer of lead are absorbed in a second layer of equal thickness. What thickness of lead will absorb all but one in 1000 of these γ rays?
72. How many Gy of exposure is needed to give a cancerous tumor a dose of 40 Sv if it is exposed to α activity?
73. A plumber at a nuclear power plant receives a whole-body dose of 30 mSv in 15 minutes while repairing a crucial valve. Find the radiation-induced yearly risk of death from cancer and the chance of genetic defect from this maximum allowable exposure.
74. Calculate the dose in rem/y for the lungs of a weapons plant employee who inhales and retains an activity of 1.00 μCi ${}^{239}\text{Pu}$ in an accident. The mass of affected lung tissue is 2.00 kg and the plutonium decays by emission of a 5.23-MeV α particle. Assume a RBE value of 20.

76. A particle physicist discovers a neutral particle with a mass of 2.02733 u that he assumes is two neutrons bound together.
(a) Find the binding energy.
(b) What is unreasonable about this result?

77. A nuclear physicist finds $1.0\ \mu\text{g}$ of ^{236}U in a piece of uranium ore ($T_{1/2} = 2.348 \times 10^7\ \text{y}$). (a) Use the decay law to determine how much ^{236}U would have had to have been on Earth when it formed $4.543 \times 10^9\ \text{y}$ ago for $1.0\ \mu\text{g}$ to be left today. (b) What is unreasonable about this result? (c) How is this unreasonable result resolved?
78. A group of scientists use carbon dating to date a piece of wood to be 3 billion years old. Why doesn't this make sense?
79. According to your lab partner, a 2.00-cm-thick sodium-iodide crystal absorbs all but 10% of rays from a radioactive source and a 4.00-cm piece of the same material absorbs all but 5%? Is this result reasonable?
80. In the science section of the newspaper, an article reports the efforts of a group of scientists to create a new nuclear reactor based on the fission of iron (Fe). Is this a good idea?
81. The ceramic glaze on a red-orange "Fiestaware" plate is U_2O_3 and contains 50.0 grams of ^{238}U , but very little ^{235}U . (a) What is the activity of the plate? (b) Calculate the total energy that will be released by the ^{238}U decay. (c) If energy is worth 12.0 cents per $\text{kW} \cdot \text{h}$, what is the monetary value of the energy emitted? (These brightly-colored ceramic plates went out of production some 30 years ago, but are still available as collectibles.)
82. Large amounts of depleted uranium (^{238}U) are available as a by-product of uranium processing for reactor fuel and weapons. Uranium is very dense and makes good counter weights for aircraft. Suppose you have a 4000-kg block of ^{238}U . (a) Find its activity. (b) How many calories per day are generated by thermalization of the decay energy? (c) Do you think you could detect this as heat? Explain.
83. A piece of wood from an ancient Egyptian tomb is tested for its carbon-14 activity. It is found to have an activity per gram of carbon of $A = 10\ \text{decay/min} \cdot \text{g}$. What is the age of the wood?

Challenge Problems

84. This problem demonstrates that the binding energy of the electron in the ground state of a hydrogen atom is much smaller than the rest mass energies of the proton and electron.
- Calculate the mass equivalent in u of the 13.6-eV binding energy of an electron in a hydrogen atom, and compare this with the known mass of the hydrogen atom.
 - Subtract the known mass of the proton from the known mass of the hydrogen atom.
 - Take the ratio of the binding energy of the electron (13.6 eV) to the energy equivalent of the electron's mass (0.511 MeV).
 - Discuss how your answers confirm the stated purpose of this problem.
85. The *Galileo* space probe was launched on its long journey past Venus and Earth in 1989, with an ultimate goal of Jupiter. Its power source is 11.0 kg of ^{238}Pu , a by-product of nuclear weapons plutonium production. Electrical energy is generated thermoelectrically from the heat produced when the 5.59-MeV α particles emitted in each decay crash to a halt inside the plutonium and its shielding. The half-life of ^{238}Pu is 87.7 years.
- What was the original activity of the ^{238}Pu in becquerels?
 - What power was emitted in kilowatts?
 - What power was emitted 12.0 y after launch? You may neglect any extra energy from daughter nuclides and any losses from escaping γ rays.
86. Find the energy emitted in the β^- decay of ^{60}Co .

- 87.** Engineers are frequently called on to inspect and, if necessary, repair equipment in nuclear power plants. Suppose that the city lights go out. After inspecting the nuclear reactor, you find a leaky pipe that leads from the steam generator to turbine chamber. (a) How do the pressure readings for the turbine chamber and steam condenser compare? (b) Why is the nuclear reactor *not* generating electricity?
- 88.** If two nuclei are to fuse in a nuclear reaction, they must be moving fast enough so that the repulsive Coulomb force between them does not prevent them from getting within $R \approx 10^{-14}$ m of one another. At this distance or nearer, the attractive nuclear force can overcome the Coulomb force, and the nuclei are able to fuse. (a) Find a simple formula that can be used to estimate the minimum kinetic energy the nuclei must have if they are to fuse. To keep the calculation simple, assume the two nuclei are identical and moving toward one another with the same speed v . (b) Use this minimum kinetic energy to estimate the minimum temperature a gas of the nuclei must have before a significant number of them will undergo fusion. Calculate this minimum temperature first for hydrogen and then for helium. (*Hint:* For fusion to occur, the minimum kinetic energy when the nuclei are far apart must be equal to the Coulomb potential energy when they are a distance R apart.)
- 89.** For the reaction, $n + {}^3\text{He} \rightarrow {}^4\text{He} + \gamma$, find the amount of energy transferred to ${}^4\text{He}$ and γ (on the right side of the equation). Assume the reactants are initially at rest. (*Hint:* Use conservation of momentum principle.)
- 90.** Engineers are frequently called on to inspect and, if necessary, repair equipment in medical hospitals. Suppose that the PET system malfunctions. After inspecting the unit, you suspect that one of the PET photon detectors is misaligned. To test your theory you position one detector at the location $(r, \theta, \phi) = (1.5, 45, 30)$ relative to a radioactive test sample at the center of the patient bed. (a) If the second photon detector is properly aligned where should it be located? (b) What energy reading is expected?

ANSWER KEY

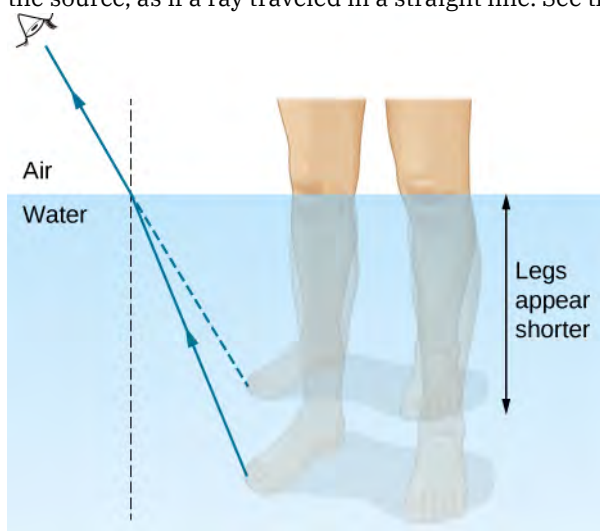
Chapter 1

Check Your Understanding

- 1.1 2.1% (to two significant figures)
- 1.2 15.1°
- 1.3 air to water, because the condition that the second medium must have a smaller index of refraction is not satisfied
- 1.4 9.3 cm
- 1.5 AA' becomes longer, $A'B'$ tilts further away from the surface, and the refracted ray tilts away from the normal.
- 1.6 also 90.0%
- 1.7 There will be only refraction but no reflection.

Conceptual Questions

1. Light can be modeled as a ray when devices are large compared to wavelength, and as a wave when devices are comparable or small compared to wavelength.
3. This fact simply proves that the speed of light is greater than that of sound. If one knows the distance to the location of the lightning and the speed of sound, one could, in principle, determine the speed of light from the data. In practice, because the speed of light is so great, the data would have to be known to impractically high precision.
5. Powder consists of many small particles with randomly oriented surfaces. This leads to diffuse reflection, reducing shine.
7. “toward” when increasing n (air to water, water to glass); “away” when decreasing n (glass to air)
9. A ray from a leg emerges from water after refraction. The observer in air perceives an apparent location for the source, as if a ray traveled in a straight line. See the dashed ray below.



11. The gemstone becomes invisible when its index of refraction is the same, or at least similar to, the water surrounding it. Because diamond has a particularly high index of refraction, it can still sparkle as a result of total internal reflection, not invisible.
13. One can measure the critical angle by looking for the onset of total internal reflection as the angle of incidence is varied. [Equation 1.5](#) can then be applied to compute the index of refraction.
15. In addition to total internal reflection, rays that refract into and out of diamond crystals are subject to dispersion due to varying values of n across the spectrum, resulting in a sparkling display of colors.
17. yes

19. No. Sound waves are not transverse waves.
21. Energy is absorbed into the filters.
23. Sunsets are viewed with light traveling straight from the Sun toward us. When blue light is scattered out of this path, the remaining red light dominates the overall appearance of the setting Sun.
25. The axis of polarization for the sunglasses has been rotated 90° .

Problems

27. 2.99705×10^8 m/s; 1.97×10^8 m/s
29. ice at 0°C
31. 1.03 ns
33. 337 m
35. proof
37. proof
39. reflection, 70° ; refraction, 45°
41. 42°
43. 1.53
45. a. 2.9 m; b. 1.4 m
47. a. 24.42° ; b. 31.33°
49. 79.11°
51. a. 1.43, fluorite; b. 44.2°
53. a. 48.2° ; b. 27.3°
55. 46.5° for red, 46.0° for violet
57. a. 0.04° ; b. 1.3 m
59. 72.8°
61. 53.5° for red, 55.2° for violet
63. 0.500
65. 0.125 or $1/8$
67. 84.3°
69. $0.250 I_0$
71. a. 0.500; b. 0.250; c. 0.187
73. 67.54°
75. 53.1°

Additional Problems

77. 114 radian/s
79. 3.72 mm
81. 41.2°
83. a. 1.92. The gem is not a diamond (it is zircon). b. 55.2°
85. a. 0.898; b. We cannot have $n < 1.00$, since this would imply a speed greater than c. c. The refracted angle is too big relative to the angle of incidence.
87. $0.707 B_1$
89. a. $1.69 \times 10^{-2}^\circ\text{C/s}$; b. yes

Challenge Problems

91. First part: 88.6° . The remainder depends on the complexity of the solution the reader constructs.
93. proof; 1.33
95. a. 0.750; b. 0.563; c. 1.33

Chapter 2

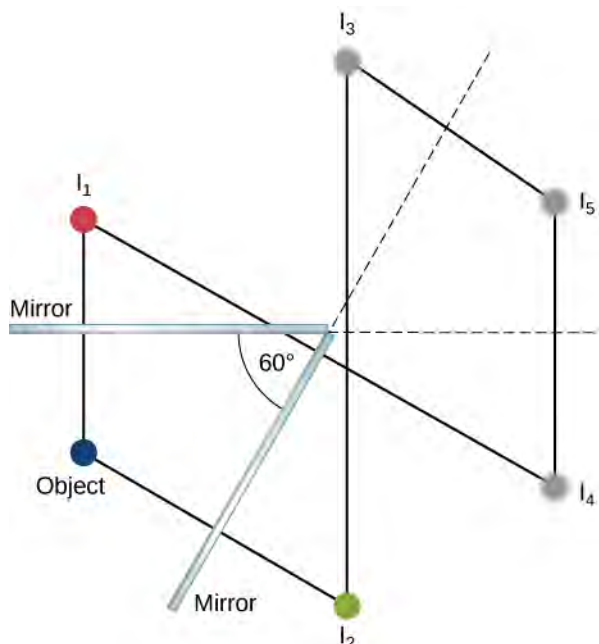
Check Your Understanding

Conceptual Questions

1. Virtual image cannot be projected on a screen. You cannot distinguish a real image from a virtual image simply by judging from the image perceived with your eye.
3. Yes, you can photograph a virtual image. For example, if you photograph your reflection from a plane mirror, you get a photograph of a virtual image. The camera focuses the light that enters its lens to form an image; whether the source of the light is a real object or a reflection from mirror (i.e., a virtual image) does not matter.
5. No, you can see the real image the same way you can see the virtual image. The retina of your eye effectively serves as a screen.
7. The mirror should be half your size and its top edge should be at the level of your eyes. The size does not depend on your distance from the mirror.
9. when the object is at infinity; see the mirror equation
11. Yes, negative magnification simply means that the image is upside down; this does not prevent the image from being larger than the object. For instance, for a concave mirror, if distance to the object is larger than one focal distance but smaller than two focal distances the image will be inverted and magnified.
13. answers may vary
15. The focal length of the lens is fixed, so the image distance changes as a function of object distance.
17. Yes, the focal length will change. The lens maker's equation shows that the focal length depends on the index of refraction of the medium surrounding the lens. Because the index of refraction of water differs from that of air, the focal length of the lens will change when submerged in water.
19. A relaxed, normal-vision eye will focus parallel rays of light onto the retina.
21. A person with an internal lens will need glasses to read because their muscles cannot distort the lens as they do with biological lenses, so they cannot focus on near objects. To correct nearsightedness, the power of the intraocular lens must be less than that of the removed lens.
23. Microscopes create images of macroscopic size, so geometric optics applies.
25. The eyepiece would be moved slightly farther from the objective so that the image formed by the objective falls just beyond the focal length of the eyepiece.

Problems

27.



29. It is in the focal point of the big mirror and at the center of curvature of the small mirror.

31. $f = \frac{R}{2} \Rightarrow R = +1.60 \text{ m}$

33. $d_o = 27.3 \text{ cm}$

35. Step 1: Image formation by a mirror is involved.

Step 2: Draw the problem set up when possible.

Step 3: Use thin-lens equations to solve this problem.

Step 4: Find f .

Step 5: Given: $m = 1.50$, $d_o = 0.120 \text{ m}$.

Step 6: No ray tracing is needed.

Step 7: Using $m = \frac{d_i}{d_o}$, $d_i = -0.180 \text{ m}$. Then, $f = 0.360 \text{ m}$.

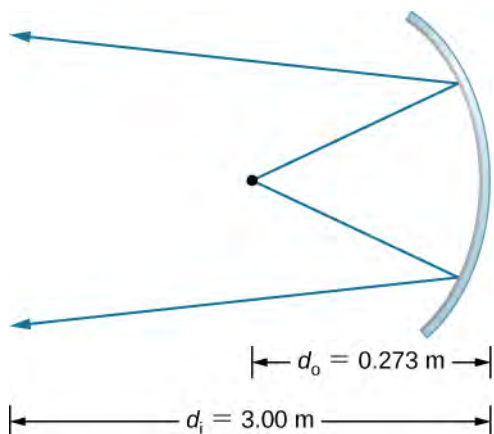
Step 8: The image is virtual because the image distance is negative. The focal length is positive, so the mirror is concave.

37. a. for a convex mirror $d_i < 0 \Rightarrow m > 0$. $m = +0.111$; b. $d_i = -0.334 \text{ cm}$ (behind the cornea);

c. $f = -0.376 \text{ cm}$, so that $R = -0.752 \text{ cm}$

39. $m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} = -\frac{-d_o}{d_o} = \frac{d_o}{d_o} = 1 \Rightarrow h_i = h_o$

41.



$$m = -11.0$$

$$A' = 0.110 \text{ m}^2$$

$$I = 6.82 \text{ kW/m}^2$$

$$x_{2m} = -x_{2m-1}, \quad (m = 1, 2, 3, \dots),$$

43. $x_{2m+1} = b - x_{2m}, \quad (m = 0, 1, 2, \dots), \text{ with } x_0 = a.$

45. $d_i = -55 \text{ cm}; m = +1.8$

47. $d_i = -41 \text{ cm}, m = 1.4$

49. proof

51. a. $\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f} \Rightarrow d_i = 3.43 \text{ m};$

b. $m = -33.33$, so that $(2.40 \times 10^{-2} \text{ m})(33.33) = 80.0 \text{ cm}$, and

$(3.60 \times 10^{-2} \text{ m})(33.33) = 1.20 \text{ m} \Rightarrow 0.800 \text{ m} \times 1.20 \text{ m} \text{ or } 80.0 \text{ cm} \times 120 \text{ cm}$

53. a. $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f};$

$$d_i = 5.08 \text{ cm}$$

b. $m = -1.695 \times 10^{-2}$, so the maximum height is $\frac{0.036 \text{ m}}{1.695 \times 10^{-2}} = 2.12 \text{ m} \Rightarrow 100\%;$

c. This seems quite reasonable, since at 3.00 m it is possible to get a full length picture of a person.

55. a. $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \Rightarrow d_o = 2.55 \text{ m};$

b. $\frac{h_i}{h_o} = -\frac{d_i}{d_o} \Rightarrow h_o = 1.00 \text{ m}$

57. a. Using $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$, $d_i = -56.67 \text{ cm}$. Then we can determine the magnification, $m = 6.67$. b.

$d_i = -190 \text{ cm}$ and $m = +20.0$; c. The magnification m increases rapidly as you increase the object distance toward the focal length.

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

59. $d_i = \frac{1}{(1/f) - (1/d_o)}$

$$\frac{d_i}{d_o} = 6.667 \times 10^{-13} = \frac{h_i}{h_o}$$

$$h_i = -0.933 \text{ mm}$$

61. $d_i = -6.7 \text{ cm}$

$$h_i = 4.0 \text{ cm}$$

63. 83 cm to the right of the converging lens, $m = -2.3$, $h_i = 6.9 \text{ cm}$

65. $P = 52.0 \text{ D}$

67. $\frac{h_i}{h_o} = -\frac{d_i}{d_o} \Rightarrow h_i = -h_o \left(\frac{d_i}{d_o} \right) = -(3.50 \text{ mm}) \left(\frac{2.00 \text{ cm}}{30.0 \text{ cm}} \right) = -0.233 \text{ mm}$

69. a. $P = +62.5 \text{ D};$

b. $\frac{h_i}{h_o} = -\frac{d_i}{d_o} \Rightarrow h_i = -0.250 \text{ mm};$

c. $h_i = -0.0800 \text{ mm}$

71. $P = \frac{1}{d_o} + \frac{1}{d_i} \Rightarrow d_o = 28.6 \text{ cm}$

73. Originally, the close vision was 51.0 D. Therefore, $P = \frac{1}{d_o} + \frac{1}{d_i} \Rightarrow d_o = 1.00 \text{ m}$

75. originally, $P = 70.0 \text{ D}$; because the power for normal distant vision is 50.0 D, the power should be decreased by 20.0 D

77. $P = \frac{1}{d_o} + \frac{1}{d_i} \Rightarrow d_o = 0.333 \text{ m}$

79. a. $P = 52.0 \text{ D};$

$$P' = 56.16 \text{ D}$$

b.

$$\frac{1}{d_o} + \frac{1}{d_i} = P \Rightarrow d_o = 16.2 \text{ cm}$$

81. We need $d_i = -18.5$ cm when $d_o = \infty$, so

$$P = -5.41 \text{ D}$$

83. Let x = far point

$$\Rightarrow P = \frac{1}{-(x-0.0175 \text{ m})} + \frac{1}{\infty} \Rightarrow -xP + (0.0175 \text{ m})P = 1$$

$$\Rightarrow x = 26.8 \text{ cm}$$

85. $M = 6 \times$

$$M = \left(\frac{25 \text{ cm}}{L} \right) \left(1 + \frac{L-l}{f} \right)$$

87.

$$L - l = d_o$$

$$d_o = 13 \text{ cm}$$

89. $M = 2.5 \times$

91. $M = -2.1 \times$

93. $M = \frac{25 \text{ cm}}{f}$

$$M_{\text{max}} = 5$$

95. $M_{\text{max}}^{\text{young}} = 1 + \frac{18 \text{ cm}}{f} \Rightarrow f = \frac{18 \text{ cm}}{M_{\text{max}}^{\text{young}} - 1}$

97. a. $M_{\text{max}}^{\text{old}} = 9.8 \times$

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \Rightarrow d_i = 4.65 \text{ cm};$$

$$\Rightarrow m = -30.0$$

$$\text{b. } M_{\text{net}} = -240$$

99. a. $\frac{1}{d_o^{\text{obj}}} + \frac{1}{d_i^{\text{obj}}} = \frac{1}{f^{\text{obj}}} \Rightarrow d_i^{\text{obj}} = 18.3 \text{ cm}$ behind the objective lens;

$$\text{b. } m^{\text{obj}} = -60.0;$$

$$d_o^{\text{eye}} = 1.70 \text{ cm}$$

$$\text{c. } d_i^{\text{eye}} = -11.3 \text{ cm}$$

in front of the eyepiece; d. $M^{\text{eye}} = 13.5$;

$$\text{e. } M_{\text{net}} = -810$$

101. $M = -40.0$

103. $f^{\text{obj}} = \frac{R}{2}$, $M = -1.67$

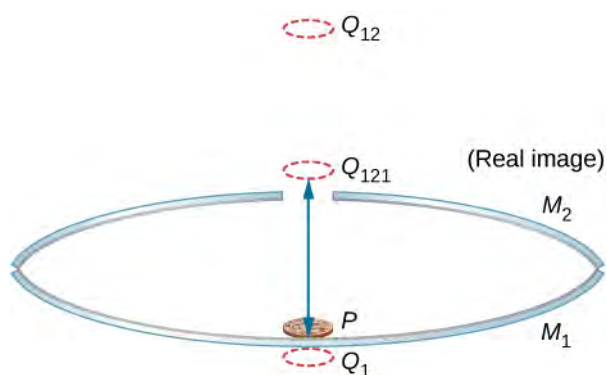
105. $M = -\frac{f^{\text{obj}}}{f^{\text{eye}}}$, $f^{\text{eye}} = +10.0 \text{ cm}$

107. Answers will vary.

109. 12 cm to the left of the mirror, $m = 3/5$

111. 27 cm in front of the mirror, $m = 0.6$, $h_i = 1.76 \text{ cm}$, orientation upright

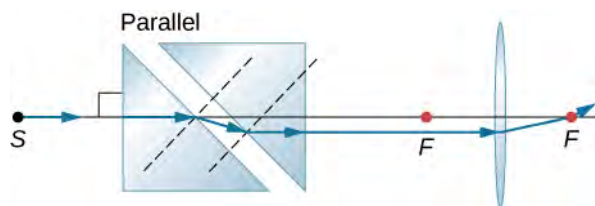
113. The following figure shows three successive images beginning with the image Q_1 in mirror M_1 . Q_1 is the image in mirror M_1 , whose image in mirror M_2 is Q_{12} whose image in mirror M_1 is the real image Q_{121} .



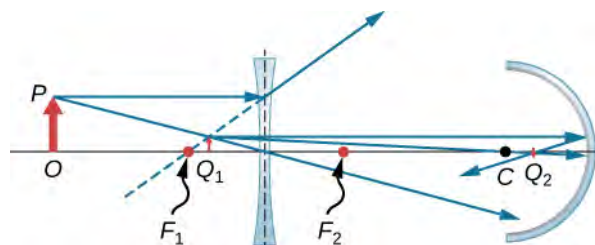
115. 5.4 cm from the axis

117. Let the vertex of the concave mirror be the origin of the coordinate system. Image 1 is at $-10/3$ cm (-3.3 cm), image 2 is at $-40/11$ cm (-3.6 cm). These serve as objects for subsequent images, which are at $-310/83$ cm (-3.7 cm), $-9340/2501$ cm (-3.7 cm), $-140,720/37,681$ cm (-3.7 cm). All remaining images are at approximately -3.7 cm.

119.



121.

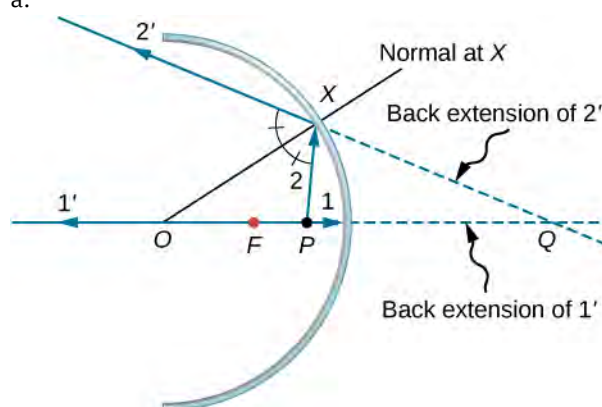


123. -5 D

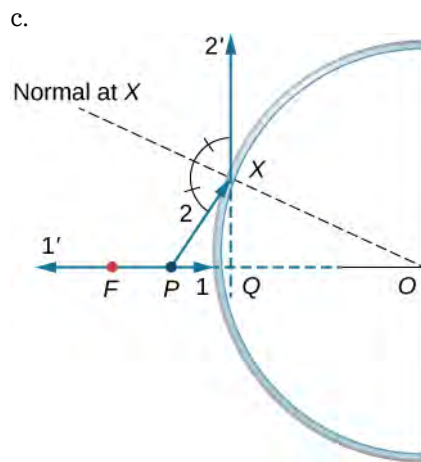
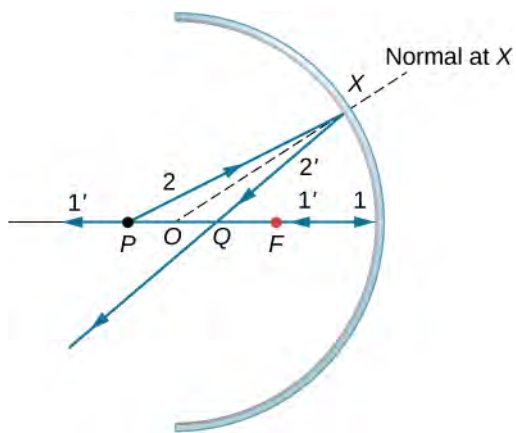
125. 11

Additional Problems

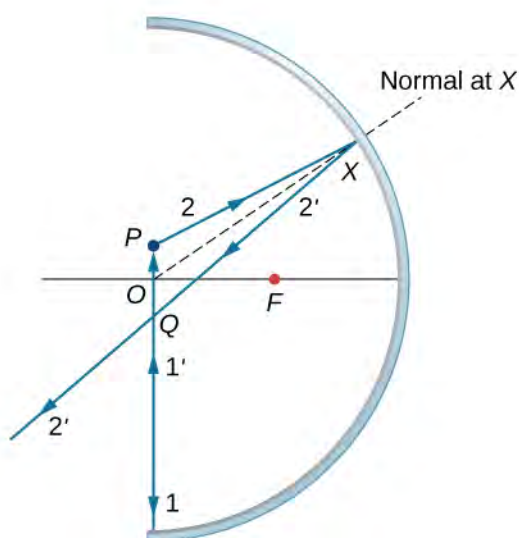
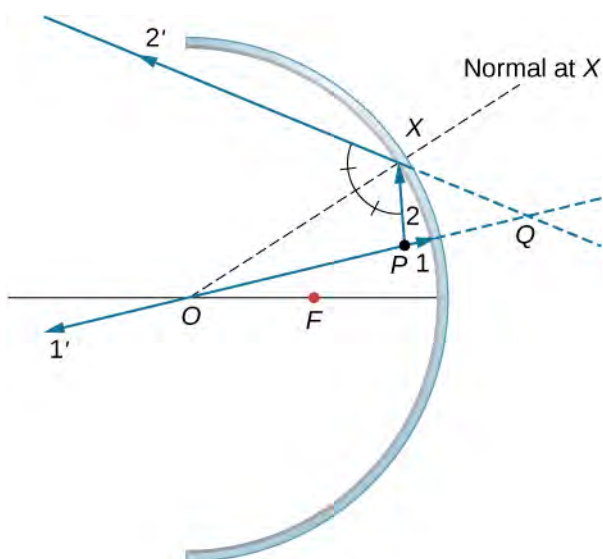
127. a.



b.



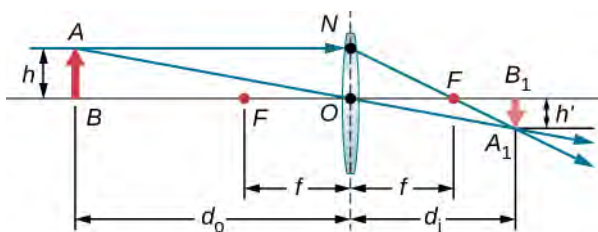
d. similar to the previous picture but with point P outside the focal length; e. Repeat (a)–(d) for a point object off the axis. For a point object placed off axis in front of a concave mirror corresponding to parts (a) and (b), the case for convex mirror left as exercises.



129. $d_i = -10/3$ cm, $h_i = 2$ cm, upright

131. proof

133.



Triangles BAO and B_1A_1O are similar triangles. Thus, $\frac{A_1B_1}{AB} = \frac{d_i}{d_o}$. Triangles NOF and B_1A_1F are similar triangles. Thus, $\frac{NO}{f} = \frac{A_1B_1}{d_i - f}$. Noting that $NO = AB$ gives $\frac{AB}{f} = \frac{A_1B_1}{d_i - f}$ or $\frac{AB}{A_1B_1} = \frac{f}{d_i - f}$. Inverting this gives $\frac{A_1B_1}{AB} = \frac{d_i - f}{f}$. Equating the two expressions for the ratio $\frac{A_1B_1}{AB}$ gives $\frac{d_i}{d_o} = \frac{d_i - f}{f}$. Dividing through by d_i gives $\frac{1}{d_o} = \frac{1}{f} - \frac{1}{d_i}$ or $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$.

135. 70 cm

137. The plane mirror has an infinite focal point, so that $d_i = -d_o$. The total apparent distance of the man in

the mirror will be his actual distance, plus the apparent image distance, or $d_o + (-d_i) = 2d_o$. If this distance must be less than 20 cm, he should stand at $d_o = 10$ cm.

139. Here we want $d_o = 25 \text{ cm} - 2.20 \text{ cm} = 0.228 \text{ m}$. If $x = \text{near point}$, $d_i = -(x - 0.0220 \text{ m})$. Thus,
 $P = \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{0.228 \text{ m}} - \frac{1}{x - 0.0220 \text{ m}}$. Using $P = 0.75 \text{ D}$ gives $x = 0.297 \text{ m}$, so the near point is 29.7 cm.
141. Assuming a lens at 2.00 cm from the boy's eye, the image distance must be
 $d_i = -(500 \text{ cm} - 2.00 \text{ cm}) = -498 \text{ cm}$. For an infinite-distance object, the required power is
 $P = \frac{1}{d_i} = -0.200 \text{ D}$. Therefore, the -4.00 D lens will correct the nearsightedness.
143. $87 \mu\text{m}$
145. Use, $M_{\text{net}} = -\frac{d_i^{\text{obj}}(f^{\text{eye}} + 25 \text{ cm})}{f^{\text{obj}}f^{\text{eye}}}$. The image distance for the objective is $d_i^{\text{obj}} = -\frac{M_{\text{net}}f^{\text{obj}}f^{\text{eye}}}{f^{\text{eye}} + 25 \text{ cm}}$. Using
 $f^{\text{obj}} = 3.0 \text{ cm}$, $f^{\text{eye}} = 10 \text{ cm}$, and $M = -10$ gives $d_i^{\text{obj}} = 8.6 \text{ cm}$. We want this image to be at the focal point of the eyepiece so that the eyepiece forms an image at infinity for comfortable viewing. Thus, the distance d between the lenses should be $d = f^{\text{eye}} + d_i^{\text{obj}} = 10 \text{ cm} + 8.6 \text{ cm} = 19 \text{ cm}$.
147. a. focal length of the corrective lens $f_c = -80 \text{ cm}$; b. -1.25 D
149. $2 \times 10^{16} \text{ km}$
151. 10^5 m

Chapter 3

Check Your Understanding

- 3.1 3.63° and 7.27° , respectively
- 3.2 a. 853 nm, 1097 nm; b. 731 nm, 975 nm
- 3.3 a. too small; b. up to 8×10^{-5}

Conceptual Questions

- No. Two independent light sources do not have coherent phase.
- Because both the sodium lamps are not coherent pairs of light sources. Two lasers operating independently are also not coherent so no interference pattern results.
- Monochromatic sources produce fringes at angles according to $d \sin \theta = m\lambda$. With white light, each constituent wavelength will produce fringes at its own set of angles, blending into the fringes of adjacent wavelengths. This results in rainbow patterns.
- Differing path lengths result in different phases at destination resulting in constructive or destructive interference accordingly. Reflection can cause a 180° phase change, which also affects how waves interfere. Refraction into another medium changes the wavelength inside that medium such that a wave can emerge from the medium with a different phase compared to another wave that travelled the same distance in a different medium.
- Phase changes occur upon reflection at the top of glass cover and the top of glass slide only.
- The surface of the ham being moist means there is a thin layer of fluid, resulting in thin-film interference. Because the exact thickness of the film varies across the piece of ham, which is illuminated by white light, different wavelengths produce bright fringes at different locations, resulting in rainbow colors.
- Other wavelengths will not generally satisfy $t = \frac{\lambda/n}{4}$ for the same value of t so reflections will result in completely destructive interference. For an incidence angle θ , the path length inside the coating will be increased by a factor $1/\cos \theta$ so the new condition for destructive interference becomes $\frac{t}{\cos \theta} = \frac{\lambda/n}{4}$.
- In one arm, place a transparent chamber to be filled with the gas. See [Example 3.6](#).

Problems

17. 0.997°
19. $0.290 \mu\text{m}$
21. $5.77 \times 10^{-7} \text{ m} = 577 \text{ nm}$
23. 62.5; since m must be an integer, the highest order is then $m = 62$.

- 25. $1.44 \mu\text{m}$
- 27. a. 20.3° ; b. 4.98° ; c. 5.76 , the highest order is $m = 5$.
- 29. a. 2.37 cm ; b. 1.78 cm
- 31. 560 nm
- 33. 1.2 mm
- 35. a. $0.40^\circ, 0.53^\circ$; b. $4.6 \times 10^{-3} \text{ m}$
- 37. $1:9$
- 39. 532 nm (green)
- 41. $8.39 \times 10^{-8} \text{ m} = 83.9 \text{ nm}$
- 43. 620 nm (orange)
- 45. 380 nm
- 47. a. Assuming n for the plane is greater than 1.20 , then there are two phase changes: 0.833 cm . b. It is too thick, and the plane would be too heavy. c. It is unreasonable to think the layer of material could be any thickness when used on a real aircraft.
- 49. $4.55 \times 10^{-4} \text{ m}$
- 51. $D = 2.53 \times 10^{-6} \text{ m}$

Additional Problems

- 53. 0.29° and 0.86°
- 55. a. 4.26 cm ; b. 2.84 cm
- 57. 6
- 59. 0.20 m
- 61. 0.0839 mm
- 63. a. $9.8, 10.4, 11.7$, and 15.7 cm ; b. 3.9 cm
- 65. 0.0575°
- 67. 700 nm
- 69. 189 nm
- 71. a. green (504 nm); b. magenta (white minus green)
- 73. 1.29
- 75. $52.7 \mu\text{m}$ and $53.0 \mu\text{m}$
- 77. 125 nm
- 79. 413 nm and 689 nm
- 81. $73.9 \mu\text{m}$
- 83. 47
- 85. $8.5 \mu\text{m}$
- 87. 0.013°C

Challenge Problems

- 89. Bright and dark fringes switch places.
- 91. The path length must be less than one-fourth of the shortest visible wavelength in oil. The thickness of the oil is half the path length, so it must be less than one-eighth of the shortest visible wavelength in oil. If we take 380 nm to be the shortest visible wavelength in air, 33.9 nm .
- 93. $4.42 \times 10^{-5} \text{ m}$
- 95. for one phase change: 950 nm (infrared); for three phase changes: 317 nm (ultraviolet); Therefore, the oil film will appear black, since the reflected light is not in the visible part of the spectrum.

Chapter 4

Check Your Understanding

- 4.1 $17.8^\circ, 37.7^\circ, 66.4^\circ$; no
- 4.2 $74.3^\circ, 0.0083 I_0$
- 4.3 From $d \sin \theta = m\lambda$, the interference maximum occurs at 2.87° for $m = 20$. From [Equation 4.1](#), this is also

the angle for the second diffraction minimum. (Note: Both equations use the index m but they refer to separate phenomena.)

- 4.4 3.332×10^{-6} m or 300 lines per millimeter
 4.5 8.4×10^{-4} rad, 3000 times broader than the Hubble Telescope
 4.6 38.4° and 68.8° ; Between $\theta = 0^\circ \rightarrow 90^\circ$, orders 1, 2, and 3, are all that exist.

Conceptual Questions

- The diffraction pattern becomes wider.
- Walkie-talkies use radio waves whose wavelengths are comparable to the size of the hill and are thus able to diffract around the hill. Visible wavelengths of the flashlight travel as rays at this size scale.
- The diffraction pattern becomes two-dimensional, with main fringes, which are now spots, running in perpendicular directions and fainter spots in intermediate directions.
- The parameter $\beta = \phi/2$ is the arc angle shown in the phasor diagram in Figure 4.7. The phase difference between the first and last Huygens wavelet across the single slit is 2β and is related to the curvature of the arc that forms the resultant phasor that determines the light intensity.
- blue; The shorter wavelength of blue light results in a smaller angle for diffraction limit.
- No, these distances are three orders of magnitude smaller than the wavelength of visible light, so visible light makes a poor probe for atoms.
- UV wavelengths are much larger than lattice spacings in crystals such that there is no diffraction. The Bragg equation implies a value for $\sin \theta$ greater than unity, which has no solution.
- Image will appear at slightly different location and/or size when viewed using 10% shorter wavelength but at exactly half the wavelength, a higher-order interference reconstructs the original image, different color.

Problems

- a. 33.4° ; b. no
- a. 1.35×10^{-6} m; b. 69.9°
- 750 nm
- 2.4 mm, 4.7 mm
- a. 1.00λ ; b. 50.0λ ; c. 1000λ
- 1.92 m
- 45.1°
- $I/I_0 = 2.2 \times 10^{-5}$
- $0.63I_0$, $0.11I_0$, $0.0067I_0$, $0.0062I_0$, $0.00088I_0$
- 0.200
- 3
- 9
- 5.97°
- 8.99×10^3
- 707 nm
- a. 11.8° , 12.5° , 14.1° , 19.2° ; b. 24.2° , 25.7° , 29.1° , 41.0° ; c. Decreasing the number of lines per centimeter by a factor of x means that the angle for the x -order maximum is the same as the original angle for the first-order maximum.
- a. using $\lambda = 700$ nm, $\theta = 5.0^\circ$; b. using $\lambda = 460$ nm, $\theta = 3.3^\circ$
- a. 26,300 lines/cm; b. yes; c. no
- 1.13×10^{-2} m
- 107 m
- a. 7.72×10^{-4} rad; b. 23.2 m; c. 590 km
- a. 2.24×10^{-4} rad; b. 5.81 km; c. 0.179 mm; d. can resolve details 0.2 mm apart at arm's length
- 2.9 μ m
- 6.0 cm
- 7.71 km
- 1.0 m

69. 1.2 cm or closer
 71. no
 73. 0.120 nm
 75. 4.51°
 77. 13.2°

Additional Problems

79. a. 2.2 mm; b. 0.172° , second-order yellow and third-order violet coincide
 81. 2.2 km
 83. 1.3 cm
 85. a. 0.28 mm; b. 0.28 m; c. 280 m; d. 113 km
 87. 33 m
 89. a. vertically; b. $\pm 20^\circ$, $\pm 44^\circ$; c. 0, $\pm 31^\circ$, $\pm 60^\circ$; d. 89 cm; e. 71 cm
 91. 0.98 cm
 93. $I/I_0 = 0.041$
 95. 340 nm
 97. a. 0.082 rad and 0.087 rad; b. 480 nm and 660 nm
 99. two orders
 101. yes and N/A
 103. 600 nm
 105. a. $3.4 \times 10^{-5}^\circ$; b. 51°
 107. 0.63 m
 109. 1
 111. 0.17 mW/cm^2 for $m = 1$ only, no higher orders
 113. 28.7°
 115. a. 42.3 nm; b. This wavelength is not in the visible spectrum. c. The number of slits in this diffraction grating is too large. Etching in integrated circuits can be done to a resolution of 50 nm, so slit separations of 400 nm are at the limit of what we can do today. This line spacing is too small to produce diffraction of light.
 117. a. 549 km; b. This is an unreasonably large telescope. c. Unreasonable to assume diffraction limit for optical telescopes unless in space due to atmospheric effects.

Challenge Problems

119. a. $I = 0.00500 I_0$, $0.00335 I_0$; b. $I = 0.00500 I_0$, $0.00335 I_0$
 121. 12,800
 123. $1.58 \times 10^{-6} \text{ m}$

Chapter 5

Check Your Understanding

- 5.1 Special relativity applies only to objects moving at constant velocity, whereas general relativity applies to objects that undergo acceleration.
- 5.2 $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.650c)^2}{c^2}}} = 1.32$
- 5.3 a. $\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.10 \times 10^{-8} \text{ s}}{\sqrt{1 - \frac{(1.90 \times 10^8 \text{ m/s})^2}{(3.00 \times 10^8 \text{ m/s})^2}}} = 2.71 \times 10^{-8} \text{ s}.$
- 5.3 b. Only the relative speed of the two spacecraft matters because there is no absolute motion through space. The signal is emitted from a fixed location in the frame of reference of A, so the proper time interval of its emission is $\tau = 1.00 \text{ s}$. The duration of the signal measured from frame of reference B is then

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.00 \text{ s}}{\sqrt{1 - \frac{(4.00 \times 10^7 \text{ m/s})^2}{(3.00 \times 10^8 \text{ m/s})^2}}} = 1.01 \text{ s}.$$

$$5.4 \quad L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = (2.50 \text{ km}) \sqrt{1 - \frac{(0.750c)^2}{c^2}} = 1.65 \text{ km}$$

5.5 Start with the definition of the proper time increment:

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - (dx^2 + dy^2 + dz^2)/c^2}.$$

where (dx, dy, dz, cdt) are measured in the inertial frame of an observer who does not necessarily see that particle at rest. This therefore becomes

$$\begin{aligned} d\tau &= \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - [(dx)^2 + (dy)^2 + (dz)^2]/c^2} \\ &= dt \sqrt{1 - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] / c^2} \\ &= dt \sqrt{1 - v^2/c^2} \end{aligned}$$

$$dt = \gamma d\tau.$$

5.6 Although displacements perpendicular to the relative motion are the same in both frames of reference, the time interval between events differ, and differences in dt and dt' lead to different velocities seen from the two frames.

5.7 We can substitute the data directly into the equation for relativistic Doppler frequency:

$$f_{\text{obs}} = f_s \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = (1.50 \text{ GHz}) \sqrt{\frac{1 - \frac{0.350c}{c}}{1 + \frac{0.350c}{c}}} = 1.04 \text{ GHz}.$$

5.8 Substitute the data into the given equation:

$$p = \gamma mu = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{(9.11 \times 10^{-31} \text{ kg})(0.985)(3.00 \times 10^8 \text{ m/s})}{\sqrt{1 - \frac{(0.985c)^2}{c^2}}} = 1.56 \times 10^{-21} \text{ kg-m/s}.$$

$$K_{\text{rel}} = (\gamma - 1) mc^2 = \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) mc^2$$

$$\begin{aligned} 5.9 \quad &= \left(\frac{1}{\sqrt{1 - \frac{(0.992c)^2}{c^2}}} - 1 \right) (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 5.67 \times 10^{-13} \text{ J} \end{aligned}$$

Conceptual Questions

1. the second postulate, involving the speed of light; classical physics already included the idea that the laws of mechanics, at least, were the same in all inertial frames, but the velocity of a light pulse was different in different frames moving with respect to each other
3. yes, provided the plane is flying at constant velocity relative to the Earth; in that case, an object with no force acting on it within the plane has no change in velocity relative to the plane and no change in velocity relative to the Earth; both the plane and the ground are inertial frames for describing the motion of the object
5. The observer moving with the process sees its interval of proper time, which is the shortest seen by any observer.
7. The length of an object is greatest to an observer who is moving with the object, and therefore measures its proper length.
9. a. No, not within the astronaut's own frame of reference. b. He sees Earth clocks to be in their rest frame moving by him, and therefore sees them slowed. c. No, not within the astronaut's own frame of reference. d. Yes, he measures the distance between the two stars to be shorter. e. The two observers agree on their

relative speed.

11. There is no measured change in wavelength or frequency in this case. The relativistic Doppler effect depends only on the relative velocity of the source and the observer, not any speed relative to a medium for the light waves.
13. It shows that the stars are getting more distant from Earth, that the universe is expanding, and doing so at an accelerating rate, with greater velocity for more distant stars.]
15. Yes. This can happen if the external force is balanced by other externally applied forces, so that the net external force is zero.
17. Because it loses thermal energy, which is the kinetic energy of the random motion of its constituent particles, its mass decreases by an extremely small amount, as described by energy-mass equivalence.
19. Yes, in principle there would be a similar effect on mass for any decrease in energy, but the change would be so small for the energy changes in a chemical reaction that it would be undetectable in practice.
21. Not according to special relativity. Nothing with mass can attain the speed of light.

Problems

23. a. 1.0328; b. 1.15
25. $5.96 \times 10^{-8} \text{ s}$
27. $0.800c$
29. $0.140c$
31. 48.6 m
33. Using the values given in [Example 5.3](#): a. 0.627 km; b. 2.00 km; c. 2.00 km
35. a. 10.0c; b. The resulting speed of the canister is greater than c, an impossibility. c. It is unreasonable to assume that the canister will move toward the earth at 1.20c.
37. The angle α approaches 45° , and the t' - and x' -axes rotate toward the edge of the light cone.
39. 15 m/s east
41. 32 m/s
43. a. The second ball approaches with velocity $-v$ and comes to rest while the other ball continues with velocity $-v$; b. This conserves momentum.
45. a. $t_1' = 0; x_1' = 0; t_2' = \tau; x_2' = 0$; b. $t_1' = 0; x_1' = 0; t_2' = \frac{\tau}{\sqrt{1-v^2/c^2}}; x_2' = \frac{-v\tau}{\sqrt{1-v^2/c^2}}$
47. $0.615c$
49. $0.696c$
51. (Proof)
53. $4.09 \times 10^{-19} \text{ kg} \cdot \text{m/s}$
55. a. $3.000000015 \times 10^{13} \text{ kg} \cdot \text{m/s}$; b. 1.000000005
57. $2.988 \times 10^8 \text{ m/s}$
59. 0.512 MeV according to the number of significant figures stated. The exact value is closer to 0.511 MeV.
61. $2.3 \times 10^{-30} \text{ kg}$; to two digits because the difference in rest mass energies is found to two digits
63. a. $1.11 \times 10^{27} \text{ kg}$; b. 5.56×10^{-5}
65. a. $7.1 \times 10^{-3} \text{ kg}$; b. $7.1 \times 10^{-3} = 7.1 \times 10^{-3}$; c. $\frac{\Delta m}{m}$ is greater for hydrogen
67. a. 208; b. 0.999988c; six digits used to show difference from c
69. a. $6.92 \times 10^5 \text{ J}$; b. 1.54
71. a. 0.914c; b. The rest mass energy of an electron is 0.511 MeV, so the kinetic energy is approximately 150% of the rest mass energy. The electron should be traveling close to the speed of light.

Additional Problems

73. a. $0.866c$; b. $0.995c$
75. a. 4.303 y to four digits to show any effect; b. 0.1434 y; c. $1/\sqrt{1-v^2/c^2} = 29.88$.
77. a. 4.00; b. $v = 0.867c$
79. a. A sends a radio pulse at each heartbeat to B, who knows their relative velocity and uses the time dilation

formula to calculate the proper time interval between heartbeats from the observed signal. b.

$$(66 \text{ beats/min}) \sqrt{1 - v^2/c^2} = 57.1 \text{ beats/min}$$

81. a. first photon: $(0, 0, 0)$ at $t = t'$; second photon:

$$t' = \frac{-vx/c^2}{\sqrt{1-v^2/c^2}} = \frac{-(c/2)(1.00 \text{ m})/c^2}{\sqrt{0.75}} = -\frac{0.577 \text{ m}}{c} = 1.93 \times 10^{-9} \text{ s}$$

$$x' = \frac{x}{\sqrt{1-v^2/c^2}} = \frac{1.00 \text{ m}}{\sqrt{0.75}} = 1.15 \text{ m}$$

b. simultaneous in A, not simultaneous in B

$$t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}} = \frac{(4.5 \times 10^{-4} \text{ s}) - (0.6c)\left(\frac{150 \times 10^3 \text{ m}}{c^2}\right)}{\sqrt{1-(0.6)^2}} \\ = 1.88 \times 10^{-4} \text{ s}$$

$$83. \quad x' = \frac{x - vt}{\sqrt{1-v^2/c^2}} = \frac{150 \times 10^3 \text{ m} - (0.60)(3.00 \times 10^8 \text{ m/s})(4.5 \times 10^{-4} \text{ s})}{\sqrt{1-(0.6)^2}}$$

$$= 8.6 \times 10^4 \text{ m} = 86 \text{ km}$$

$$y = y' = 15 \text{ km}$$

$$z = z' = 1 \text{ km}$$

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1-v^2/c^2}}$$

- 85.

$$0 = \frac{\Delta t' + v(500 \text{ m})/c^2}{\sqrt{1-v^2/c^2}};$$

since $v \ll c$, we can ignore the term v^2/c^2 and find

$$\Delta t' = -\frac{(50 \text{ m/s})(500 \text{ m})}{(3.00 \times 10^8 \text{ m/s})^2} = -2.78 \times 10^{-13} \text{ s}$$

The breakdown of Newtonian simultaneity is negligibly small, but not exactly zero, at realistic train speeds of 50 m/s.

$$\Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1-v^2/c^2}}$$

$$87. \quad 0 = \frac{(0.30 \text{ s}) - \frac{(v)(2.0 \times 10^9 \text{ m})}{(3.00 \times 10^8 \text{ m/s})^2}}{\sqrt{1-v^2/c^2}}$$

$$v = \frac{(0.30 \text{ s})}{(2.0 \times 10^9 \text{ m})} (3.00 \times 10^8 \text{ m/s})^2$$

$$v = 1.35 \times 10^7 \text{ m/s}$$

89. Note that all answers to this problem are reported to five significant figures, to distinguish the results. a. 0.99947c; b. $1.2064 \times 10^{11} \text{ y}$; c. $1.2058 \times 10^{11} \text{ y}$

91. a. $-0.400c$; b. $-0.909c$

93. a. 1.65 km/s; b. Yes, if the speed of light were this small, speeds that we can achieve in everyday life would be larger than 1% of the speed of light and we could observe relativistic effects much more often.

95. 775 MHz

97. a. $1.12 \times 10^{-8} \text{ m/s}$; b. The small speed tells us that the mass of a protein is substantially smaller than that of even a tiny amount of macroscopic matter.

99. a.

$$F = \frac{dp}{dt} = \frac{d}{dt} \left(\frac{mu}{\sqrt{1-u^2/c^2}} \right) \\ = \frac{du}{dt} \left(\frac{m}{\sqrt{1-u^2/c^2}} \right) - \frac{1}{2} \frac{mu^2}{(1-u^2/c^2)^{3/2}} 2 \frac{du}{dt} ; \\ = \frac{m}{(1-u^2/c^2)^{3/2}} \frac{du}{dt}$$

b.

$$\begin{aligned}
 F &= \frac{m}{(1-u^2/c^2)^{3/2}} \frac{du}{dt} \\
 &= \frac{1 \text{ kg}}{\left(1-\left(\frac{1}{2}\right)^2\right)^{3/2}} (1 \text{ m/s}^2) \\
 &= 1.53 \text{ N}
 \end{aligned}$$

101. 90.0 MeV

103. a. $\gamma^2 - 1$; b. yes105. 1.07×10^3 107. a. $6.56 \times 10^{-8} \text{ kg}$; b. $m = (200 \text{ L}) (1 \text{ m}^3/1000 \text{ L}) (750 \text{ kg/m}^3) = 150 \text{ kg}$; therefore,

$$\frac{\Delta m}{m} = 4.37 \times 10^{-10}$$

109. a. 0.314c; b. 0.99995c (Five digits used to show difference from c)

111. a. 1.00 kg; b. This much mass would be measurable, but probably not observable just by looking because it is 0.01% of the total mass.

113. a. $6.06 \times 10^{11} \text{ kg/s}$; b. $4.67 \times 10^{10} \text{ y}$; c. $4.27 \times 10^9 \text{ kg}$; d. 0.32%

Chapter 6

Check Your Understanding

6.1 Bunsen's burner

6.2 The wavelength of the radiation maximum decreases with increasing temperature.

6.3 $T_\alpha/T_\beta = 1/\sqrt{3} \cong 0.58$, so the star β is hotter.6.4 $3.3 \times 10^{-19} \text{ J}$ 6.5 No, because then $\Delta E/E \approx 10^{-21}$ 6.6 -0.91 V ; 1040 nm6.7 $h = 6.40 \times 10^{-34} \text{ J} \cdot \text{s} = 4.0 \times 10^{-15} \text{ eV} \cdot \text{s}$; -3.5% 6.8 $(\Delta\lambda)_{\min} = 0 \text{ m}$ at a 0° angle; $71.0 \text{ pm} + 0.5\lambda_c = 72.215 \text{ pm}$

6.9 121.5 nm and 91.1 nm; no, these spectral bands are in the ultraviolet

6.10 $v_2 = 1.1 \times 10^6 \text{ m/s} \cong 0.0036c$; $L_2 = 2\hbar$ $K_2 = 3.4 \text{ eV}$

6.11 29 pm

6.12 $\lambda = 2\pi na_0 = 2(3.324 \text{ \AA}) = 6.648 \text{ \AA}$ 6.13 $\lambda = 2.14 \text{ pm}$; $K = 261.56 \text{ keV}$ 6.14 0.052°

6.15 doubles it

Conceptual Questions

1. yellow

3. goes from red to violet through the rainbow of colors

5. would not differ

7. human eye does not see IR radiation

9. No

11. from the slope

13. Answers may vary

15. the particle character

17. Answers may vary

19. no; yes

21. no

23. right angle

25. no

27. They are at ground state.

- 29. Answers may vary
- 31. increase
- 33. for larger n
- 35. Yes, the excess of 13.6 eV will become kinetic energy of a free electron.
- 37. no
- 39. X-rays, best resolving power
- 41. proton
- 43. negligibly small de Broglie's wavelengths
- 45. to avoid collisions with air molecules
- 47. Answers may vary
- 49. Answers may vary
- 51. yes
- 53. yes

Problems

- 55. a. 0.81 eV; b. 2.1×10^{23} ; c. 2 min 20 sec
- 57. a. 7245 K; b. 3.62 μm
- 59. about 3 K
- 61. 4.835×10^{18} Hz; 0.620 $\text{\AA}$
- 63. 263 nm; no
- 65. 3.68 eV
- 67. 4.09 eV
- 69. 5.60 eV
- 71. a. 1.89 eV; b. 459 THz; c. 1.21 V
- 73. 264 nm; UV
- 75. 1.95×10^6 m/s
- 77. 1.66×10^{-32} kg $\cdot$ m/s
- 79. 56.21 eV
- 81. 6.63×10^{-23} kg $\cdot$ m/s; 124 keV
- 83. 82.9 fm; 15 MeV
- 85. (Proof)
- 87. $\Delta\lambda_{30}/\Delta\lambda_{45} = 45.74\%$
- 89. 121.5 nm
- 91. a. 0.661 eV; b. -10.2 eV; c. 1.511 eV
- 93. 3038 THz
- 95. 97.33 nm
- 97. a. h/π ; b. 3.4 eV; c. -6.8 eV; d. -3.4 eV
- 99. $n = 4$
- 101. 365 nm; UV
- 103. no
- 105. 7
- 107. 145.5 pm
- 109. 20 fm; 9 fm
- 111. a. 2.103 eV; b. 0.846 nm
- 113. 80.9 pm
- 115. 2.21×10^{-19} m/s
- 117. 9.929×10^{32}
- 119. $\gamma = 1060$; 0.00124 fm
- 121. 24.11 V
- 123. a. $P = 2I/c = 8.67 \times 10^{-6}$ N/m²; b. $a = PA/m = 8.67 \times 10^{-4}$ m/s²; c. 74.91 m/s
- 125. $x = 4.965$

Additional Problems

127. $7.124 \times 10^{16} \text{ W/m}^3$
 129. 1.034 eV
 131. 5.93×10^{18}
 133. 387.8 nm
 135. a. 4.02×10^{15} ; b. 0.533 mW
 137. a. 4.02×10^{15} ; b. 0.533 mW; c. 0.644 mA; d. 2.57 ns
 139. a. 0.132 pm; b. 9.39 MeV; c. 0.047 MeV
 141. a. 2 kJ; b. $1.33 \times 10^{-5} \text{ kg} \cdot \text{m/s}$; c. $1.33 \times 10^{-5} \text{ N}$; d. yes
 143. a. 0.003 nm; b. 105.56°
 145. $n = 3$
 147. a. $a_0/2$; b. $-54.4 \text{ eV}/n^2$; c. $a_0/3, -122.4 \text{ eV}/n^2$
 149. a. 36; b. 18.2 nm; c. UV
 151. 396 nm; 5.23 neV
 153. 7.3 keV
 155. 728 m/s; $1.5 \mu\text{V}$
 157. $\lambda = hc / \sqrt{K(2E_0 + K)} = 3.705 \times 10^{-12} \text{ m}$, $K = 100 \text{ keV}$
 159. $\Delta\lambda_c^{(\text{electron})} / \Delta\lambda_c^{(\text{proton})} = m_p / m_e = 1836$
 161. (Proof)
 163. $5.1 \times 10^{17} \text{ Hz}$

Chapter 7

Check Your Understanding

- 7.1 $(3 + 4i)(3 - 4i) = 9 - 16i^2 = 25$
 7.2 $A = \sqrt{2/L}$
 7.3 $(1/2 - 1/\pi)/2 = 9\%$
 7.4 $4.1 \times 10^{-8} \text{ eV}$; $1.1 \times 10^{-5} \text{ nm}$
 7.5 $0.5m\omega^2 x^2 \psi(x)^* \psi(x)$
 7.6 None. The first function has a discontinuity; the second function is double-valued; and the third function diverges so is not normalizable.
 7.7 a. 9.1%; b. 25%
 7.8 a. 295 N/m; b. 0.277 eV
 7.9 $\langle x \rangle = 0$
 7.10 $L_{\text{proton}}/L_{\text{electron}} = \sqrt{m_e/m_p} = 2.3\%$

Conceptual Questions

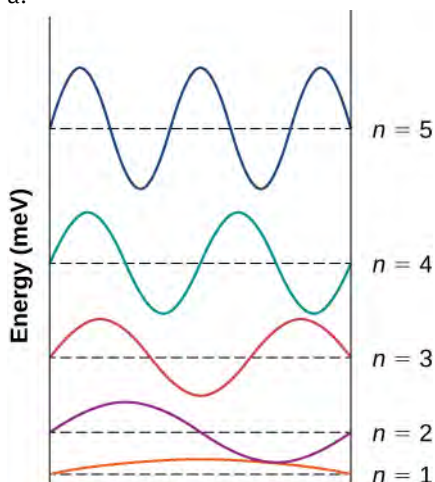
- $1/\sqrt{L}$, where $L = \text{length}$; $1/L$, where $L = \text{length}$
- The wave function does not correspond directly to any measured quantity. It is a tool for predicting the values of physical quantities.
- The average value of the physical quantity for a large number of particles with the same wave function.
- Yes, if its position is completely unknown. Yes, if its momentum is completely unknown.
- No. According to the uncertainty principle, if the uncertainty on the particle's position is small, the uncertainty on its momentum is large. Similarly, if the uncertainty on the particle's position is large, the uncertainty on its momentum is small.
- No, it means that predictions about the particle (expressed in terms of probabilities) are time-independent.
- No, because the probability of the particle existing in a narrow (infinitesimally small) interval at the discontinuity is undefined.
- No. For an infinite square well, the spacing between energy levels increases with the quantum number n . The *smallest* energy measured corresponds to the transition from $n = 2$ to 1, which is three times the

ground state energy. The largest *energy* measured corresponds to a transition from $n = \infty$ to 1, which is infinity. (Note: Even particles with extremely large energies remain bound to an infinite square well—they can never “escape”)

17. No. This energy corresponds to $n = 0.25$, but n must be an integer.
19. Because the smallest allowed value of the quantum number n for a simple harmonic oscillator is 0. No, because quantum mechanics and classical mechanics agree only in the limit of large n .
21. Yes, within the constraints of the uncertainty principle. If the oscillating particle is localized, the momentum and therefore energy of the oscillator are distributed.
23. doubling the barrier width
25. No, the restoring force on the particle at the walls of an infinite square well is infinity.

Problems

27. $|\psi(x)|^2 \sin^2 \omega t$
29. (a) and (e), can be normalized
31. a. $A = \sqrt{2\alpha/\pi}$; b. probability = 29.3%; c. $\langle x \rangle = 0$; d. $\langle p \rangle = 0$; e. $\langle K \rangle = \alpha^2 \hbar^2 / 2m$
33. a. $\Delta p \geq 2.11 \times 10^{-34} \text{ N} \cdot \text{s}$; b. $\Delta v \geq 6.31 \times 10^{-8} \text{ m}$; c. $\Delta v / \sqrt{k_B T / m_\alpha} = 5.94 \times 10^{-11}$
35. $\Delta \tau \geq 1.6 \times 10^{-25} \text{ s}$
37. a. $\Delta f \geq 1.59 \text{ MHz}$; b. $\Delta \omega / \omega_0 = 3.135 \times 10^{-9}$
39. Carrying out the derivatives yields $k^2 = \frac{\omega^2}{c^2}$.
41. Carrying out the derivatives (as above) for the sine function gives a cosine on the right side the equation, so the equality fails. The same occurs for the cosine solution.
43. $E = \hbar^2 k^2 / 2m$
45. $\hbar^2 k^2$; The particle has definite momentum and therefore definite momentum squared.
47. 9.4 eV, 64%
49. 0.38 nm
51. 1.82 MeV
53. 24.7 nm
55. 6.03 Å
57. a.



;

b. $\lambda_{5 \rightarrow 3} = 12.9 \text{ nm}$, $\lambda_{3 \rightarrow 1} = 25.8 \text{ nm}$, $\lambda_{4 \rightarrow 3} = 29.4 \text{ nm}$

59. proof
61. $6.662 \times 10^{14} \text{ Hz}$
63. $n \approx 2.037 \times 10^{30}$

65. $\langle x \rangle = 0.5m\omega^2 \left\langle x^2 \right\rangle = \hbar\omega/4$; $\langle K \rangle = \langle E \rangle - \langle U \rangle = \hbar\omega/4$
67. proof
69. A complex function of the form, $Ae^{i\phi}$, satisfies Schrödinger's time-independent equation. The operators for kinetic and total energy are linear, so any linear combination of such wave functions is also a valid solution to Schrödinger's equation. Therefore, we conclude that [Equation 7.68](#) satisfies [Equation 7.61](#), and [Equation 7.69](#) satisfies [Equation 7.63](#).
71. a. 4.21%; b. 0.84%; c. 0.06%
73. a. 0.13%; b. close to 0%
75. 0.38 nm

Additional Problems

77. proof
79. a. 4.0 %; b. 1.4 %; c. 4.0%; d. 1.4%
81. a. $t = mL^2/h = 2.15 \times 10^{26}$ years; b. $E_1 = 1.46 \times 10^{-66}$ J, $K = 0.4$ J
83. proof
85. 1.2 N/m
87. 0

Challenge Problems

89. 19.2 μm ; 11.5 μm
91. 3.92%
93. proof

Chapter 8

Check Your Understanding

- 8.1 No. The quantum number $m = -l, -l + 1, \dots, 0, \dots, l - 1, l$. Thus, the magnitude of L_z is always less than L because $< \sqrt{l(l+1)}$
- 8.2 $s = 3/2 <$
- 8.3 frequency quadruples

Conceptual Questions

- n (principal quantum number) $\rightarrow$ total energy
 l (orbital angular quantum number) $\rightarrow$ total absolute magnitude of the orbital angular momentum m (orbital angular projection quantum number) $\rightarrow$ z-component of the orbital angular momentum
- The Bohr model describes the electron as a particle that moves around the proton in well-defined orbits. Schrödinger's model describes the electron as a wave, and knowledge about the position of the electron is restricted to probability statements. The total energy of the electron in the ground state (and all excited states) is the same for both models. However, the orbital angular momentum of the ground state is different for these models. In Bohr's model, L (ground state) = 1, and in Schrödinger's model, L (ground state) = 0.
- a, c, d; The total energy is changed (Zeeman splitting). The work done on the hydrogen atom rotates the atom, so the z-component of angular momentum and polar angle are affected. However, the angular momentum is not affected.
- Even in the ground state ($l = 0$), a hydrogen atom has magnetic properties due the intrinsic (internal) electron spin. The magnetic moment of an electron is proportional to its spin.
- For all electrons, $s = 1/2$ and $m_s = \pm 1/2$. As we will see, not all particles have the same spin quantum number. For example, a photon as a spin 1 ($s = 1$), and a Higgs boson has spin 0 ($s = 0$).
- An electron has a magnetic moment associated with its intrinsic (internal) spin. Spin-orbit coupling occurs when this interacts with the magnetic field produced by the orbital angular momentum of the electron.

13. Elements that belong in the same column in the periodic table of elements have the same fillings of their outer shells, and therefore the same number of valence electrons. For example:
 Li: $1s^2 2s^1$ (one valence electron in the $n = 2$ shell)
 Na: $1s^2 2s^2 2p^6 3s^1$ (one valence electron in the $n = 2$ shell)
 Both, Li and Na belong to first column.
15. Atomic and molecular spectra are said to be “discrete,” because only certain spectral lines are observed. In contrast, spectra from a white light source (consisting of many photon frequencies) are continuous because a continuous “rainbow” of colors is observed.
17. UV light consists of relatively high frequency (short wavelength) photons. So the energy of the absorbed photon and the energy transition (ΔE) in the atom is relatively large. In comparison, visible light consists of relatively lower-frequency photons. Therefore, the energy transition in the atom and the energy of the emitted photon is relatively small.
19. For macroscopic systems, the quantum numbers are very large, so the energy difference (ΔE) between adjacent energy levels (orbits) is very small. The energy released in transitions between these closely spaced energy levels is much too small to be detected.
21. Laser light relies on the process of stimulated emission. In this process, electrons must be prepared in an excited (upper) metastable state such that the passage of light through the system produces de-excitations and, therefore, additional light.
23. A Blu-Ray player uses blue laser light to probe the bumps and pits of the disc and a CD player uses red laser light. The relatively short-wavelength blue light is necessary to probe the smaller pits and bumps on a Blu-ray disc; smaller pits and bumps correspond to higher storage densities.

Problems

25. $(r, \theta, \phi) = (\sqrt{6}, 66^\circ, 27^\circ)$.
27. $\pm 3, \pm 2, \pm 1, 0$ are possible
29. 18
31. $F = -k \frac{Qq}{r^2}$
33. (1, 1, 1)
35. For the orbital angular momentum quantum number, l , the allowed values of:
 $m = -l, -l + 1, \dots, 0, \dots, l - 1, l$.
 With the exception of $m = 0$, the total number is just $2l$ because the number of states on either side of $m = 0$ is just l . Including $m = 0$, the total number of orbital angular momentum states for the orbital angular momentum quantum number, l , is: $2l + 1$. Later, when we consider electron spin, the total number of angular momentum states will be found to twice this value because each orbital angular momentum states is associated with two states of electron spin: spin up and spin down).
37. The probability that the $1s$ electron of a hydrogen atom is found outside of the Bohr radius is

$$\int_{a_0}^{\infty} P(r) dr \approx 0.68$$
39. For $n = 2$, $l = 0$ (1 state), and $l = 1$ (3 states). The total is 4.
41. The $3p$ state corresponds to $n = 3$, $l = 2$. Therefore, $\mu = \mu_B \sqrt{6}$
43. The ratio of their masses is $1/207$, so the ratio of their magnetic moments is 207. The electron's magnetic moment is more than 200 times larger than the muon.
45. a. The $3d$ state corresponds to $n = 3$, $l = 2$. So,
 $I = 4.43 \times 10^{-7}$ A.
 b. The maximum torque occurs when the magnetic moment and external magnetic field vectors are at right angles ($\sin \theta = 1$). In this case:
 $|\vec{\tau}| = \mu B$.
 $\tau = 5.70 \times 10^{-26}$ N · m.
47. A $3p$ electron is in the state $n = 3$ and $l = 1$. The minimum torque magnitude occurs when the magnetic moment and external magnetic field vectors are most parallel (antiparallel). This occurs when $m = \pm 1$. The torque magnitude is given by

$$|\vec{\tau}| = \mu B \sin \theta,$$

Where

$$\mu = (1.31 \times 10^{-24} \text{ J/T}).$$

For $m = \pm 1$, we have:

$$|\vec{\tau}| = 2.32 \times 10^{21} \text{ N} \cdot \text{m}.$$

49. An infinitesimal work dW done by a magnetic torque τ to rotate the magnetic moment through an angle $-d\theta$:

$$dW = \tau (-d\theta),$$

where $\tau = |\vec{\mu} \times \vec{B}|$. Work done is interpreted as a drop in potential energy U , so

$$dW = -dU.$$

The total energy change is determined by summing over infinitesimal changes in the potential energy:

$$U = -\mu B \cos \theta$$

$$U = -\vec{\mu} \cdot \vec{B}.$$

51. Spin up (relative to positive z -axis):

$$\theta = 55^\circ.$$

Spin down (relative to positive z -axis):

$$\theta = \cos^{-1} \left(\frac{S_z}{S} \right) = \cos^{-1} \left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) = \cos^{-1} \left(\frac{-1}{\sqrt{3}} \right) = 125^\circ.$$

53. The spin projection quantum number is $m_s = \pm 1/2$, so the z -component of the magnetic moment is

$$\mu_z = \pm \mu_B.$$

The potential energy associated with the interaction between the electron and the external magnetic field is

$$U = \mp \mu_B B.$$

The energy difference between these states is $\Delta E = 2\mu_B B$, so the wavelength of light produced is

$$\lambda = 8.38 \times 10^{-5} \text{ m} \approx 84 \text{ } \mu\text{m}$$

55. It is increased by a factor of 2.

57. a. 32; b.

ℓ	$2(2\ell + 1)$	
0	s	2(0 + 1) = 2
1	p	2(2 + 1) = 6
2	d	2(4 + 1) = 10
3	f	2(6 + 1) = 14
		32

59. a. and e. are allowed; the others are not allowed.

b. $l = 3$ not allowed for $n = 1$, $l \leq (n - 1)$.

c. Cannot have three electrons in s subshell because $3 > 2(2l + 1) = 2$.

d. Cannot have seven electrons in p subshell (max of 6) $2(2l + 1) = 2(2 + 1) = 6$.

61. $[\text{Ar}] 4s^2 3d^6$

63. a. The minimum value of ℓ is $l = 2$ to have nine electrons in it.

b. $3d^9$.

65. $[\text{He}] 2s^2 2p^2$

67. For He^+ , one electron "orbits" a nucleus with two protons and two neutrons ($Z = 2$). Ionization energy refers to the energy required to remove the electron from the atom. The energy needed to remove the electron in the ground state of He^+ ion to infinity is negative the value of the ground state energy, written:

$$E = -54.4 \text{ eV}.$$

Thus, the energy to ionize the electron is $+54.4 \text{ eV}$.

Similarly, the energy needed to remove an electron in the first excited state of Li^{2+} ion to infinity is negative the value of the first excited state energy, written:

$$E = -30.6 \text{ eV}.$$

The energy to ionize the electron is 30.6 eV .

69. The wavelength of the laser is given by:

$$\lambda = \frac{hc}{-\Delta E},$$

where E_γ is the energy of the photon and ΔE is the magnitude of the energy difference. Solving for the latter, we get:

$$\Delta E = -2.795 \text{ eV}.$$

The negative sign indicates that the electron lost energy in the transition.

71. $\Delta E_{L \rightarrow K} \approx (Z - 1)^2 (10.2 \text{ eV}) = 3.68 \times 10^3 \text{ eV}.$

73. According to the conservation of the energy, the potential energy of the electron is converted completely into kinetic energy. The initial kinetic energy of the electron is zero (the electron begins at rest). So, the kinetic energy of the electron just before it strikes the target is:

$$K = e\Delta V.$$

If *all* of this energy is converted into braking radiation, the frequency of the emitted radiation is a maximum, therefore:

$$f_{\max} = \frac{e\Delta V}{h}.$$

When the emitted frequency is a maximum, then the emitted wavelength is a minimum, so:

$$\lambda_{\min} = 0.1293 \text{ nm}.$$

75. A muon is 200 times heavier than an electron, but the minimum wavelength does not depend on mass, so the result is unchanged.

77. $4.13 \times 10^{-11} \text{ m}$

79. 72.5 keV

81. The atomic numbers for Cu and Au are $Z = 29$ and 79, respectively. The X-ray photon frequency for gold is greater than copper by a factor:

$$\left(\frac{f_{\text{Au}}}{f_{\text{Cu}}}\right)^2 = \left(\frac{79-1}{29-1}\right)^2 \approx 8.$$

Therefore, the X-ray wavelength of Au is about eight times shorter than for copper.

83. a. If flesh has the same density as water, then we used 1.34×10^{23} photons. b. 2.52 MW

Additional Problems

85. The smallest angle corresponds to $l = n - 1$ and $m = l = n - 1$. Therefore $\theta = \cos^{-1} \left(\sqrt{\frac{n-1}{n}} \right).$

87. a. According to [Equation 8.1](#), when $r = 0$, $U(r) = -\infty$, and when $r = +\infty$, $U(r) = 0$. b. The former result suggests that the electron can have an infinite negative potential energy. The quantum model of the hydrogen atom avoids this possibility because the probability density at $r = 0$ is zero.

89. A formal solution using sums is somewhat complicated. However, the answer easily found by studying the mathematical pattern between the principal quantum number and the total number of orbital angular momentum states.

For $n = 1$, the total number of orbital angular momentum states is 1; for $n = 2$, the total number is 4; and, when $n = 3$, the total number is 9, and so on. The pattern suggests the total number of orbital angular momentum states for the n th shell is n^2 .

(Later, when we consider electron spin, the total number of angular momentum states will be found to be $2n^2$, because each orbital angular momentum states is associated with two states of electron spin; spin up and spin down).

91. 50

93. The maximum number of orbital angular momentum electron states in the n th shell of an atom is n^2 . Each of these states can be filled by a spin up and spin down electron. So, the maximum number of electron states in the n th shell is $2n^2$.

95. a., c., and e. are allowed; the others are not allowed. b. $l > n$ is not allowed.
d. $7 > 2(2l + 1)$

97. $f = 1.8 \times 10^9 \text{ Hz}$

99. The atomic numbers for Cu and Ag are $Z = 29$ and 47, respectively. The X-ray photon frequency for silver is greater than copper by the following factor:

$$\left(\frac{f_{\text{Ag}}}{f_{\text{Cu}}}\right)^2 = 2.7.$$

Therefore, the X-ray wavelength of Ag is about three times shorter than for copper.

101. a. 3.24; b. n_i is not an integer. c. The wavelength must not be correct. Because $n_i > 2$, the assumption that the line was from the Balmer series is possible, but the wavelength of the light did not produce an integer value for n_i . If the wavelength is correct, then the assumption that the gas is hydrogen is not correct; it might be sodium instead.

Chapter 9

Check Your Understanding

- 9.1 It corresponds to a repulsive force between core electrons in the ions.
 9.2 the moment of inertia
 9.3 more difficult
 9.4 It decreases.
 9.5 The forward bias current is much larger. To a good approximation, diodes permit current flow in only one direction.
 9.6 a low temperature and low magnetic field

Conceptual Questions

- An ionic bond is formed by the attraction of a positive and negative ion. A covalent bond is formed by the sharing of one or more electrons between atoms. A van der Waals bond is formed by the attraction of two electrically polarized molecules.
1. An electron is removed from one atom. The resulting atom is a positive ion. 2. An electron is absorbed from another atom. The result atom is a negative ion. 3. The positive and negative ions are attracted together until an equilibrium separation is reached.
- Bonding is associated with a spatial function that is symmetric under exchange of the two electrons. In this state, the electron density is largest between the atoms. The total function must be antisymmetric (since electrons are fermions), so the spin function must be antisymmetric. In this state, the spins of the electrons are antiparallel.
- rotational energy, vibrational energy, and atomic energy
- Each ion is in the field of multiple ions of the other opposite charge.
- 6, 6
- 0.399 nm
- increases by a factor of $\sqrt[3]{8^2} = 4$
- For larger energies, the number of accessible states increases.
- (1) Solve Schrödinger's equation for the allowed states and energies. (2) Determine energy levels for the case of a very large lattice spacing and then determine the energy levels as this spacing is reduced.
- For N atoms spaced far apart, there are N different wave functions, all with the same energy (similar to the case of an electron in the double well of H_2). As the atoms are pushed together, the energies of these N different wave functions are split. By the exclusion principle, each electron must each have a unique set of quantum numbers, so the N atoms bringing N electrons together must have at least N states.
- For a semiconductor, there is a relatively large energy gap between the lowest completely filled band and the next available unfilled band. Typically, a number of electrons traverse the gap and therefore the electrical conductivity is small. The properties of a semiconductor are sensitivity to temperature: As the temperature is increased, thermal excitations promote charge carriers from the valence band across the gap and into the conduction band.
- a. Germanium has four valence electrons. If germanium doped with *arsenic* (five valence electrons), four are used in bonding and one electron will be left for conduction. This produces an n -type material. b. If germanium is doped with *gallium* (three valence electrons), all three electrons are used in bonding, leaving one hole for conduction. This results in a p -type material.
- The Hall effect is the production of a potential difference due to motion of a conductor through an external

magnetic field. This effect can be used to determine the drift velocity of the charge carriers (electrons or hole). If the current density is measured, this effect can also determine the number of charge carriers per unit volume.

29. It produces new unfilled energy levels just above the filled valence band. These levels accept electrons from the valence band.
31. The electric field produced by the uncovered ions reduces further diffusion. In equilibrium, the diffusion and drift currents cancel so the net current is zero. Therefore, the resistance of the depletion region is large.
33. The positive terminal is applied to the n -side, which uncovers more ions near the junction (widens the depletion layer), increases the junction voltage difference, and therefore reduces the diffusion of holes across the junction.
35. Sound moves a diaphragm in and out, which varies the input or base current of the transistor circuit. The transistor amplifies this signal (p - n - p semiconductor). The output or collector current drives a speaker.
37. BSC theory explains superconductivity in terms of the interactions between electron pairs (Cooper pairs). One electron in a pair interacts with the lattice, which interacts with the second electron. The combine electron-lattice-electron interaction binds the electron pair together in a way that overcomes their mutual repulsion.
39. As the magnitude of the magnetic field is increased, the critical temperature decreases.

Problems

41. $U = -5.16 \text{ eV}$
43. $-4.43 \text{ eV} = -4.69 \text{ eV} + U_{\text{ex}}, U_{\text{ex}} = 0.26 \text{ eV}$
45. The measured value is 0.484 nm , and the actual value is close to 0.127 nm . The laboratory results are the same order of magnitude, but a factor 4 high.
47. 0.110 nm
49. a. $E = 2.2 \times 10^{-4} \text{ eV}$; b. $\Delta E = 4.4 \times 10^{-4} \text{ eV}$
51. 0.65 nm
53. $r_0 = 0.240 \text{ nm}$
55. 2196 kcal
57. 11.5
59. a. 4% ; b. $4.2 \times 10^{-4}\%$; for very large values of the quantum numbers, the spacing between adjacent energy levels is very small ("in the continuum"). This is consistent with the expectation that for large quantum numbers, quantum and classical mechanics give approximately the same predictions.
61. 10.0 eV
63. 4.55×10^9
65. Fermi energy, $E_F = 7.03 \text{ eV}$, Temperature, $T_F = 8.2 \times 10^4 \text{ K}$
67. For an insulator, the energy gap between the valence band and the conduction band is larger than for a semiconductor.
69. 4.13 keV
71. $n = 1.56 \times 10^{19} \text{ holes/m}^3$
73. 5 T
75. $V_b = 0.458 \text{ V}$
77. $T = 829 \text{ K}$
79. $T = 0.707 T_c$
81. 61 kV

Additional Problems

83. $U_{\text{coul}} = -5.65 \text{ eV}$
 $E_{\text{form}} = -4.71 \text{ eV}, E_{\text{diss}} = 4.71 \text{ eV}$
85. $E_{0r} = 7.43 \times 10^{-3} \text{ eV}$
87. $E_{0r} = 7.43 \times 10^{-3} \text{ eV}; l = 0; E_r = 0 \text{ eV}$ (no rotation);
 $l = 1; E_r = 1.49 \times 10^{-2} \text{ eV}; l = 2; E_r = 4.46 \times 10^{-2} \text{ eV}$

89. i. They are fairly hard and stable.
 ii. They vaporize at relatively high temperatures (1000 to 2000 K).
 iii. They are transparent to visible radiation, because photons in the visible portion of the spectrum are not energetic enough to excite an electron from its ground state to an excited state.
 iv. They are poor electrical conductors because they contain effectively no free electrons.
 v. They are usually soluble in water, because the water molecule has a large dipole moment whose electric field is strong enough to break the electrostatic bonds between the ions.
91. No, He atoms do not contain valence electrons that can be shared in the formation of a chemical bond.
93. $\sum_1^{N/2} n^2 = \frac{1}{3} \left(\frac{N}{2} \right)^3$, so $\bar{E} = \frac{1}{3} E_F$
95. An impurity band will be formed when the density of the donor atoms is high enough that the orbits of the extra electrons overlap. We saw earlier that the orbital radius is about 50 Angstroms, so the maximum distance between the impurities for a band to form is 100 Angstroms. Thus if we use 1 Angstrom as the interatomic distance between the Si atoms, we find that 1 out of 100 atoms along a linear chain must be a donor atom. And in a three-dimensional crystal, roughly 1 out of 10^6 atoms must be replaced by a donor atom in order for an impurity band to form.
97. a. $E_F = 7.11$ eV; b. $E_F = 3.24$ eV; c. $E_F = 9.46$ eV
99. $9.15 \approx 9$

Challenge Problems

101. In three dimensions, the energy of an electron is given by:
 $E = R^2 E_1$, where $R^2 = n_1^2 + n_2^2 + n_3^2$. Each allowed energy state corresponds to node in N space (n_1, n_2, n_3) . The number of particles corresponds to the number of states (nodes) in the first octant, within a sphere of radius, R . This number is given by: $N = 2 \left(\frac{1}{8} \right) \left(\frac{4}{3} \right) \pi R^3$, where the factor 2 accounts for two states of spin. The density of states is found by differentiating this expression by energy:
 $g(E) = \frac{\pi V}{2} \left(\frac{8m_e}{h^2} \right)^{3/2} E^{1/2}$. Integrating gives: $\bar{E} = \frac{3}{5} E_F$.

Chapter 10

Check Your Understanding

- 10.1 eight
 10.2 harder
 10.3 Half-life is inversely related to decay rate, so the half-life is short. Activity depends on both the number of decaying particles and the decay rate, so the activity can be great or small.
 10.4 Neither; it stays the same.
 10.5 the same
 10.6 the conversion of mass to energy
 10.7 power

Conceptual Questions

- The nucleus of an atom is made of one or more nucleons. A nucleon refers to either a proton or neutron. A nuclide is a stable nucleus.
- A bound system should have less mass than its components because of energy-mass equivalence ($E = mc^2$). If the energy of a system is reduced, the total mass of the system is reduced. If two bricks are placed next to one another, the attraction between them is purely gravitational, assuming the bricks are electrically neutral. The gravitational force between the bricks is relatively small (compared to the strong nuclear force), so the mass defect is much too small to be observed. If the bricks are glued together with cement, the mass defect is likewise small because the electrical interactions between the electrons involved in the bonding are still relatively small.
- Nucleons at the surface of a nucleus interact with fewer nucleons. This reduces the binding energy per nucleon, which is based on an average over all the nucleons in the nucleus.

7. That it is constant.
9. Gamma (γ) rays are produced by nuclear interactions and X-rays and light are produced by atomic interactions. Gamma rays are typically shorter wavelength than X-rays, and X-rays are shorter wavelength than light.
11. Assume a rectangular coordinate system with an xy -plane that corresponds to the plane of the paper. α bends into the page (trajectory parabolic in the xz -plane); β^+ bends into the page (trajectory parabolic in the xz -plane); and γ is unbent.
13. Yes. An atomic bomb is a fission bomb, and a fission bomb occurs by splitting the *nucleus* of atom.
15. Short-range forces between nucleons in a nucleus are analogous to the forces between water molecules in a water droplet. In particular, the forces between nucleons at the surface of the nucleus produce a surface tension similar to that of a water droplet.
17. The nuclei produced in the fusion process have a larger binding energy per nucleon than the nuclei that are fused. That is, nuclear fusion decreases average energy of the nucleons in the system. The energy difference is carried away as radiation.
19. Alpha particles do not penetrate materials such as skin and clothes easily. (Recall that alpha radiation is barely able to pass through a thin sheet of paper.) However, when produce inside the body, neighboring cells are vulnerable.

Problems

21. Use the rule $A = Z + N$.

	Atomic Number (Z)	Neutron Number (N)	Mass Number (A)
(a)	29	29	58
(b)	11	13	24
(c)	84	126	210
(d)	20	25	45
(e)	82	124	206

23. a. $r = r_0 A^{1/3}$, $\rho = \frac{3u}{4\pi r_0^3}$;
b. $\rho = 2.3 \times 10^{17} \text{ kg/m}^3$
25. side length = $1.6 \mu\text{m}$
27. 92.4 MeV
29. $8.790 \text{ MeV} \approx \text{graph's value}$
31. a. 7.570 MeV ; b. $7.591 \text{ MeV} \approx \text{graph's value}$
33. The decay constant is equal to the negative value of the slope or 10^{-9} s^{-1} . The half-life of the nuclei, and thus the material, is $T_{1/2} = 693 \text{ million years}$.
35. a. The decay constant is $\lambda = 1.99 \times 10^{-5} \text{ s}^{-1}$. b. Since strontium-91 has an atomic mass of 90.90 g , the number of nuclei in a 1.00-g sample is initially $N_0 = 6.63 \times 10^{21}$ nuclei.
The initial activity for strontium-91 is
 $A_0 = \lambda N_0$
 $= 1.32 \times 10^{17} \text{ decays/s}$
The activity at $t = 15.0 \text{ h} = 5.40 \times 10^4 \text{ s}$ is
 $A = 4.51 \times 10^{16} \text{ decays/s}$.
37. $1.20 \times 10^{-2} \text{ mol}$; $6.00 \times 10^{-3} \text{ mol}$; $3.75 \times 10^{-4} \text{ mol}$
39. a. 0.988 Ci ; b. The half-life of ^{226}Ra is more precisely known than it was when the Ci unit was established.

41. a. $2.73\mu\text{g}$; b. $9.76 \times 10^4 \text{ Bq}$
43. a. $7.46 \times 10^5 \text{ Bq}$; b. $7.75 \times 10^5 \text{ Bq}$
45. a. 4.273 MeV ; b. 1.927×10^{-5} ; c. Since ^{238}U is a slowly decaying substance, only a very small number of nuclei decay on human timescales; therefore, although those nuclei that decay lose a noticeable fraction of their mass, the change in the total mass of the sample is not detectable for a macroscopic sample.
47. a. $^{90}_{38}\text{Sr}_{52} \rightarrow ^{90}_{39}\text{Y}_{51} + \beta^{-1} + \bar{\nu}_e$; b. 0.546 MeV
49. $^3_1\text{H}_2 \rightarrow ^3_2\text{He}_1 + \beta^{-} + \bar{\nu}_e$
51. a. $^7_4\text{Be}_3 + e^{-} \rightarrow ^7_3\text{Li}_4 + \nu_e$; b. 0.862 MeV
53. a. $X = ^{208}_{82}\text{Pb}_{126}$; b. 33.05 MeV
55. a. 177.1 MeV ; b. This value is approximately equal to the average BEN for heavy nuclei. c.
 $n + ^{238}_{92}\text{U}_{146} \rightarrow ^{96}_{38}\text{Sr}_{58} + ^{140}_{54}\text{Xe}_{86} + 3n$,
 $A_i = 239 = A_f$,
 $Z_i = 92 = 38 + 54 = Z_f$
57. a. $2.57 \times 10^3 \text{ MW}$; b. $8.04 \times 10^{19} \text{ fissions/s}$; c. 991 kg
 $^1_1\text{H} + ^1_1\text{H} \rightarrow ^2_1\text{H} + e^{+} + \nu_e$
59. i. $A_i = 1 + 1 = 2$; $A_f = 2$ $Z_i = 1 + 1 = 2$;
 $Z_f = 1 + 1 = 2$
 $^1_1\text{H} + ^1_1\text{H} \rightarrow ^2_1\text{H} + \gamma$
- ii. $A_i = 1 + 2 = 3$; $A_f = 3 + 0 = 3$ $Z_i = 1 + 1 = 2$;
 $Z_E = 1 + 1 = 2$
 $^3_2\text{H} + ^3_2\text{H} \rightarrow ^4_2\text{H} + ^1_1\text{H} + ^1_1\text{H}$
- iii. $A_i = 3 + 3 = 6$; $A_f = 4 + 1 + 1 = 6$ $Z_i = 2 + 2 = 4$
 $Z_f = 2 + 1 + 1 = 4$
61. 26.73 MeV
63. a. $3 \times 10^{38} \text{ protons/s}$; b. $6 \times 10^{14} \text{ neutrinos/m}^2 \cdot \text{s}$;
This huge number is indicative of how rarely a neutrino interacts, since large detectors observe very few per day.
65. a. The atomic mass of deuterium (^2H) is 2.014102 u , while that of tritium (^3H) is 3.016049 u , for a total of 5.032151 u per reaction. So a mole of reactants has a mass of 5.03 g , and in 1.00 kg , there are $(1000 \text{ g}) / (5.03 \text{ g/mol}) = 198.8 \text{ mol}$ of reactants. The number of reactions that take place is therefore $(198.8 \text{ mol}) (6.02 \times 10^{23} \text{ mol}^{-1}) = 1.20 \times 10^{26} \text{ reactions}$.
The total energy output is the number of reactions times the energy per reaction:
 $E = 3.37 \times 10^{14} \text{ J}$;
b. Power is energy per unit time. One year has $3.16 \times 10^7 \text{ s}$, so
 $P = 10.7 \text{ MW}$.
We expect nuclear processes to yield large amounts of energy, and this is certainly the case here. The energy output of $3.37 \times 10^{14} \text{ J}$ from fusing 1.00 kg of deuterium and tritium is equivalent to 2.6 million gallons of gasoline and about eight times the energy output of the bomb that destroyed Hiroshima. Yet the average backyard swimming pool has about 6 kg of deuterium in it, so that fuel is plentiful if it can be utilized in a controlled manner.
67. $\text{Gy} = \frac{\text{Sv}}{\text{RBE}}$: a. 0.01 Gy ; b. 0.0025 Gy ; c. 0.16 Gy
69. 1.24 MeV
71. 1.69 mm
73. For cancer: $(3 \text{ rem}) \left(\frac{10}{10^6 \text{ rem}\cdot\text{y}} \right) = \frac{30}{10^6 \text{ y}}$, The risk each year of dying from induced cancer is 30 in a million. For genetic defect: $(3 \text{ rem}) \left(\frac{3.3}{10^6 \text{ rem}\cdot\text{y}} \right) = \frac{9.9}{10^6 \text{ y}}$, The chance each year of an induced genetic defect is 10 in a million.

Additional Problems

75. atomic mass (Cl) = 35.5 g/mol
77. a. 1.71×10^{58} kg; b. This mass is impossibly large; it is greater than the mass of the entire Milky Way galaxy. c. ^{236}U is not produced through natural processes operating over long times on Earth, but through artificial processes in a nuclear reactor.
79. If 10% of rays are left after 2.00 cm, then only $(0.100)^2 = 0.01 = 1\%$ are left after 4.00 cm. This is much smaller than your lab partner's result (5%).
81. a. 1.68×10^{-5} Ci; (b) From [Appendix B](#), the energy released per decay is 4.27 MeV, so 8.65×10^{10} J; (c) The monetary value of the energy is $\$2.9 \times 10^3$
83. We know that $\lambda = 3.84 \times 10^{-12} \text{ s}^{-1}$ and $A_0 = 0.25 \text{ decays/s} \cdot \text{g} = 15 \text{ decays/min} \cdot \text{g}$.
Thus, the age of the tomb is

$$t = -\frac{1}{3.84 \times 10^{-12} \text{ s}^{-1}} \ln \frac{10 \text{ decays/min} \cdot \text{g}}{15 \text{ decays/min} \cdot \text{g}} = 1.06 \times 10^{11} \text{ s} \approx 3350 \text{ y}.$$

Challenge Problems

85. a. 6.97×10^{15} Bq; b. 6.24 kW; c. 5.67 kW
87. a. Due to the leak, the pressure in the turbine chamber has dropped significantly. The pressure difference between the turbine chamber and steam condenser is now very low. b. A large pressure difference is required for steam to pass through the turbine chamber and turn the turbine.
89. The energies are
 $E_\gamma = 20.6 \text{ MeV}$
 $E_{4\text{He}} = 5.68 \times 10^{-2} \text{ MeV}$. Notice that most of the energy goes to the γ ray.

Chapter 11

Check Your Understanding

- 11.1 1
 11.2 0
 11.3 0
 11.4 0
 11.5 1 eV
 11.6 The radius of the track is cut in half.
 11.7 The colliding particles have identical mass but opposite vector momenta.
 11.8 blueshifted
 11.9 about the same

Conceptual Questions

- Strong nuclear force: interaction between quarks, mediated by gluons. Electromagnetic force: interaction between charge particles, mediated photons. Weak nuclear force: interactions between fermions, mediated by heavy bosons. Gravitational force: interactions between material (massive) particle, mediate by hypothetical gravitons.
- electron, muon, tau; electron neutrino, muon neutrino, tau neutrino; down quark, strange quark, bottom quark; up quark, charm quark, top quark
- Conservation energy, momentum, and charge (familiar to classical and relativistic mechanics). Also, conservation of baryon number, lepton number, and strangeness—numbers that do not change before and after a collision or decay.
- It means that the theory that requires the conservation law is not understood. The failure of a long-established theory often leads to a deeper understanding of nature.
- 3 quarks, 2 quarks (a quark-antiquark pair)
- Baryons with the same quark composition differ in rest energy because this energy depends on the internal energy of the quarks ($m = E/c^2$). So, a baryon that contains a quark with a large angular

momentum is expected to be more massive than the same baryon with less angular momentum.

13. the “linac” to accelerate the particles in a straight line, a synchrotron to accelerate and store the moving particles in a circular ring, and a detector to measure the products of the collisions
15. In a colliding beam experiment, the energy of the colliding particles goes into the rest mass energy of the new particle. In a fix-target experiment, some of this energy is lost to the momentum of the new particle since the center-of-mass of colliding particles is not fixed.
17. The Standard Model is a model of elementary particle interactions. This model contains the electroweak theory and quantum chromodynamics (QCD). It describes the interaction of leptons and quarks through the exchange of photons (electromagnetism) and bosons (weak theory), and the interaction of quark through the exchange of gluons (QCD). This model does not describe gravitational interactions.
19. To explain particle interactions that involve the strong nuclear, electromagnetic, and weak nuclear forces in a unified way.
21. No, however it will explain why the W and Z bosons are massive (since the Higgs “imparts” mass to these particles), and therefore why the weak force is short ranged.
23. Cosmological expansion is an expansion of space. This expansion is different than the explosion of a bomb where particles pass rapidly *through* space. A plot of the recessional speed of a galaxy is proportional to its distance. This speed is measured using the red shift of distant starlight.
25. With distance, the absolute brightness is the same, but the apparent brightness is inversely proportional to the square of its distance (or by Hubble’s law recessional velocity).
27. The observed expansion of the universe and the cosmic background radiation spectrum.
29. If light slow down, it takes long to reach Earth than expected. We conclude that the object is much closer than it really is. Thus, for every recessional velocity (based on the frequency of light, which we assume is not disturbed by the slowing), the distance is smaller than the “true” value, Hubble’s constant is larger than the “true” value, and the age of the universe is smaller than the “true” value.

Problems

31. 1.022 MeV
33. 0.511 MeV, $2.73 \times 10^{-22} \text{ kg} \cdot \text{m/s}$, $1.23 \times 10^{20} \text{ Hz}$
35. a, b, and c
37. a. $\bar{p}_e^+ \nu_e$; b. $\bar{p}\pi^+$ or $\bar{p}\pi^0$; c. $\Xi^0 \pi^0$ or $\Lambda^0 K^+$; d. $\mu^- \bar{\nu}_\mu$ or $\pi^- \pi^0$; e. $\bar{p}\pi^0$ or $\bar{n}\pi^-$
39. A proton consists of two up quarks and one down quark. The total charge of a proton is therefore $+\frac{2}{3} + \frac{2}{3} + -\frac{1}{3} = +1$.
41. The K^+ meson is composed of an up quark and a strange antiquark ($u\bar{s}$). Since the charges of this quark and antiquark are $2e/3$ and $e/3$, respectively, the net charge of the K^+ meson is e , in agreement with its known value. Two spin $-1/2$ particles can combine to produce a particle with spin of either 0 or 1, consistent with the K^+ meson’s spin of 0. The net strangeness of the up quark and strange antiquark is $0 + 1 = 1$, in agreement with the measured strangeness of the K^+ meson.
43. a. color; b. quark-antiquark
45. $d \rightarrow u + e^- + \bar{\nu}_e$; $u \rightarrow d + e^+ + \nu_e$
47. 965 GeV
49. According to [Example 11.7](#),
 $W = 2E_{\text{beam}} = 9.46 \text{ GeV}$,
 $M = 9.46 \text{ GeV}/c^2$.
 This is the mass of the upsilon ($1S$) meson first observed at Fermi lab in 1977. The upsilon meson consists of a bottom quark and its antiparticle $(b\bar{b})$.
51. 0.135 fm; Since this distance is too short to make a track, the presence of the W^- must be inferred from its decay products.
53. 3.33 MV
55. The graviton is massless, so like the photon is associated with a force of infinite range.
57. 67.5 MeV

59. a. 33.9 MeV; b. By conservation of momentum, $|p_\mu| = |p_\nu| = p$. By conservation of energy, $E_\nu = 29.8 \text{ MeV}$, $E_\mu = 4.1 \text{ MeV}$
61. $(0.99)(299792 \text{ km/s}) = \left(70 \frac{\text{km}}{\text{s}}\right) / \text{Mpc} (d)$, $d = 4240 \text{ Mpc}$
63. $1.0 \times 10^4 \text{ km/s}$ away from us.
65. $2.26 \times 10^8 \text{ y}$
67. a. $1.5 \times 10^{10} \text{ y} = 15 \text{ billion years}$; b. Greater, since if it was moving slower in the past it would take less more to travel the distance.
69. $v = \sqrt{\frac{GM}{r}}$

Additional Problems

71. a. $\bar{n}$; b. K^+ ; c. K^+ ; d. π^- ; e. $\bar{\nu}_\tau$; f. e^+
73. $14.002 \text{ TeV} \approx 14.0 \text{ TeV}$
75. 964 rev/s
77. a. $H_0 = \frac{30 \text{ km/s}}{1 \text{ Mly}} = 30 \text{ km/s} \cdot \text{Mly}$; b. $H_0 = \frac{15 \text{ km/s}}{1 \text{ Mly}} = 15 \text{ km/s} \cdot \text{Mly}$

Challenge Problems

79. a. 5×10^{10} ; b. divide the number of particles by the area they hit: $5 \times 10^4 \text{ particles/m}^2$
81. a. 2.01; b. $2.50 \times 10^{-8} \text{ s}$; c. 6.50 m

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow$$

83.

$$v = \left(\frac{GM}{r}\right)^{1/2} = \left[\frac{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(3 \times 10^{41} \text{ kg})}{(30,000 \text{ ly})(9.46 \times 10^{15} \text{ m/ly})}\right] = 2.7 \times 10^5 \text{ m/s}$$

85. a. 938.27 MeV; b. 1.84×10^3
87. a. $3.29 \times 10^{18} \text{ GeV} \approx 3 \times 10^{18} \text{ GeV}$; b. 0.3; Unification of the three forces breaks down shortly after the separation of gravity from the unification force (near the Planck time interval). The uncertainty in time then becomes greater. Hence the energy available becomes less than the needed unification energy.

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