Physics SN3

Problem Set #1: Simple Harmonic Motion

Solutions

Part A

1. C
2. C
3. D
4. D
5. E
6. B
7. D
8. C
9. B
10. D

11. Object in *second quarter* of motion → moving toward maximum displacement
    - Kinetic energy decreasing
    - Moving away from equilibrium
    - v greatest when x = 0
    - at x = -2.5 (which is -A)

12. **period** (T) no change
    k: no change
    $\mathbf{E_{tot}}$: four times
    $\mathbf{v_{max}}$: two times
    $\mathbf{a_{max}}$: two times
    ω: no change

13. a) T increases (like being on planes with smaller g)
    b) T decreases (like being on planes with larger g)
    c) no change

Part B

1. a) the angular frequency and frequency are both related together by the following equation:

$$\omega = 2\pi f = 12.56 rad/s \qquad f = \frac{12.56}{2\pi} = 2.00 Hz$$

b) The maximum velocity: $v_{\max} = \omega A = 12.56 \times 15.0 = 188 cm/s$

c) From the simple harmonic equation of the wave, we can find the velocity equation as a function of time

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$v = -(12.56)(15)\underbrace{\sin(12.56 \times 0.25 + \frac{\pi}{2})}_{=-1}$$

$v = +188 cm/s$

t = 0.25s is ½ T

object is at phase $\frac{3\pi}{2}$ rad

d) Because the velocity is max, the position and the acceleration will have to be min at that time.

a = 0; again, solve by inspection or plug into $a = \frac{d^2x}{dt^2} - \omega^2 A \cos(\omega t + \phi)$

e). From the angular frequency equation, we can obtain the spring constant

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow k = \omega^2 m = (12.56)^2 \times 0.2 = 31.6 N/m$$

2. a) The total energy of the system is given by:

$E_{tot} = \frac{1}{2}kA^2$ with the spring constant $k = \omega^2 m = (\frac{\pi}{3})^2 \times 3 = 3.29 N/m$

$$E_{tot} = \frac{1}{2}(3.29)(0.5)^2 = 0.411 J$$

b) To find the potential energy of the system, we have to find the position at a specific time.

$$x = 0.5\cos\left(\frac{\pi}{3}\times 1.3 - \frac{\pi}{4}\right) = 0.419m$$

$$U_{elastic} = \frac{1}{2}kx^2 = \frac{1}{2}(3.29)(0.419)^2 = 0.289J$$

c) The total energy is the sum of the kinetic and potential energy

$$K + U_{elastic} = E_{tot} \Rightarrow K = E_{tot} - U_{elastic}$$

$$K = 0.411 - 0.289 = 0.122J$$

d) When the potential energy is half the total energy ($U_{elastic} = \frac{1}{2}E_{tot}$)

$$\frac{1}{2}kx^2 = \frac{1}{2}\left(\frac{1}{2}kA^2\right)$$

$$x^2 = \frac{1}{2}A^2 \Rightarrow x = \frac{A}{\sqrt{2}} = 0.354m$$

3. a) The period and the frequency are both related together by the following equation:

$$T = \frac{1}{f} = \frac{1}{0.25} = 4s$$

b) The angular frequency:

$$\omega = 2\pi f = 2\pi(0.25) = \frac{\pi}{2}$$

c) The total energy is the sum of the potential energy and kinetic energy

$$E_{total} = U_{elastic} + K$$

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

with $\omega = \sqrt{\frac{k}{m}} \Rightarrow k = \omega^2 m = (\frac{\pi}{2})^2(0.1) = 0.247N/m$

$$E = \frac{1}{2}(0.247)(2.8\times10^{-3})^2 + \frac{1}{2}(0.1)(4\times10^{-3})^2 = 1.77\times10^{-6}J$$

d) To can obtain the amplitude of motion with the total energy

$$E_{total} = \frac{1}{2}kA^2 \Rightarrow A = \sqrt{\frac{2E_{total}}{k}} = \sqrt{\frac{2(1.77\times10^{-6})}{0.247}} = 3.79\times10^{-3}\,m$$

e) The maximum velocity can be obtain from the maximum kinetic energy equation or from the maximum velocity equation

$$K_{\max} = E_{total} = \frac{1}{2}mv_{\max}^2 \Rightarrow v_{\max} = \sqrt{\frac{2E_{total}}{m}}$$

$$v_{\max} = \sqrt{\frac{2(1.77\times10^{-6})}{0.1}} = 5.95\times10^{-3}\,m/s$$

or $$v_{\max} = \omega A = \left(\frac{\pi}{2}\right)(3.79\times10^{-3}) = 5.95\times10^{-3}\,m/s$$

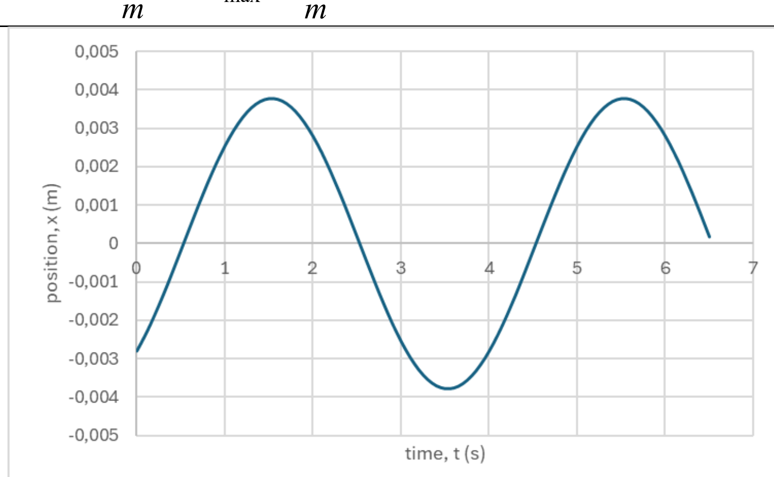
f) The maximum acceleration is found from the equation of the acceleration as a function of time

$$a_{\max} = \omega^2 A$$

$$a_{\max} = \left(\frac{\pi}{2}\right)^2(3.79\times10^{-3}) = 9.35\times10^{-3}\,m/s^2$$

or $$a = \frac{kx}{m} \qquad a_{\max} = \frac{kA}{m}$$

g) You should be able to sketch the position-time graph, with initial negative position and initial positive velocity.
Also since you are asked to sketch for 6 seconds, your reader should see a little bit more that one cycle.



Enrichment:
To find the phase constant:

$$-0.28\cdot10^{-2} = 0.379\cdot10^{-2}cos(\phi)$$

$$\phi = cos^{-1}\left(\frac{-0.28\cdot10^{-2}}{0.379\cdot10^{-2}}\right)$$

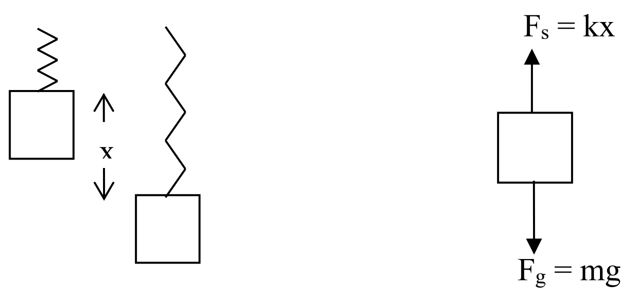
There are 2 possibilities for ϕ:
2.40 rad or $(2\pi-2.40) = 3.88$rad

We note that the initial velocity is positive, making the phase constant in the 3rd quadrant of the trigonometric circle.
In a similar line of reasoning, the positive initial velocity suggests that the body is in the 3rd “quarter" of the cycle.
Therefore, we choose the solution:
$\phi = 3.88$ rad for the cosine function

(A choice of sine function would give a phase constant shifted by $\pi/2$: $\phi_{sine}= 5.45$ rad).

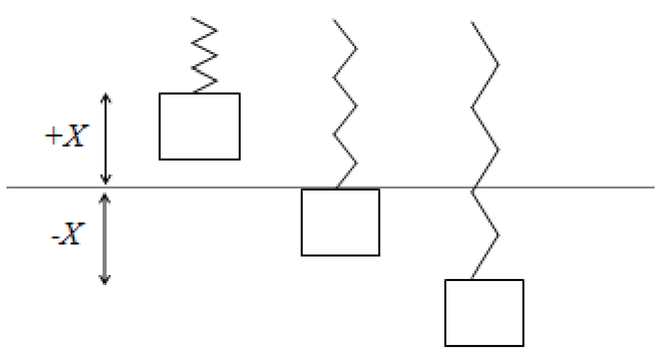
4. a) At equilibrium position,

$$F_s = F_g \Rightarrow x = \frac{mg}{k} = \frac{2(9.81)}{150} = 0.131m$$



Free body diagram

b) Amplitude $A = 0.131$ m, therefore, it will fall 0.262 m below initial position.



c) To find the position as a function of time, we need to find the phase and the angular frequency.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{150}{2}} = 8.66 rad/s$$

$$\phi = 0 \quad \text{(starts at } +A \text{ at t = 0)}$$

$$A = 0.131\, m$$

$$y = 0.131\cos(8.66t + 0)$$

5. From the position equation, we can obtain the acceleration equation as a function of time

$$x = A\cos(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \underbrace{\cos(\omega t + \phi)}_{=\pm 1}$$

$$a_{\max} = \omega^2 A \qquad \text{with} \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{1.7} = 3.70 rad/s$$

$$a_{\max} = (3.70)^2(0.25) = 3.42 m/s^2$$

The maximum force

$$F_{\max} = \mu N = ma$$

$$\mu mg = ma \Rightarrow \mu = \frac{a}{g} = \frac{3.42}{9.81} = 0.349$$

6. a) The period of oscillation depends on the length of the string and the gravity.

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow g = \frac{4\pi^2 L}{T^2} = \frac{4\pi^2(2.3)}{(4.25)^2} = 5.03 m/s^2$$

b) The angular frequency:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4.25} = 1.48 rad/s$$

c) The angular position equation is obtained by finding the phase, the angular frequency and the amplitude.

$$\phi = 0 \qquad \omega = 1.48 rad/s \qquad A = 12°$$

$$\theta = (12°)\cos(1.48t + 0)$$

d) At t = 1.5, the angular position of the pendulum is:

$$\theta = (12°)\cos(1.48 \times 1.5) = -7.25°$$

e) The angular velocity can be obtained by taking the derivative as a function of time of the angular position equation (note that ω_z represents the angular velocity and ω represents the angular frequency… which of course are not the same!)

$$\omega_z = \frac{d\theta}{dt} = -\omega A \sin(\omega t + \phi)$$

$$\omega_z = -1.48 \times 12° \sin(1.48 \times 1.5 + 0) = -14.1°/s$$

7. a) The amplitude is obtained by finding the total energy when the velocity is max

$$\frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2 \Rightarrow A = \sqrt{\frac{mv_{\max}^2}{k}} = \sqrt{\frac{(0.3)(2)^2}{12}} = 0.316m$$

You could also use the equation for the mass' maximum velocity ($v_{max} = \omega A$), noting that this equals the elevator's velocity (since this is the velocity the mass has when the elevator stops).

$$v_{max} = \omega A$$
$$2 = (k/m)^{1/2} A$$
$$2 = (12/0.3)^{1/2} A$$
$$2 = 6.32A$$
$$A = 0.316 \text{ m}$$

b) the position equation depends on the angular frequency, the amplitude and the phase

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{0.3}} = 6.32 rad/s$$

$\phi = \frac{\pi}{2}$ (at t = 0, it is at equilibrium and moving downward)

$$y = 0.316\cos(6.32t + \frac{\pi}{2})$$

8. If plate acceleration is greater than g downward, then the block will not accelerate quickly enough to keep contact with it. The only forces on block are the normal N and the weight and N can't be less than 0.

$$\omega = 2\pi f = 4\pi rad/s$$

$$a_{\max} = \omega^2 A \Rightarrow A = \frac{9.81}{(4\pi)^2} = 0.0621m = 6.21cm$$

9. a) The angular frequency is found from the period: $T = 0.06s$ $\quad \omega = \frac{2\pi}{T} = \frac{2\pi}{0.06} = 105 rad/s$

b) A = 0.06m

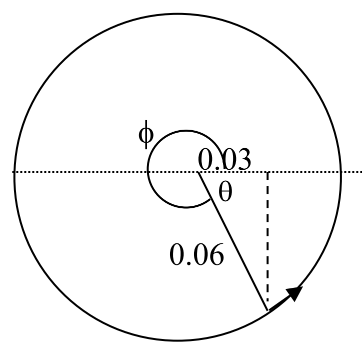
ϕ: For a cosine function, it is 5/6 of the way through its cycle. At t = 0, $\phi = \frac{5}{6}(2\pi) = 5.24 rad$

or

$$\theta = \cos^{-1}\left(\frac{0.03}{0.06}\right) = 1.047 rad$$

$$\phi = 2\pi - \theta = 5.24 rad$$

$$x = 0.06\cos(105t + 5.24)$$



c) at t = 0.1s, the position is:

$$x = 0.06\cos(10.5 + 5.24) = -0.06\,m$$

Also by inspection: time is 1(T) after 0.04s.

d) From the acceleration equation, we can find the maximum acceleration. When ...

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \underbrace{\cos(\omega t + \phi)}_{=1}$$

$$a = -105^2(0.06) = -662 m/s^2$$

e) The spring constant is found from the angular frequency

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow k = \omega^2 m = (105)^2(4) = 4.41\times10^4\, N/m$$

or from the force equation of a spring

$$k = \frac{F}{x} = \frac{ma}{x} = \frac{4(662)}{0.06} = 4.41\times10^4\, N/m$$