Supplementary Problems 2: Circuits and Capacitors

1. The quantity of charge, q, that has passed through any cross-section of a certain conductor in the time interval 0 sec. to t sec. is given by the equation

$$q(t) = 4t^3 + 5t + 6$$

where q is in coulombs.

(a) What is the average current through the conductor from $t = 2$s to $t = 4$s?

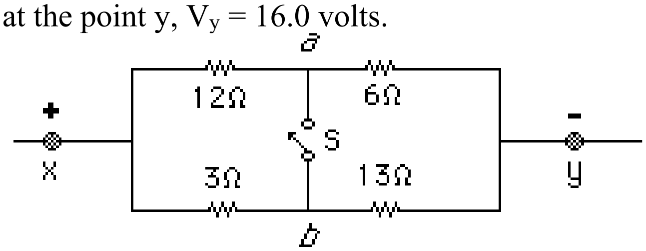
Since current is defined as $I = \frac{dQ}{dt}$, the average current over a time interval will be $I = \frac{\Delta Q}{\Delta t}$. Plugging in the equation for $q(t)$,

$$I_{ave} = \frac{Q(4) - Q(2)}{4 - 2} = \frac{1}{2}[(4 \cdot 4^3 + 5 \cdot 4 + 6) - (4 \cdot 2^3 + 5 \cdot 2 + 6)] = 117\ A$$

(b) If the resistance of the conductor is 12Ω, what is the rate at which it is absorbing electrical energy at $t = 2$s?

$$I = \frac{dQ}{dt} = 12t^2 + 5$$
$$I(t = 2s) = 12 \cdot 2^2 + 5 = 53\ A$$
$$P = I^2R = 53^2 \cdot 12 = 3.37 \times 10^4 W$$

2. The figure below shows a section of a circuit where the electric potential at the point x, $V_x = 64.0$ volts and at the point y, $V_y = 16.0$ volts.



(a) With the switch S open determine the current in the 6 Ω resistor **and** the potential difference V_{ab} .

With the switch open we have two parallel branches, the top with the 12 Ω and 6 Ω resistors in series and the bottom with the 3 Ω and 13 Ω resistors in series.

The potential difference across both branches is $V_{xy} = 64.0 - 16.0 = 48.0$ V.

The equivalent resistance of the top branch is $R_{top} = 12 + 6 = 18$ Ω.

The current through the top branch, and so the 6 Ω resistor, is $I_{top} = \frac{V}{R} = \frac{48.0}{18} = 2.67\ A$

The potential at a can be found by starting at either end: $V_x - V_{12\Omega}$ or $V_y + V_{6\Omega}$. Either way, $V_a = 64 - 2.67 \cdot 12 = 16 + 2.67 \cdot 6 = 32\ V$

To find the potential at b, we also need to find the current in the bottom branch with equivalent resistance $R_{bottom} = 3 + 13 = 16$ Ω. This is $I_{bottom} = \frac{V}{R} = \frac{48}{16} = 3\ A$.

Like above, $V_b = V_x - V_{3\Omega} = V_y + V_{13\Omega} = 64 - 3 \cdot 3 = 16 + 3 \cdot 13 = 55\ V$

The potential difference is $V_a - V_b = 32 - 55 = -23\ V$

(b) With the switch S closed determine the current through the switch (include direction) **and** V_{ab} .

With the switch closed, we now have two segments in series. One the left the 12 Ω and 3 Ω resistors are connected in parallel, and on the right the 6 Ω and 13 Ω resistors are connected in parallel.

Points a and b are now connected by an empty wire with no resistance, so $V_{ab} = 0$ V!

To find the current through the switch, we need to first find the equivalent resistance of the entire circuit:

$R_{left} = \left(\frac{1}{12} + \frac{1}{3}\right)^{-1} = 2.4\ \Omega$

$R_{right} = \left(\frac{1}{6} + \frac{1}{13}\right)^{-1} = 4.1\ \Omega$

$R_{eq} = 2.4 + 4.1 = 6.5\ \Omega$

$I_{tot} = \frac{V_{tot}}{R_{eq}} = \frac{48.0}{6.5} = 7.38\ A$

We can now find the voltage drop in each segment of the circuit, and then the current through each resistor.

$V_{left} = I_{tot} R_{left} = 7.38 \cdot 2.4 = 17.7$ V

$V_{right} = I_{tot} R_{right} = 7.38 \cdot 4.1 = 30.3$ V

$I_{12} = \frac{17.7}{12} = 1.48$ A

$I_3 = \frac{17.7}{3} = 5.90$ A

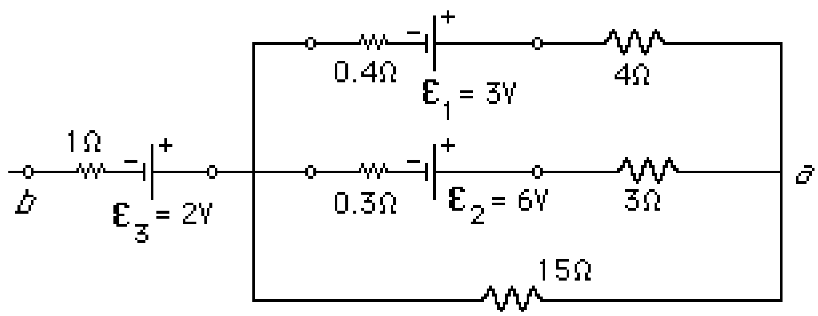
$I_6 = \frac{30.3}{6} = 5.05$ A

$I_{13} = \frac{30.3}{13} = 2.33$ A

To find the current through the switch, we need to apply Kirchhoff's junction rule at either a or b. At a: $I_{12} + I_{switch} = I_6$ and at b: $I_3 = I_{switch} + I_{13}$. Either way, we see that the current through the switch is directed up with a value

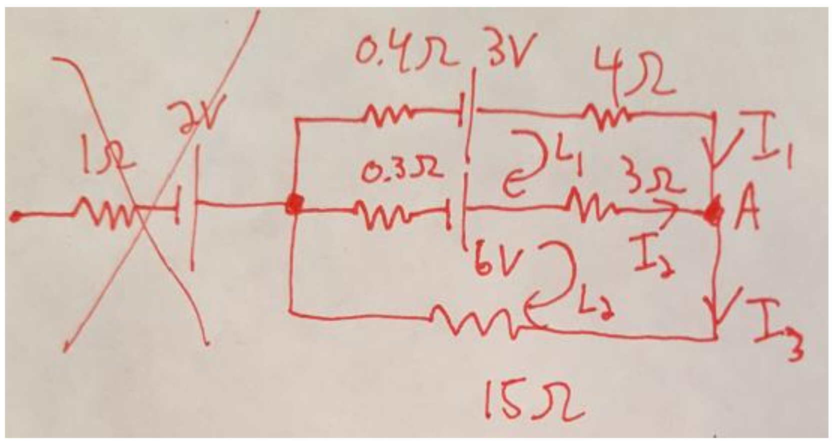
$I_{switch} = I_6 - 1_{12} = I_3 - I_{13} = 5.05 - 1.48 = 5.90 - 2.33 = 3.57$ A

3. In the circuit below solve for



(a) the current in each part of the circuit,

Note that the question specifies this is an entire circuit, not a section of circuit. On the far left, the branch ends at b and does not from any closed loop, meaning there is no current in this branch and the emf $\mathcal{E}_3$ does not send any power to the rest of the circuit. We can simply ignore this far left branch in our analysis.



I've chosen to define currents I_1, I_2 and I_3 as shown above. Applying Kirchhoff's rules:
Junction: $I_1 + I_2 = I_3$
Top loop: $+3 - 4I_1 + 3I_2 - 6 + 0.3I_2 - 0.4I_1 = 0 = -3 - 4.4I_1 + 3.3I_2$
Bottom loop: $+6 - 3I_2 - 15I_3 - 0.3I_2 = 0 = 6 - 3.3I_2 - 15I_3$
Outside loop: $+3 - 4I_1 - 15I_3 - 0.4I_1 = 0 = 3 - 4.4I_1 - 15I_3$

We only need two of the three loop equations to solve. I'll use the top loop to isolate I_1 in terms of I_2 and the bottom loop to isolate I_3 in terms of I_2. Then, we substitute into the junction equation:

$I_1 = \frac{-3+3.3I_2}{4.4}$

$I_3 = \frac{6-3.3I_2}{15}$

$I_2 = I_3 - I_1 = \left(\frac{6-3.3I_2}{15}\right) - \left(\frac{-3+3.3I_2}{4.4}\right) = 1.08 - 0.97I_2$

We can now find the three currents:

$I_2 = \frac{1.08}{1.97} = 0.548$ A (to the right)

$I_1 = \frac{-3+3.3\cdot 0.548}{4.4} = -0.271$ A (opposite the initial guess, so to the left)

$I_3 = \frac{6-3.3\cdot 0.548}{15} = 0.279$ A (to the left)

(b) the potential difference V_{ab} ,

We can use any of the three branches to find V_{ab}, being careful not to forget $\mathcal{E}_3$.

Top: $V_b + \mathcal{E}_3 - 4.4I_1 + \mathcal{E}_1 = V_a$
Middle: $V_b + \mathcal{E}_3 - 3.3I_2 + \mathcal{E}_2 = V_a$
Bottom: $V_b + \mathcal{E}_3 + 15I_3 = V_a$

In all cases,

$$V_a - V_b = 2 - 4.4 \cdot -0.271 + 3 = 2 - 3.3 \cdot 0.548 + 6 = 2 + 15 \cdot 0.279 = 6.19\ V$$

(b) the terminal voltage of the seat $\mathcal{E}_1$, and

The terminal voltage of the seat $\mathcal{E}_1$ includes its internal resistance of 0.4 Ω.
$V_T = -0.4I_1 + \mathcal{E}_1 = (-0.4)(-0.271) + 3 = 3.11\ V$

(d) the rate at which this seat supplies/absorbs electrical energy.
This seat has current going through it "backwards", so it **absorbs** electrical energy at a rate
$P = I_1 V_T = (-0.271)(3.11) = -0.84$ W

We could of course sum the power of the emf and the resistance separately:
$|P| = |I_1\mathcal{E}_1| + |I_2 r_1| = (0.271)(3) + (0.271)^2(0.4) = 0.84$ W

4. For the circuit shown in the figure below, determine the current through each resistor
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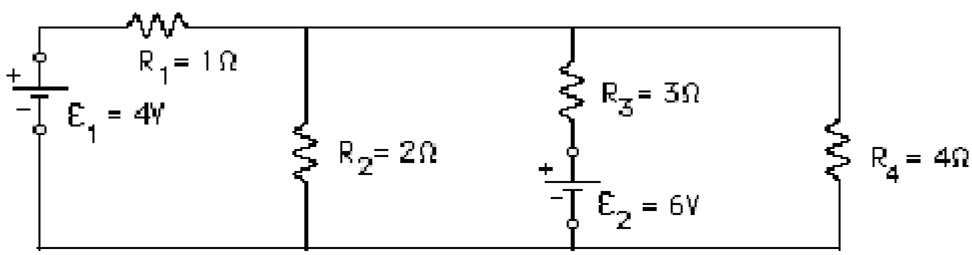
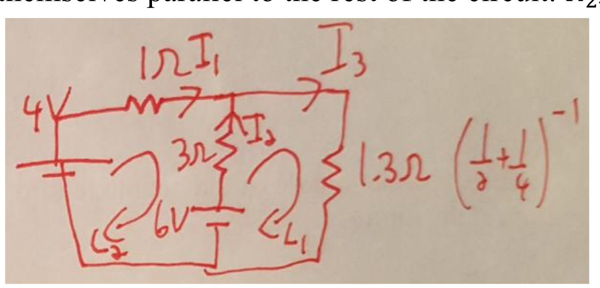


Before applying Kirchhoff's rules, we can simplify the circuit by replacing R_2 and R_4 connected in parallel, themselves parallel to the rest of the circuit. $R_{24} = \left(\frac{1}{2} + \frac{1}{4}\right)^{-1} = 1.33\ \Omega$



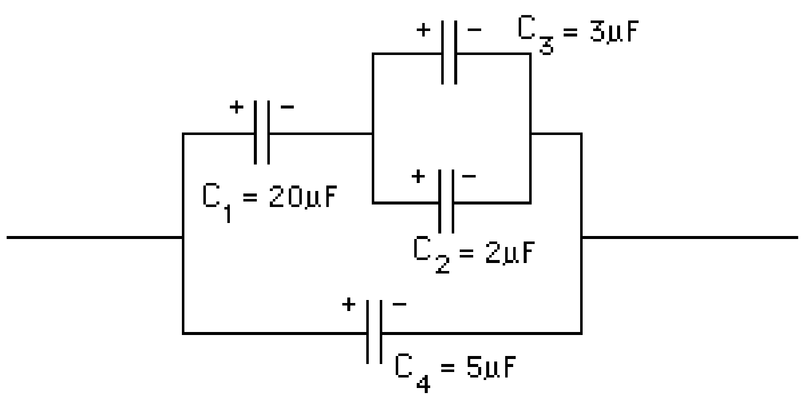
I've chosen to define currents I_1, I_2 and I_3 as shown above. Applying Kirchhoff's rules:
Junction: $I_1 + I_2 = I_3$
Left loop: $+4 - 1I_1 + 3I_2 - 6 = 0 = -2 - I_1 + 3I_2$
Right loop: $+6 - 3I_2 - 1.33I_3 = 0$
Outside loop: $+4 - 1I_1 - 1.33I_3 = 0$

We only need two of the three loop equations to solve. I'll use the left loop to isolate I_1 in terms of I_2 and the right loop to isolate I_3 in terms of I_2. Then, we substitute into the junction equation:
$I_1 = -2 + 3I_2$
$I_3 = \frac{6-3I_2}{1.33}$
$I_2 = I_3 - I_1 = \left(\frac{6-3I_2}{1.33}\right) - (-2 + 3I_2) = 8 - 5.26I_2$
We can now find the three currents:
$I_2 = \frac{8}{6.26} = 1.28$ A (up, this is the current through R_3)
$I_1 = -2 + 3 \cdot 1.28 = 1.84$ A (up, so to the right through R_1)
$I_3 = \frac{6-3\cdot 1.28}{1.33} = 1.62$ A (down, so this is the sum of the currents through R_2 and R_4)

To find the current through R_2 and R_4, we must find the voltage drop across them. This can be done with Ohm's law or with the voltage loops, with small differences from rounding:
$V_{24} = I_3R_{24} = 1.62 \cdot 1.33 = 2.15$ V
$V_{24} = \mathcal{E}_1 - I_1R_1 = 4 - 1.84 \cdot 1 = 2.16$ V
$V_{24} = \mathcal{E}_2 - I_2R_3 = 6 - 1.28 \cdot 3 = 2.16$ V

Through R_2, $I = \frac{2.16}{2} = 1.08$ A and through R_4, $I = \frac{2.16}{4} = 0.54$ A, both downwards.

5. In the combination of capacitors shown below, all are fully charged.



(a) Determine the equivalent capacitance of the combination.

First, C_2 and C_3 are connected in parallel, so $C_{23} = 2 + 3 = 5$ μF. These are connected to C_1 in series, so $C_{123} = \left(\frac{1}{20} + \frac{1}{5}\right)^{-1} = 4$ μF. This is finally in parallel with C_4, so $C_{eq} = 4 + 5 = 9$ μF.

(b) If the charge on C_3 is 90 μC, determine the charge on C_4.

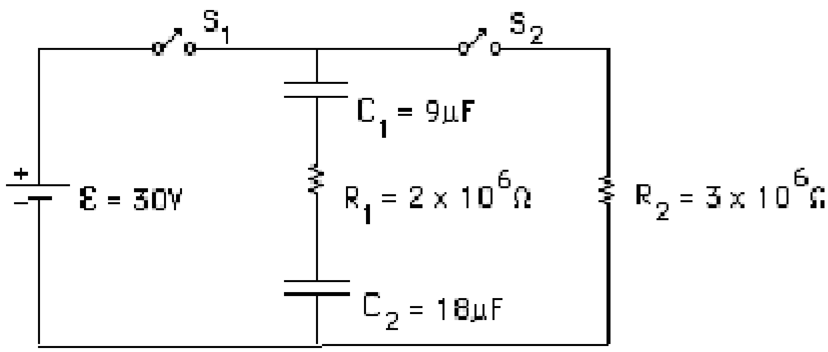
Since we know the charge on C_3, we can find the voltage across it $V_3 = \frac{Q_3}{C_3} = \frac{90\times10^{-6}}{3\times10^{-6}} = 30$ V.

This is also the voltage across C_2 : $V_3 = V_2 = V_{23} = 30$ V. We can find the charge on C_2 and C_3 with their equivalent capacitance C_{23} : $Q_{23} = C_{23}V_{23} = 5 \cdot 30 = 150$ μC.

This is also the charge on C_1: $Q_1 = Q_{23} = Q_{123} = 150$ μC. We can now find the voltage across C_1, C_2 and C_3 : $V_{123} = \frac{Q_{123}}{C_{123}} = \frac{150}{4} = 37.5$ V.

This is also the voltage across C_4 : $V_4 = V_{123} = V_{tot}$. Finally, we can find the charge on C_4 : $Q_4 = C_4 V_4 = 5 \cdot 37.5 = 188$ μC

6. In the circuit shown below the switches S_1 and S_2 are initially open and the capacitors are uncharged.



(a) Switch S_1 is now closed. Six seconds after S_1 is closed :

(i) what is the charge on C_1?

C_1 and C_2 are connected in series, so $C_{eq} = \left(\frac{1}{9} + \frac{1}{18}\right)^{-1} = 6$ μF. The time constant of the circuit is thus $\tau = R_{eq}C_{eq} = (2 \times 10^{-6})(6 \times 10^{-6}) = 12$ s.

The maximum charge that can be accumulated on the capacitors (equal charge on C_1 and C_2 as they are in series) is $Q_{max} = C_{eq}\mathcal{E} = 6 \times 30 = 180$ μF.

This charge increases over times as $Q(t) = Q_{max}\left(1 - e^{-\frac{t}{\tau}}\right)$

$Q(t = 6\text{s}) = 180\ \left(1 - e^{-\frac{6}{12}}\right) = 70.8$ μC

(ii) what is the current through R_1?

We could find the initial current through R_1 and the find $I(t) = I_{max}e^{-\frac{t}{\tau}}$.

Instead, since we already know the accumulated charge, we know that the voltage across the capacitors is $V_C = \frac{Q(t)}{C_{eq}} = \frac{70.8}{6} = 11.8$ V.

The voltage across the resistor is thus $V_R = \mathcal{E} - V_C = 30 - 11.8 = 18.2$ V, and the current through it is $I(t = 6\text{s}) = \frac{V_R}{R} = \frac{18.2}{2 \times 10^6} = 9.10$ μA

(iii) what is the electric potential energy stored in C_2?

No need to find the voltage across only C_2, since we know its capacitance and charge. $U = \frac{Q^2}{2C} = \frac{(70.8 \times 10^{-6})^2}{2(18 \times 10^{-6})} = 1.39 \times 10^{-4}$ J.

(b) **If now, at six seconds after switch S_1 has been closed, it is opened and S_2 is closed.**

(i) Determine the charge on C_2 18 seconds after S_2 is closed and

The discharging circuit has a new time constant
$\tau = R_{eq}C_{eq} = (3 \times 10^{-6})(6 \times 10^{-6}) = 18$ s

The charge on C_2 is the same as on C_1 and begins as $Q_{max} = 70.8$ μC and then decreases over time: $Q(t) = Q_{max}e^{-\frac{t}{\tau}}$
$Q(t = 18 \text{ s}) = 70.8\, e^{-18/18} = 26.0$ μC

(ii) determine the potential difference across R_2 at this time.

In this discharging circuit, the voltage across R_2 is the same as over the two capacitors together
$V_R = V_{C_{eq}} = \frac{Q(t)}{C_{eq}} = \frac{26.0}{6} = 4.33$ V