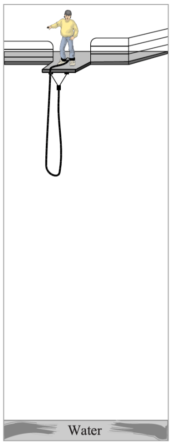
PROBLEM SET 7
Multiple-concept problems

Conceptual Questions

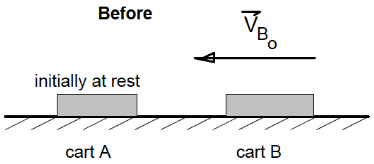


Question 1. One end of an elastic bungee cord is attached to a bungee jumper's ankles, and the other end is attached to a high platform. The cord stretches when a force is applied to it. The bungee jumper steps off the edge of the platform, comes to rest just above the water, and rises back up. The "jump" is considered over when the jumper is hanging head down from the end of the cord and is no longer moving.

a) Explain what happens to the bungee jumper from the time he steps off the edge until the time when he is closest to the water. In your answer: fully describe the mechanical energy transformations (gravitational potential energy, elastic potential energy, and kinetic energy) that occur. Fully describe the forces that act on the bungee jumper.
ANS: When you are at the top, you can consider yourself to have gravitational potential energy, using the surface of the water as the "zero point" or the location of zero height. As you begin to fall, but the cord is still slack, you lose gravitational potential energy and gain kinetic energy. During this period, the only force acting on the jumper is the gravitational force. Once you reach the point where there is non-zero tension in the bungee cord, your gravitational potential continues to decrease, but your kinetic energy begins to decrease as well. To compensate, elastic potential energy begins to build up in the bungee as it stretches. Now, there are two forces acting on the jumper: the gravitational force pulling down and the bungee force pulling upward. When the jumper reaches the bottom (the "end" of the jump). All the energy initially in the system is now stored in the bungee as both the gravitational potential energy and the kinetic energy are now zero. There is still both the gravitational force and bungee force acting on the jumper. The two forces at this point are NOT equal. The bungee force will be stronger than the gravitational force, which will pull the jumper back upward.

b) Clearly explain why an elastic bungee cord must be used rather than a standard rope. Use appropriate formulas to support your answer.
ANS: Elastic bungees are stretchable and as such can readily store potential energy. So, they are able to slow down the jumper at a slower rate, reducing the deceleration as the bungee starts stretching. This reduces the tension on the bungee at any given time, allowing the bungee to remain intact. A standard rope has very little stretch. As the rope begins to go taut, the jumper slows down very quickly, resulting in a large deceleration as the rope begins to go taut, resulting in a large tension in the rope at that time. This large tension could break the rope, or break the jumper, or both, making the rope unsafe to use.

Question 2. In the figure at the right are shown two carts on an air track. The carts have equal masses. At the time shown, cart B is moving in the negative x direction and the center of mass of cart A is at the origin and at rest. When the carts collide, they stick together. Friction with the track is small and may be neglected.



In the graphs below are shown a number of possible plots for the various physical parameters associated with one of the two carts. For each property (a)-(f) select the number of a graph that could be a plot of the property as a function of time. You may use a graph more than once or not at all.

a) the momentum of cart A
ANS: Graph 1

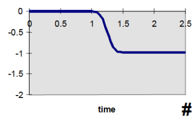
b) the momentum of cart B
ANS: Graph 3

c) the kinetic energy of cart B
ANS: Graph 4

d) the position of (the center of mass) of cart B
ANS: Graph 9 (no graph works)

e) the acceleration of cart A
ANS: Graph 8

f) the total momentum of carts A+B
ANS: Graph 9 (no graph works) should be a constant graph with a negative value



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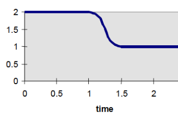

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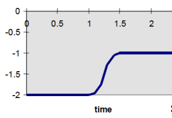

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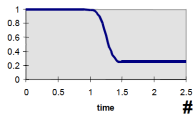

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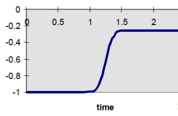

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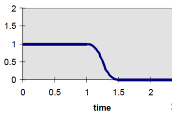

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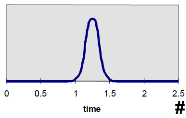

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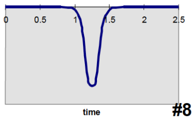
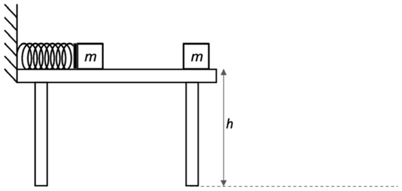


#8

None of the graphs shown

#9



Question3. A block of mass *m,* initially compressed by a distance *x*, is launched horizontally from an ideal linear spring with force constant k on a smooth table. After it separates from the spring, it collides elastically with an identical block of mass m.

a) If the blocks stuck together, would the horizontal displacement increase, decrease, or stay the same? Why?

If the experiment were repeated with a spring with spring force constant $4k$, by what factor will the horizontal displacement from the edge of the table change?
[Hint: Derive an expression for the horizontal displacement from the edge of the table for the block on the right in terms of h, k, x, m and any fundamental constants.]

ANS: To find the horizontal displacement of the right block, we need to know its velocity as it slides off the table. To find this, we must conserve momentum in the collision of the two blocks, and to solve THAT problem, we need to know what velocity the left block comes in with, which we can find by conserving energy in the spring/block interaction:

$$K_i + U_{sp_i} = K_f + U_{sp_f}$$

$$0 + \frac{1}{2}kx^2 = \frac{1}{2}mv_{l_i}^2 + 0$$

$$v_{l_i} = \sqrt{\frac{k}{m}}x$$

Now the momentum conservation:

$$p_{l_i} + p_{r_i} = p_{l_f} + p_{r_f}$$

$$mv_{l_i} = mv_{l_f} + mv_{r_f}$$

$$v_{l_i} = v_{l_f} + v_{r_f}$$

We have two unknowns, so we need another equation. We know that the collision is elastic, so:

$$K_i = K_f$$

$$\frac{1}{2}mv_{l_i}^2 = \frac{1}{2}mv_{l_f}^2 + \frac{1}{2}mv_{r_f}^2$$

$$v_{l_i}^2 = v_{l_f}^2 + v_{r_f}^2$$

We plug in our momentum conservation result:

$$\left(v_{l_f} + v_{r_f}\right)^2 = v_{l_f}^2 + v_{r_f}^2$$

$$v_{l_f}^2 + 2v_{l_f}v_{r_f} + v_{r_f}^2 = v_{l_f}^2 + v_{r_f}^2$$

$$2v_{l_f}v_{r_f} = 0$$

This gives us two possibilities: $v_{l_f} = v_{l_i}$, $v_{r_f} = 0$ and $v_{l_f} = 0$, $v_{r_f} = v_{l_i}$. The first possibility is when the left block whiffs past the right one and doesn't collide (which we know doesn't happen) and the second possibility is when the left block comes to a stop and the right block continues with the same velocity (this is our scenario). So:

$$v_{r_f} = v_{l_i} = \sqrt{\frac{k}{m}}x$$

Now we can finally find the horizontal displacement of the right block as it falls. We can set its initial horizontal position to 0 and its final horizontal position as l, that is: $x_0 = 0,\ x_f = l$. For the vertical direction, we set the initial vertical position to h and the final vertical position to 0, that is: $y_0 = h,\ y_f = 0$. We can now set up our kinematic equations for constant acceleration $\vec{a} = 0\hat{\imath} - g\hat{\jmath}$:

$$x_f = x_0 + v_{x_0}t + \frac{1}{2}a_x t^2, \qquad y_f = y_0 + v_{y_0}t + \frac{1}{2}a_y t^2$$

$$l = v_{r_f}t, \qquad 0 = h - \frac{1}{2}gt^2$$

We can solve the second equation for t and plug into the first equation to find l:

$$t = \sqrt{\frac{2h}{g}}$$

$$l = \sqrt{\frac{k}{m}}x\sqrt{\frac{2h}{g}}$$

$$l = \sqrt{\frac{2kh}{mg}}x$$

Now we want to know what the distance would be if the collision were completely inelastic, that is, the blocks stick together. Going back to the conservation of momentum step:

$$mv_{l_i} = (m+m)v_f = 2mv_f$$

$$v_f = \frac{1}{2}v_{l_i} = \frac{1}{2}\sqrt{\frac{k}{m}}x$$

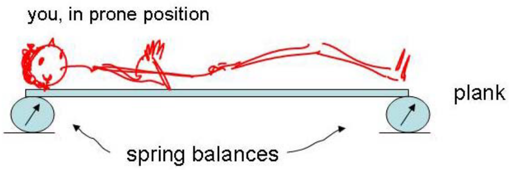
The kinematics step is exactly the same only with this new value of v_{x_0}, so:

$$l = \frac{1}{2}\sqrt{\frac{k}{m}}x\sqrt{\frac{2h}{g}}$$

$$l = \sqrt{\frac{kh}{2mg}}x$$

This is smaller than it was when the collision was elastic.

Question4. Explain how you could measure both your mass and the position of your body's centre of mass if all you had as experimental equipment was two spring balances (calibrated of course!), and a light rigid plank, set up as shown in the figure.



ANS: Lying on the plank, you are in static equilibrium (you are neither moving linearly, nor are you rotating in place). There are a total of 3 forces acting on you (all in the vertical direction): gravity (F_g) acting on your center of mass as well as the left spring force (F_{s_l}) acting at the tip of your head as well as the right spring force (F_{s_r}) acting at the bottom of your feet. Call your total height h and the distance from the bottom of your feet to your center of mass l. Our goal is to find l. Now we can apply the equations of static equilibrium. First, the total force acting on you is equal to zero. So:

$$\sum F = 0$$
$$F_{s_l} + F_{s_r} - mg = 0$$
$$F_{s_l} + F_{s_r} = mg$$

Second, the total torque acting on you is equal to zero. We can set the pivot to be wherever we want. There are 3 places that make sense to put it: the tip of the head, the bottom of the feet, or at the center of mass. We will put ours at the bottom of the feet, but all 3 will yield the same answer. We choose counterclockwise (looking at the diagram) to be the positive direction of rotation. So:

$$\sum \tau = 0$$
$$mgl - F_{s_l}h = 0$$
$$l = \frac{F_{s_l}h}{mg}$$

Finally, keep in mind that our known quantities are the readings of the two scales and our height, F_{s_l}, F_{s_r} and h, and our unknown quantities are our total mass and the height of our center of mass over our feet, m and l. So we can use our $\sum F = 0$ result to substitute into our $\sum \tau = 0$ equation:

$$l = \left(\frac{F_{s_l}}{F_{s_l} + F_{s_r}}\right)h = \left(\frac{1}{1 + F_{s_r}/F_{s_l}}\right)h$$

We can see that the fraction in brackets is less than 1, and so $l < h$, which makes perfect sense.

Question5. This question has two parts – make sure to explain your reasoning fully for both!

a) In a modified Atwood's machine, a pulley is connected to a mass. The mass is released from rest and descends to the ground, causing the pulley to rotate. Which mass will descend faster: a mass attached to a pulley which is a solid cylinder of mass m and radius r or a pulley which has the same mass and radius, but is composed of light spokes connected to a cylindrical shell (that is, all the mass is at the outside radius r)? Assume both pulleys are on frictionless bearings.

ANS: The key difference in the situations is the moment of inertia for the pulleys. For the solid cylinder, $I_{solid} = \frac{1}{2}mr^2$ and for the shell with light spokes, we can consider the pulley to be a collection of many particles all at the same distance r to the pivot, so $I_{spoke} = \sum_i m_i r_i^2 = \sum_i m_i r^2 = (\sum_i m_i) r^2 = mr^2$. That is to say, $I_{spoke} = 2I_{solid}$.

We can quickly solve the atwood machine with the equations for the lighter mass m_1, the heavier mass m_2 and the pulley with moment of inertia I:

$$T_1 - m_1 g = m_1 a$$
$$m_2 g - T_2 = m_2 a$$
$$T_2 r - T_1 r = I\alpha = \frac{Ia}{r} \rightarrow T_2 - T_1 = \frac{I}{r^2}a$$
$$(m_2 g - m_2 a) - (m_1 g + m_1 a) = \frac{I}{r^2}a$$
$$(m_2 - m_1)g = \left(m_1 + m_2 + \frac{I}{r^2}\right)a$$
$$a = \left(\frac{m_2 - m_1}{m_1 + m_2 + \frac{I}{r^2}}\right)g$$

From this expression, we can see that larger values of I will result in smaller values of a. So the machine in which the masses descend faster is the machine for which I is smaller, that is, $I_{solid} = \frac{1}{2}mr^2$.

b) Two solid cylinders, one of mass m and radius r, and the other of mass $M > m$ and radius $R > r$, are rolled down a long inclined plane. Which arrives first?

ANS: We are looking for the acceleration as the cylinders roll. The cylinders then have two forces acting on them: the force of gravity at the center of mass and the force of static friction acting at the rim of the cylinder. Because we don't know this force of friction, we will put the pivot for our analysis at the rim. To find the moment of inertia for this analysis, we need to use the parallel axis theorem:

$$I = I_{cyl} + mh^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

Now, we can find the acceleration by:

$$\sum \tau = I\alpha$$
$$mgr\sin(\theta) = \frac{3}{2}mr^2\left(\frac{a}{r}\right)$$
$$a = \frac{2}{3}g\sin(\theta)$$

As we can see, the final result does not depend on the total mass or radius of the cylinder, so both cylinders will arrive at the same time. Beware: the result DOES depend on the specific moment of inertia of a cylinder, so if we had a cylinder racing, say, a ball, the two would NOT arrive at the same time.

Problems

1. A compressed spring launches a 20.0 g metal ball at a 30.0° angle. Compressing the spring 20.0 cm causes the ball to hit the floor 1.50m below the point at which it leaves the spring after traveling 5.00m horizontally. What is the spring constant?

   ANS: This resembles a normal projectile-off-a-table problem, only now we will have an initial velocity that has both an x and a y component. First, we set up the energy conservation equation for the launching of the ball that will allow us to solve for our spring constant k:

$$K_i + U_{sp_i} + U_{g_i} = K_f + U_{sp_f} + U_{g_f}$$

$$0 + \frac{1}{2}kl^2 + 0 = \frac{1}{2}mv^2 + 0 + mgl\sin(30°)$$

$$k = \frac{mv^2}{l^2} + \frac{2mg\sin(30°)}{l}$$

   We have all the quantities we need, except for the launch speed v. We can find it by solving the kinematics part of the analysis:

$$x_f = x_0 + v_{x_0}t + \frac{1}{2}a_x t^2$$

$$\Delta x = v\cos(30°)\,t$$

$$t = \frac{\Delta x}{v\cos(30°)} = \frac{5}{v\cos(30°)}$$

$$y_f = y_0 + v_{y_0}t + \frac{1}{2}a_y t^2$$

$$\Delta y = v\sin(30°)\,t - \frac{g}{2}t^2$$

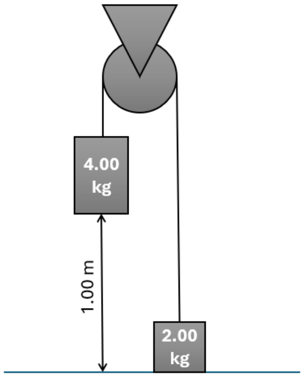
$$-1.5 = 5\tan(30°) - 4.9\left(\frac{5}{v\cos(30°)}\right)^2$$

$$-4.387 = -\frac{163.3}{v^2}$$

$$v^2 = 37.22$$

   Now we can plug this back into the energy conservation equation:

$$k = \frac{(0.0200)(37.22)}{0.200^2} + \frac{2(0.0200)(9.81)\sin(30°)}{0.200} = 18.61 + 0.981 = 19.6\,N/m$$



2. The two blocks in the figure are connected by a massless rope that passes over a pulley. The pulley, a solid cylinder, is 12.0 cm in diameter and has a mass of 2.00 kg. As the pulley turns, friction at the axle exerts a torque of magnitude 0.500 Nm. If the blocks are released from rest, how long does it take for the 4.00 kg block to reach the floor?
ANS: We can solve this like a normal Atwood machine problem, and then use the acceleration we find to do the kinematics for the heavier left block. We set up the equations for the left block, right block and the pulley:

$$m_1 g - T_1 = m_1 a$$
$$T_2 - m_2 g = m_2 a$$
$$T_1 r - T_2 r - \tau_{axle} = I\alpha$$
$$T_1 - T_2 - \frac{\tau_{axle}}{r} = \left(\frac{1}{r}\right)\frac{1}{2}Mr^2\left(\frac{a}{r}\right) = \frac{1}{2}Ma$$

We can isolate T_1 in the first equation and T_2 in the second in order to plug them into the third equation:

$$T_1 = m_1 g - m_1 a$$
$$T_2 = m_2 g + m_2 a$$

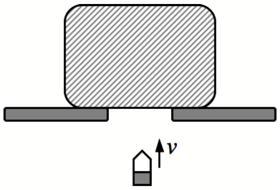
Now, plugging them into the third equation:

$$m_1 g - m_1 a - (m_2 g + m_2 a) - \frac{\tau_{axle}}{r} = \frac{1}{2}Ma$$
$$m_1 g - m_2 g - \frac{\tau_{axle}}{r} = a\left(m_1 + m_2 + \frac{1}{2}M\right)$$
$$a = \frac{m_1 g - m_2 g - \frac{\tau_{axle}}{r}}{m_1 + m_2 + \frac{1}{2}M} = \frac{(4.00 - 2.00)9.81 - \frac{0.500}{0.0600}}{4.00 + 2.00 + \frac{1}{2}2.00} = 1.612\frac{m}{s^2}$$

We can now use this in the 1D kinematic equation:

$$y_f = y_0 + v_{y_0}t + \frac{1}{2}a_y t^2$$
$$0 = 1.00 + 0t + \frac{1}{2}(-1.612)t^2$$
$$0.806t^2 = 1.00$$
$$t = 1.11\ s$$

3. A 10 g bullet moving 1000 m/s strikes and passes through a 2.0 kg block initially at rest, as shown. The bullet emerges from the block with a speed of 400 m/s. To what maximum height will the block rise above its initial position?



ANS: This problem will use a combination of the conservation of momentum (when the bullet hits the block) and the conservation of energy (when the block flies in the air). We begin with the conservation of momentum:

$$p_{bullet_i} + p_{block_i} = p_{bullet_f} + p_{block_f}$$

$$m_{bullet}v_{bullet_i} + m_{block}v_{block_i} = m_{bullet}v_{bullet_f} + m_{block}v_{block_f}$$

$$(0.010)(1000) + (2.0)(0) = (0.010)(400) + (2.0)v_{block_f}$$

$$2.0v_{block_f} = (0.010)(1000 - 400) = 6.0$$

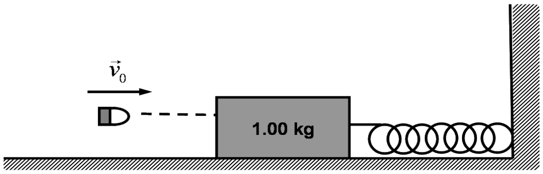
$$v_{block_f} = 3.0\ m/s$$

With this result, we can now find the maximum height attained by the block by the conservation of energy:

$$K_i + U_{g_i} = K_f + U_{g_f}$$

$$\frac{1}{2}m_{block}v_{block}^2 + 0 = 0 + m_{block}gh$$

$$h = \frac{v_{block}^2}{2g} = \frac{3.0^2}{2 \times 9.81} = 0.459\ m$$



4. A 1.00 kg block of wood is attached to a spring of force constant 200 N/m and rests on a smooth surface as shown in the figure. A 20 gram bullet is fired into the block and becomes embedded. If the spring compresses 13.3 cm answer the following:

a) Find the original speed, v_o, of the bullet before the collision.
ANS: This is a combination of conservation of momentum during the collision and conservation of energy in the compression of the spring. We will work backwards from the compression:

$$K_i + U_{sp_i} = K_f + U_{sp_f}$$
$$\frac{1}{2}m_{b+b}v_{b+b}^2 + 0 = 0 + \frac{1}{2}kx^2$$
$$v_{b+b} = \sqrt{\frac{k}{m_{b+b}}}x = \sqrt{\frac{200}{1.020}}(0.133) = 1.862\ m/s$$

With this information, we can move backwards to the collision and the conservation of momentum:

$$m_{bullet}v_0 + m_{block}(0) = m_{b+b}v_{b+b}$$
$$0.020v_0 = 1.020(1.862)$$
$$v_0 = 95.0\ m/s$$

b) What fraction of the original kinetic energy of the bullet is lost in this collision?
ANS: We find the fraction:

$$\frac{-\Delta K}{K_i} = \frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{\frac{1}{2}m_{b+b}v_{b+b}^2}{\frac{1}{2}m_{bullet}v_0^2} = 1 - \frac{m_{b+b}}{m_{bullet}}\left(\frac{v_{b+b}}{v_0}\right)^2 = 1 - \left(\frac{1.02}{0.02}\right)\left(\frac{1.862}{95.0}\right)^2$$
$$= 0.980$$

So, the bullet has lost approximately 98% of its initial kinetic energy.

5. A steel ball of mass m_1 = 1.0 kg is fastened to a cord 5.0 meters long and is released when the cord is horizontal. At the bottom of its path, the ball strikes a 4.0 kg steel block initially at rest on a smooth horizontal table. After collision, the steel ball will swing up 1.8 m

a) Find the velocity of the ball right before collision.

ANS: This is a conservation of energy problem. Initially, the ball has gravitational potential energy and after, the ball has kinetic energy:

$$U_{g_i} + K_i = U_{g_f} + K_f$$

$$m_1 g h_1 + 0 = 0 + \frac{1}{2} m_1 v_{ball_i}^2$$

$$v_{ball_i} = \sqrt{2gh_1} = \sqrt{2(9.81)(5.0)} = 9.90\ m/s$$

b) Find the velocities of the ball and the block just after collision.

ANS: We can find the final velocity of the ball (just after the collision) using conservation of energy again:

$$U_{g_i} + K_i = U_{g_f} + K_f$$

$$0 + \frac{1}{2} m_1 v_{1_f}^2 = m_1 g h_2$$

$$v_{1_f} = \sqrt{2gh_2} = \sqrt{2(9.81)(1.8)} = -5.94\ m/s$$

We must use the negative solution, because the ball bounces back to the left.

To find the velocity of the block just after collision, we can use the velocities of the ball we just calculated and use conservation of momentum through the collision:

$$m_1 v_{1_i} + m_2 v_{2_i} = m_1 v_{1_f} + m_2 v_{2_f}$$

$$(1.0)(9.90) + (4.0)(0) = (1.0)(-5.94) + (4.0) v_{2_f}$$

$$v_{2_f} = 3.96\ m/s$$

c) Is the collision elastic or inelastic?

ANS: To see if the collision is elastic or not, we compare the total kinetic energy of the system before the collision and after the collision:

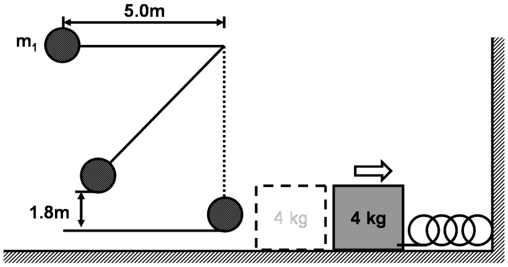
$$K_i \overset{?}{\Leftrightarrow} K_f$$

$$\frac{1}{2} m_1 v_{1_i}^2 + \frac{1}{2} m_2 v_{2_i}^2 \overset{?}{\Leftrightarrow} \frac{1}{2} m_1 v_{1_i}^2 + \frac{1}{2} m_2 v_{2_f}^2$$

$$0.5(1.0)(9.90^2) + 0.5(4.0)(0)$$
$$= 0.5(1.0)(5.94^2) + 0.5(4.0)(3.96^2)$$

$$49 = 49$$

So kinetic energy is conserved. This is an ELASTIC collision.



d) A massless spring with a spring constant of k = 1000 N/m is attached to the table as shown in the figure. Find the maximum compression of the spring.

ANS: This is a final conservation of energy problem:

$$U_{sp_i} + K_i = U_{sp_f} + K_f$$

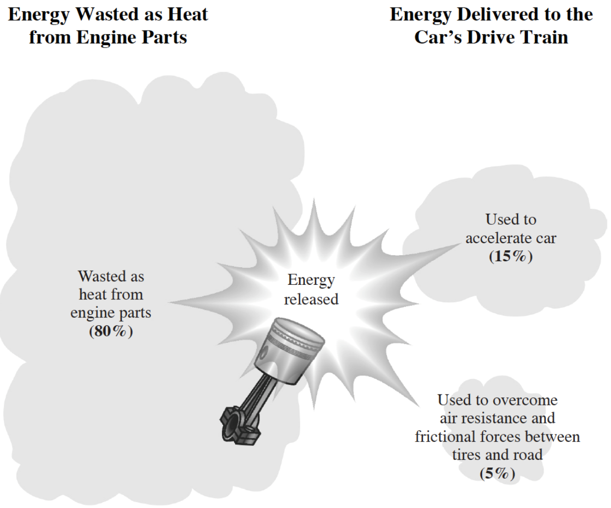
$$0 + \frac{1}{2} m_2 v_{2_f}^2 = \frac{1}{2} k x^2 + 0$$

$$x = \sqrt{\frac{m_2}{k}} v_{2_f}$$

$$x = \sqrt{\frac{4.0}{1000}} (3.96)$$

$$x = 0.250\ m = 25.0\ cm$$

6. The distribution of energy released during the burning of gasoline in a car is illustrated on the right. Gasoline releases 30.2 MJ/L during burning. A particular car has a mass of 1.60×10^3 kg. In a test drive, the car accelerated from 3.00 m/s to 15.0 m/s over a distance of 115 m.



a) Find the maximum amount of energy that would be delivered to the drive train when 65.0 L of gasoline is burned.

ANS: Since the gas releases 30.2 MJ for every L consumed, 65.0 L of gas would release:

$$E = 30.2 \times 65.0 = 1963\ MJ$$
$$= 1.963 \times 10^9\ J$$

of energy. Since only 20% of this is used to accelerate the car (delivered to the drive train), then this amounts to

$$0.20(1.963 \times 10^9) = 2.94 \times 10^8\ J$$

b) Find the change in the kinetic energy of the car during the test drive.

ANS: The change in kinetic energy is calculated using the initial and final speeds:

$$\Delta K = \frac{1}{2}m\left(v_f^2 - v_i^2\right)$$
$$\Delta K = 0.5(1.6 \times 10^3)(15^2 - 3^2)$$
$$\Delta K = 1.72 \times 10^5\ J$$

c) Find the magnitude of the impulse on the car during the test drive.

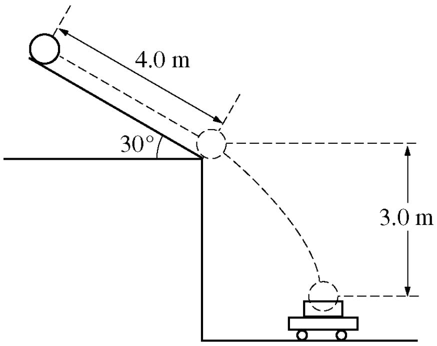
ANS: The impulse is equal to the change in the car's momentum:

$$J = \Delta p = p_f - p_i = m\left(v_f - v_i\right)$$
$$J = (1.60 \times 10^3)(15.0 - 3.00)$$
$$J = 1.92 \times 10^4\ N \cdot s$$

d) Find the average net force on the car during the test drive

ANS: If we find the average acceleration, we can calculate the average force:

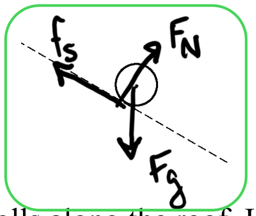
$$v_f^2 = v_i^2 + 2a_{av}(\Delta x)$$
$$15^2 = 3^2 + 2a_{av}(115)$$
$$a_{av} = 0.939\frac{m}{s^2}$$
$$F_{av} = ma_{av} = (1.6 \times 10^3)(0.939)$$
$$F_{av} = 1.50 \times 10^3\ N$$



Note: Figure not drawn to scale.

7. A bowling ball of mass 6.0 kg is released from rest from the top of a slanted roof that is 4.0 m long and angled at 30°, as shown in the diagram. The ball rolls along the roof without slipping. The rotational inertia of a sphere of mass M and radius R about its center of mass is $\frac{2}{5}MR^2$.

a) On the figure below, draw and label the forces (not components) acting on the ball at their points of application as it rolls along the roof.



b) Calculate the force due to friction acting on the ball as it rolls along the roof. If you need to draw anything other than what you have shown in part a) to assist in your solution, use the space below. Do NOT add anything to the figure in part a).
ANS: We can use the fact that the ball rolls without slipping to relate the center of mass acceleration a_{CM} to the angular acceleration of the ball α:

$$a_{CM} = R\alpha$$

$$\frac{F_{net}}{M} = R\left(\frac{\tau_{net}}{I}\right)$$

$$\frac{F_g\sin(30°) - f_s}{M} = R\left(\frac{f_sR\sin(90°)}{\frac{2}{5}MR^2}\right)$$

$$Mg\sin(30°) - f_s = \frac{5f_s}{2}$$

$$\frac{7f_s}{2} = Mg\sin(30°) = (6.0)(9.81)\sin(30°) = 29.43$$

$$f_s = 8.41\ N$$

c) Calculate the linear speed of the center of mass of the ball when it reaches the bottom edge of the roof.
ANS: Now that we know the force of static friction, we can find the center of mass acceleration of the ball:

$$a_{CM} = \frac{F_{net}}{M} = \frac{Mg\sin(30°) - f_s}{M} = \frac{(6.0)(9.81)\sin(30°) - 8.41}{6.0} = 3.50\ m/s^2$$

With this acceleration, we can calculate the final speed as it leaves the roof:

$$v_f^2 = v_i^2 + 2a_{CM}\Delta x = 0 + 2(3.50)(4.0) = 28$$

$$v_f = 5.29\ m/s$$

d) A wagon containing a box is at rest on the ground below the roof so that the ball falls a vertical distance of 3.0 m and lands and sticks in the center of the box. The total mass of the wagon and the box is 12 kg. Calculate the horizontal speed of the wagon immediately after the ball lands in it.
ANS: This is a conservation of momentum problem. When the ball hits the cart, momentum is conserved in all directions. We now select our coordinate axes as x in the horizontal direction and y in the vertical direction. It is difficult to calculate the conservation of momentum in the y direction

since the cart is sitting on the ground, so the entire earth is part of the calculation. However, in the x direction, we can only consider the ball and the wagon. So, (using m as the mass of the wagon):

$$Mv_{b_{x_i}} + mv_{w_{x_i}} = (M+m)v_{x_f}$$

Since the wagon starts at rest, $v_{w_{x_i}} = 0$ and so:

$$v_{x_f} = \left(\frac{M}{M+m}\right)v_{b_{x_i}}$$

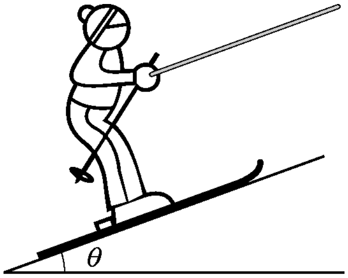
Now, we find that the horizontal component of the ball's velocity, which DOES NOT CHANGE as is falls down the 3 m, is:

$$v_{b_{x_i}} = v\cos(30°) = 5.29\cos(30°) = 4.58\ m/s$$

And so:

$$v_{x_f} = \left(\frac{6}{6+12}\right)(4.58) = 1.53\ m/s$$

8. A skier of mass m will be pulled up a hill by a rope, as shown in the diagram. The magnitude of the acceleration of the skier as a function of time t can be modeled by the equations:



$$a = a_{max} \sin\left(\frac{\pi t}{T}\right) \qquad (0 < t < T)$$

$$a = 0 \qquad (t \geq T)$$

where a_{max} and T are constants. The hill is inclined at an angle θ above the horizontal, and friction between the skis and the snow is negligible. Express your answers in terms of given quantities and fundamental constants.

a) Derive an expression for the velocity of the skier as a function of time during the acceleration. Assume the skier starts from rest.
ANS: We know that the acceleration is the time rate of change, or DERIVATIVE, of the velocity:

$$a = \frac{dv}{dt}$$

So, we need to find a function for which we get the given acceleration function when we take its derivative. Since we know that the derivative of $\cos(x)$ is $-\sin(x)$ we can intuit the form of the velocity function for $0 < t < T$:

$$v = -v_{max} \cos\left(\frac{\pi t}{T}\right) + C$$

We can find the value of v_{max} by actually taking the derivative and comparing:

$$\frac{dv}{dt} = -v_{max}\left(-\sin\left(\frac{\pi t}{T}\right) \cdot \frac{\pi}{T}\right) = \frac{\pi v_{max}}{T} \sin\left(\frac{\pi t}{T}\right)$$

So, we have that $v_{max} = a_{max}T/\pi$. This gives us our velocity for $0 < t < T$. To find C, we use the initial condition, which tells us the skier starts at rest. So:

$$v(0) = -v_{max} + C = 0 \;\; \rightarrow \;\; C = v_{max} = a_{max}T/\pi$$

To find the velocity after T, we will insist that the velocity function is continuous. Since the acceleration after T is zero, the velocity after T is constant, and should be equal to the velocity when we plug in $t = T$ in our function for $0 < t < T$:

$$v(T) = \frac{a_{max}T}{\pi}\left(1 - \cos\left(\frac{\pi T}{T}\right)\right) = \frac{2a_{max}T}{\pi}$$

Putting it all together:

$$v(t) = \begin{cases} \dfrac{a_{max}T}{\pi}\left(1 - \cos\left(\dfrac{\pi t}{T}\right)\right) & 0 < t < T \\ \dfrac{2a_{max}T}{\pi} & t \geq T \end{cases}$$

b) Derive an expression for the work done by the net force on the skier from rest until terminal speed is reached.
ANS: The work done on the skier is equal to the change in the skier's kinetic energy:

$$W = \Delta K = K_f - K_i = \frac{1}{2}m(v(T)^2 - v(0)^2)$$

$$W = \frac{1}{2}m\left(\left(\frac{2a_{max}T}{\pi}\right)^2 - (0)^2\right) = 2m\left(\frac{a_{max}T}{\pi}\right)^2$$

c) Determine the magnitude of the force exerted by the rope on the skier at terminal speed.

ANS: At terminal speed, $t = T$ and $a(T) = 0$, so $F_{net}(T) = ma(T) = 0$. On the other hand, in the direction along the slope:

$$F_{net}(t) = F_{rope}(t) - mg\sin(\theta)$$

So:

$$F_{net}(T) = F_{rope}(T) - mg\sin(\theta) = 0$$

$$\therefore F_{rope}(T) = mg\sin(\theta)$$

Note for part e): we find similarly that $F_{net}(0) = ma(0) = 0$ and so $F_{rope}(0) = mg\sin(\theta)$

d) Derive an expression for the total impulse imparted to the skier during the acceleration.

ANS: This calculation can be done by integrating the force over the total time, but there is actually a much easier way, now that we've already integrated the acceleration to find the form of the velocity equation:

$$J = \Delta p = p(T) - p(0) = m\big(v(T) - v(0)\big)$$

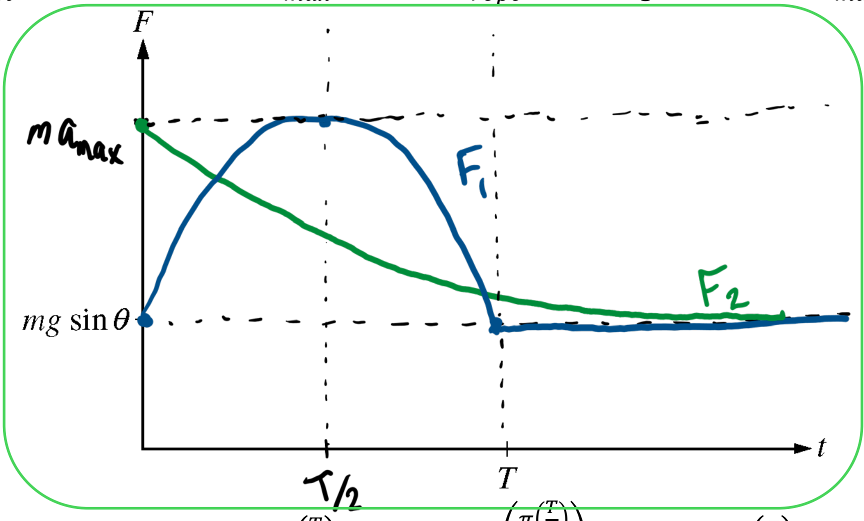
$$J = m\left(\left(\frac{2a_{max}T}{\pi}\right) - 0\right) = \frac{2ma_{max}T}{\pi}$$

e) Suppose that the magnitude of the acceleration is instead modeled as $a = a_{max}e^{-\frac{\pi t}{2T}}$ for all $t > 0$, where a_{max} and T are the same as in the original model. On the axes below, sketch the graphs of the force exerted by the rope on the skier for the two models, from $t = 0$ to a time $t > T$. Label the original model F_1 and the new model F_2.

ANS: Before we graph anything, we do a similar analysis for this new model as we did in part c)

$$F_{net}(0) = ma(0) = ma_{max} \rightarrow \quad F_{rope}(0) = mg\sin(\theta) + ma_{max}$$

$$F_{net}(T) = ma(T) = ma_{max}e^{-\frac{\pi}{2}} \rightarrow \quad F_{rope}(T) = mg\sin(\theta) + ma_{max}e^{-\frac{\pi}{2}}$$



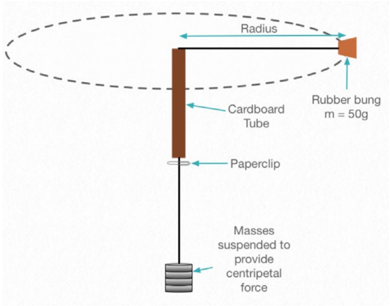
Note that F_1 peaks when $t = T/2$ and $a\left(\frac{T}{2}\right) = a_{max}\sin\left(\frac{\pi\left(\frac{T}{2}\right)}{T}\right) = a_{max}\sin\left(\frac{\pi}{2}\right) = a_{max}$ and so:

$$F_1 = mg\sin(\theta) + ma_{max}$$

For F_2, we note that $a(t) \rightarrow 0$ as $t \rightarrow \infty$ and so $F_2 \rightarrow mg\sin(\theta)$. In both cases, we've used the fact that $F_{net}(t) = ma(t) = F_{rope}(t) - mg\sin(\theta)$ and so $F_{rope}(t) = mg\sin(\theta) + ma(t)$.

9. A mini experiment!

A student is investigating how the velocity of a whirling bung is affected by the radius of rotation. A bung of mass $m = 50\ g$ is attached to a long string and is passed through a tube. Masses are suspended on the string to provide the centripetal force. This force is kept constant throughout.



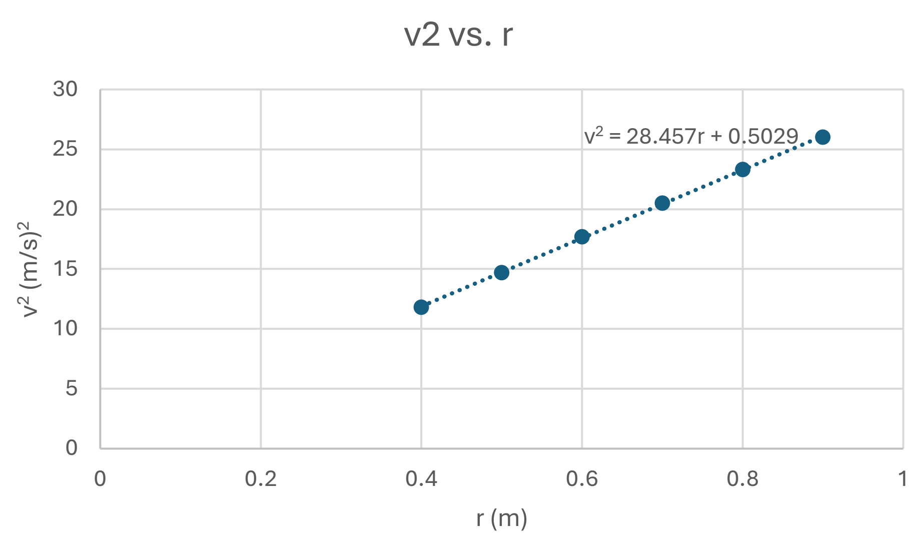
The student collects the following data:

Radius (m)	Time taken for 10 complete rotations $t_{10}\ (s)$			Average time taken for 1 complete rotation, $t_1(s)$	Tangential speed $v\ \left(\frac{m}{s}\right)$	$v^2\ (m^2s^{-2})$
	$t_{10(1)}\ (s)$	$t_{10(2)}\ (s)$	$t_{10(3)}\ (s)$			
0.4	7.26	7.34	7.31	0.730	3.44	11.8
0.5	8.23	8.24	8.15	0.821	3.83	14.7
0.6	8.99	9.01	8.84	0.895	4.21	17.7
0.7	9.70	9.68	9.74	0.971	4.53	20.5
0.8	10.36	10.41	10.45	1.041	4.83	23.3
0.9	11.05	11.23	10.95	1.108	5.10	26.0

a) Complete the table.

ANS: Note: to fill in the table, we used $t_1 = \frac{t_{10(1)}+t_{10(2)}+t_{10(3)}}{30}$ and $v = \frac{2\pi r}{t_1}$

b) Draw a graph of radius (x-axis) against v^2 (y-axis). Draw a line of best fit.



c) From the graph (and the slope!) determine the value of the centripetal force.

ANS: The centripetal force is $F_c = \frac{mv^2}{r} = m(slope) = (0.05)(28.46) = 1.423\ N$

d) Now, $3 \times 50g$ are suspended on the bottom of the string to provide the centripetal force. Determine the % difference between the experimental value of the centripetal force and the actual value.

ANS: Since the hanging weights are not moving up or down, $F_{net_y} = 0 = F_c - F_g$. So:

$$F_c = F_g = m_h g = 3(0.05)(9.81) = 1.472\ N$$

Now we can calculate the percent difference:

$$\%diff = \frac{\left|F_{c_1} - F_{c_2}\right|}{\frac{F_{c_1} + F_{c_2}}{2}} \times 100 = \frac{|1.423 - 1.472|}{\frac{1.423 + 1.472}{2}} \times 100 = 3.38\%$$

10. A student is given the orbital data for some of the moons on Saturn shown below and is asked to use the data to determine the mass M_S of Saturn. Assume the orbits of these moons are circular.

Orbital Period, *T* (s)	**Orbital Radius, *R* (m)**	**T^2 (s^2)**	**R^3 (m^3)**
8.14×10^4	1.85×10^8	6.63×10^9	6.33×10^{24}
1.18×10^5	2.38×10^8	1.39×10^{10}	1.35×10^{25}
1.63×10^5	2.95×10^8	2.66×10^{10}	2.57×10^{25}
2.37×10^5	3.77×10^8	5.62×10^{10}	5.36×10^{25}

a) Write an algebraic expression (that means, using variables) for the gravitational force between Saturn and one of its moons.
ANS: The expression for the gravitational force is Newton's law of gravitation:

$$F_g = \frac{GM_SM_m}{R^2}$$

b) Use your expression from part a) and the assumption of circular orbits to derive an equation for the orbital period *T* of a moon as a function of its orbital radius *R*.
ANS: If we assume the orbits are circular, then the gravitational force is a centripetal force. So:

$$F_g = F_c$$

$$\frac{GM_SM_m}{R^2} = M_mR\left(\frac{2\pi}{T}\right)^2$$

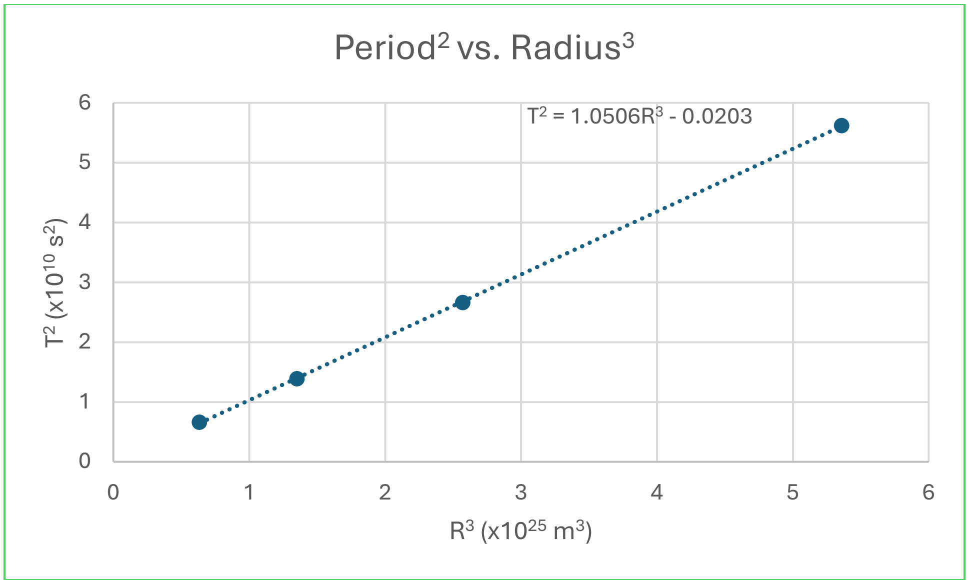
$$T^2 = \frac{4\pi^2}{GM_S}R^3$$

$$T = \frac{2\pi}{\sqrt{GM_S}}R^{3/2}$$

c) Which quantities should be graphed to yield a straight line whose slope could be used to determine Saturn's mass?
ANS: We can either graph T against $R^{3/2}$ or we can graph T^2 against R^3. Either one will result in a linear relationship whose slope would allow us to solve for M_S. Note that the slope in either of these graphs would NOT be M_S itself. We would have to compare the equation of the graph to the relationship between the variables to allow us to find M_S using the slope.

d) Complete the data table by calculating the two quantities to be graphed. Label the top of each column, including units.
ANS: In completing the graph, note that we are choosing to plot T^2 against R^3

e) Plot a graph. Label the axes with the variables used and appropriate numbers to indicate the scale.
ANS: The graph is on the following page. Note the powers of 10 in each scale, meaning that the slope is actually $1.0506 \times 10^{-15} s^2/m^3$



f) Using the graph, calculate the mass of Saturn.

ANS: Comparing the equation from the graph to the equation derived in part b), we find that:

$$slope = \frac{4\pi^2}{GM_S}$$

$$1.0506 \times 10^{-15}\,\frac{s^2}{m^3} = \frac{4\pi^2}{GM_S}$$

$$M_S = \frac{4\pi^2}{(6.67 \times 10^{-11})(1.0506 \times 10^{-15})}$$

$$M_S = 5.63 \times 10^{26}\ kg$$

Note: A simple internet search shows that in actuality, $M_S = 5.68 \times 10^{26}\ kg$, which gives a percent error of:

$$\%err = \frac{|theo - ex|}{theo} \times 100 = \frac{|5.68 \times 10^{26} - 5.63 \times 10^{26}|}{5.68 \times 10^{26}} \times 100 = 0.880\%$$

This is quite a good result.

11. A student places a 0.40 kg car on a frictionless track and holds it so that it touches an uncompressed ideal spring, as shown in the figure-1. The student then pushes the cart back to compress the spring by 0.25 m, as shown in figure-2. At time $t = 0$, the student releases the cart, and a motion sensor begins recording the velocity of the reflector at the front of the cart as a function of time. The data points are shown in the table below. At time $t = 0.79\ s$, the cart loses contact with the spring.

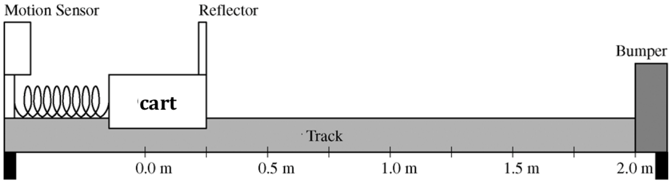


Figure 1

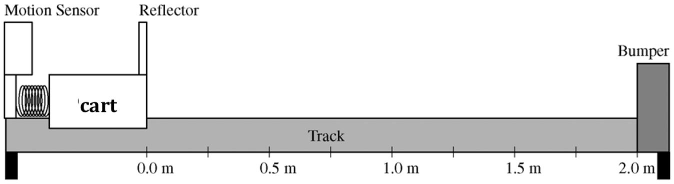
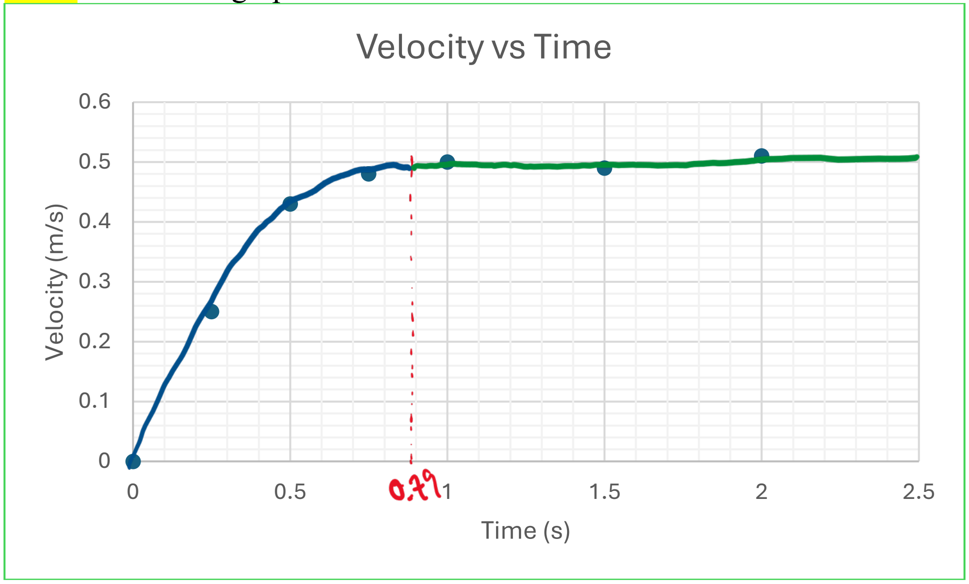


Figure 2

Times (s)	0	0.25	0.50	0.75	1.00	1.50	2.00
Velocity (m/s)	0	0.25	0.43	0.48	0.50	0.49	0.51

a) Sketch the velocity as a function of time for the cart, and draw a smooth curve that best fit the data. Be sure to label your axes!
ANS: Here is the graph. We've denoted the time at which the car loses



b) The student wishes to use the data to plot position x as a function of time for the cart.

i) Describe a method the student could use to do this.
ANS: The student could find the area under the velocity time graph between successive times to find the displacements during that time interval. They would then add that displacement to the position at the time at the beginning of the interval to find the approximate position at the end of the time interval. Using the following equation, to calculate the area of the trapezoid under the $v - t$ graph between t_n and t_{n+1}:

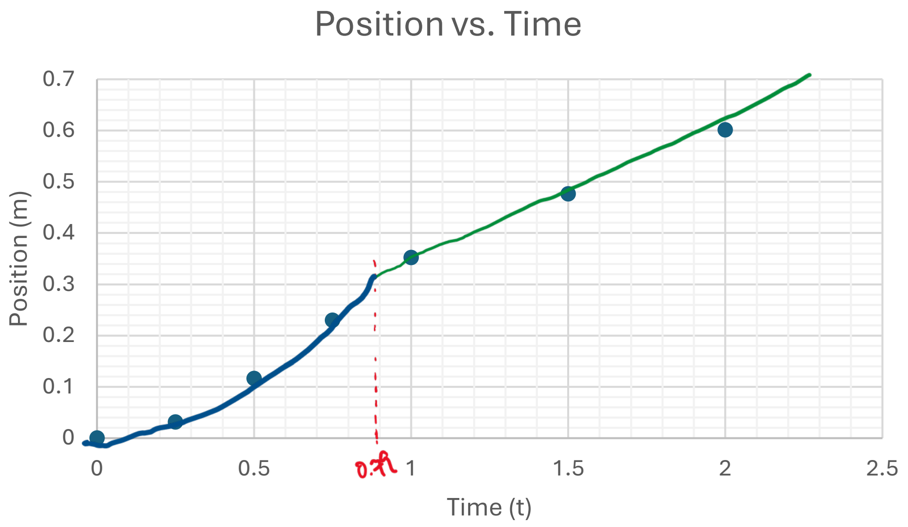
$$x_{n+1} - x_n = \left(\frac{v_{n+1} + v_n}{2}\right)(t_{n+1} - t_n)$$

Since we know that the cart begins at the origin, we can set $x_0 = 0$ at $t_0 = 0$. We can find all the other x_n from the above equation and initial conditions. This yields the table:

Times (s)	0	0.25	0.50	0.75	1.00	1.50	2.00
Velocity (m/s)	0	0.25	0.43	0.48	0.50	0.49	0.51
Position (m)	0	0.03125	0.11625	0.23000	0.35250	0.47625	0.60125

ii) Sketch the position as a function of time for the cart. Explicitly label any intercepts, asymptotes, maxima, or minima with numerical values or algebraic expression, as appropriate.

ANS: Note again that we have shown where the car loses contact with the spring:



c) Calculate the time at which the cart makes contact with the bumper at the far right.

ANS: We notice that once the car leaves the spring, it travels at constant velocity (roughly $0.5\ m/s$), which makes perfect sense as there are no forces acting on the car in the x-direction. So, the function for it's position is:

$$x_2(t) = 0.5(t - 0.79) + C$$

We can see here that C is the position of the car when it leaves the spring: $x_2(0.79) = C$. If we could find C, we could then solve $x_2(t) = 2.0\ m$ to find the time at which it arrives at the bumper. Finding the equation of the car's motion while still in contact with the spring is beyond the scope of the course (this would be a good Waves problem), so we will find $x_2(0.79)$ using another method. We find that $x_2(0.79) = C \approx 0.32\ m$. So:

$$x_2(t) = 0.5(t - 0.79) + 0.32 = 0.5t - 0.075$$

Now we can calculate the time the car reaches the bumper t_b:

$$2.0 = x_2(t_b) = 0.5t_b - 0.075$$

$$t_b = 4.15\ s$$

d) Calculate the force constant of the spring.

ANS: We can calculate this using conservation of energy. We know that the spring is initially compressed by $0.25\ m$ and it is released from the spring with a velocity of $\sim 0.5\ m/s$. So:

$$U_{sp_i} + K_i = U_{sp_f} + K_f$$

$$\frac{1}{2}kx_i^2 + 0 = 0 + \frac{1}{2}mv_f^2$$
$$\frac{1}{2}k(-0.25)^2 = \frac{1}{2}(0.40)(0.5)^2$$
$$k = 1.6\ N/m$$

e) The experiment is run again, but this time the cart is attached to the spring rather than simply pushed against it.

i) Determine the maximum stretch (amplitude) of the spring

ANS: This is a simple solve using conservation of energy:

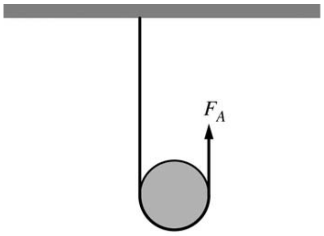
$$U_{sp_i} + K_i = U_{sp_f} + K_f$$
$$\frac{1}{2}kx_i^2 + 0 = \frac{1}{2}kx_f^2 + 0$$
$$x_f = x_i = 0.25$$

Note that these are NOT positions on the meter stick, they are the maximum stretches and compressions of the spring, that is, they are the amplitude: $A = 0.25\ m$

ii) Determine the time (period) to complete a cycle.

ANS: Since we know that the amplitude is $A = 0.25\ m$, we can see how much time it took for the block to move that distance when it was not attached to the spring. Because it was still in contact with the spring, we can assume that this would be the same amount of time it takes for the block to reach that point when it IS attached to the spring. From the graph, we see this is approximately $0.8\ s$. Because this is the time it takes for the spring to move from the amplitude to the equilibrium, this is equal to one quarter of a full cycle (one full cycle has the block move from one amplitude to the equilibrium, to the other amplitude, back through equilibrium and back to the starting point). So the total period is $T \approx 3.2\ s$. Using techniques you will learn in your Waves class, we can calculate the period exactly as $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.4}{1.6}} = 3.14\ s$, so we can see our approximation is quite good.

12. A disk of mass $M = 2.0\ kg$ and radius $R = 0.10\ m$ is supported by a rope of negligible mass, as shown. The rope is attached to the ceiling at one end and passes under the disk. The other end of the rope is pulled upward with a force F_A. The rotational inertia of the disk around its center is $\frac{1}{2}MR^2$.



Note: Figure not drawn to scale.

a) Calculate the magnitude of the force F_A necessary to hold the disk at rest.

ANS: Since the disk is neither translating nor rotating, we see that it is in static equilibrium and so $F_{net_y} = 0$ and $\tau_{net} = 0$. (Also, trivially, $F_{net_x} = 0$, but this won't help us).

We will call the tension in the left side of the rope F_T and so:

$$F_{net_y} = 0$$

$$F_T + F_A - Mg = 0$$

And:

$$\tau_{net} = 0$$

$$F_A R \sin(90°) - F_T R \sin(90°) = 0$$

From the second equation, we see that $F_T = F_A$. Plugging this into the first equation, we see that:

$$F_A = \frac{Mg}{2} = \frac{2(9.81)}{2} = 9.81\ N$$

At time $t = 0$, the force F_A is increased to 12 N, causing the disk to accelerate upward. The rope does not slip on the disk as the disk rotates.

b) Calculate the linear acceleration of the disk.

ANS: We now set up the same two equations as in a), only now we have non-zero linear and angular acceleration that are related via the non-slip rule: $\alpha = a_{CM}/R$:

$$F_{net_y} = Ma_{CM}$$

$$F_A + F_T - Mg = Ma_{CM}$$

And:

$$\tau_{net} = I\alpha$$

$$F_A R \sin(90°) - F_T R \sin(90°) = \frac{1}{2}MR^2\left(\frac{a_{CM}}{R}\right)$$

$$F_A - F_T = \frac{1}{2}Ma_{CM}$$

Adding the left and right hand sides of the two equations:

$$2F_A - Mg = \frac{3}{2}Ma_{CM}$$

$$a_{CM} = \frac{4F_A}{3M} - \frac{2g}{3} = \frac{4(12)}{3(2)} - \frac{2(9.81)}{3} = 1.46\ m/s^2$$

c) Calculate the angular speed of the disk at $t = 3.0\ s$.

ANS: With $a_{CM} = 1.46\ m/s^2$, we find that $\alpha = a_{CM}/R = 1.46/0.10 = 14.6\ rad/s^2$. Using this, we can solve the kinematic equation:

$$\omega = \omega_i + \alpha t$$

$$\omega = 0 + 14.6(3.0) = 43.8\ rad/s$$

d) Calculate the increase in total mechanical energy of the disk from $t = 0$ to $t = 3.0\ s$.

ANS: Because the disk rolls without slipping, we know that $v_{CM} = \omega R$ and so at $t = 3.0\ s$:

$$v_{CM} = \omega R = 43.8(0.10) = 4.38\ m/s$$

With these, we can calculate the increase in kinetic energy of the disk:

$$\Delta K = K_f - K_i = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2 - 0$$

$$\Delta K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2$$

$$\Delta K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{4}Mv_{CM}^2 = \frac{3}{4}Mv_{CM}^2$$

$$\Delta K = 0.75(2.0)(4.38^2) = 28.8\ J$$

However, this is NOT the total increase in mechanical energy, as the disk has lifted higher in these 3.0 s. We begin by calculating the total angular displacement of the wheel:

$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = 0(3.0) + \frac{1}{2}(14.6)(3.0^2) = 65.7\ rad$$

Because the disk rolls without slipping:

$$\Delta y_{CM} = R\Delta\theta = 0.10(65.7) = 6.57\ m$$

And so, the disk also gains gravitational potential energy:

$$\Delta U_g = Mg\Delta y_{CM}$$

$$\Delta U_g = (2.0)(9.81)(6.57) = 128.9\ J$$

So, the total increase in mechanical energy is:

$$\Delta E_{mech} = \Delta K + \Delta U_g = 28.8 + 128.9 = 157.7\ J$$

e) The disk is replaced by a hoop of the same mass and radius. Compare the linear acceleration of the hoop with the one of the disk ($a_{disk} > = or < a_{hoop}$). Explain your answer.

ANS: As the hoop's mass is essentially distributed at the rim, we have that $I_{hoop} \approx MR^2$. Even if this is not exact, we certainly have that $I_{hoop} > I_{disk}$. Since the total torque exerted is the same, this means that $\alpha_{hoop} < \alpha_{disk}$ and by extension, we have that $a_{hoop} < a_{disk}$.