SOLUTIONS TO PROBLEM SET # 4

Centre of Mass, Torque and Equilibrium

CONCEPTUAL QUESTIONS

Question 1.

After the explosion, the center of mass will continue to follow its parabolic trajectory, provided that air resistance is negligible. The horizontal and vertical positions will be found using the kinematics equations with a constant acceleration $\vec{a} = (0\,\hat{\imath} - 9.81\,\hat{\jmath})\ \mathrm{m/s^2}$.

In the scenario where the projectile of mass m does not explode, the horizontal range given is 10.0 m. In the scenario where the projectile splits into two fragments of equal mass ($m_1 = m_2 = m/2$), one of the two pieces lands at a horizontal range of 15.0 m. We must apply conservation of linear momentum in that case to solve the question.

The momentum for the system before the explosion is:

$$\vec{P}_i = m\vec{v}_i$$

The momentum for the system after the explosion is:

$$\vec{P}_f = \left(\frac{m}{2}\right)\vec{v}_1 + \left(\frac{m}{2}\right)\vec{v}_2$$

Air resistance is neglected, and so momentum for the system is constant so we can write:

$$m\vec{v}_i = \left(\frac{m}{2}\right)\vec{v}_1 + \left(\frac{m}{2}\right)\vec{v}_2$$

Assuming that the time it takes for both fragments to reach the ground and denoting it Δt:

$$m\left(\frac{10.0}{\Delta t}\right) = \left(\frac{m}{2}\right)\left(\frac{15.0}{\Delta t}\right) + \left(\frac{m}{2}\right)\left(\frac{\Delta x}{\Delta t}\right)$$

We can now solve for the unknown distance Δx:

$$\Delta x = 5.00\ \mathrm{m}$$

Question 2.

a. 1.50 cm.

The center of mass is obtained using the following formula:

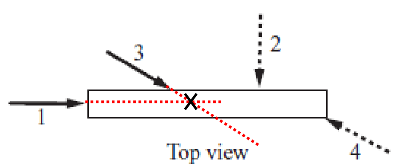
$$x_{cm} = \frac{\sum_i m_i r_i}{\sum_i m_i}$$

$$x_{cm} = \frac{(2.00\text{ mg}) \cdot (0.00\text{ cm}) + (6.00\text{ mg}) \cdot (6.00\text{ cm})}{(2.00\text{ mg}) + (6.00\text{ mg})} = 4.50\text{ cm}$$

This represents the center of mass as measured from the lighter particle (located at $x = 0.00$ cm in the calculation above). So, the center of mass is at $(6.00 - 4.50) = 1.50$ cm from the heavier particle.

Question 3.

We must assume that the board is of uniform density. If we extend the arrows corresponding to forces 1 and 3, they will intersect at a point corresponding to the center of mass.



We can then confirm that forces 1 and 3 will only make the board to translate, while forces 2 and 4 will make the board rotate and translate.

Question 4.

The center of mass is located at a point where the knife will not rotate. At that position, equal masses will be on both sides of the pivot point (your finger) so that torques will be equal and opposite ensuring the state of rotational equilibrium.

Question 5.
We can choose the right end of the platform as a pivot to simplify the calculations. In doing so, here is the corrected solution.

Applying the static rotational equilibrium equation ($\vec{\tau}_{\text{net}} = 0$):

$$-T_{left}(10.0\ m) + 200\ N(5.0\ m) + 800\ N\ (3.0\ m) + T_{right}(0) = 0\ \cdot$$
$$T_{left} = 340\ N$$

Applying the static translational equilibrium equation ($\vec{F}_{\text{net}} = 0$):

$$+T_{left} - 200 - 800 + T_{right} = 0$$
$$T_{right} = 660\ N$$

Question 6.
a)

$$\vec{\tau}_{\text{net}} = \vec{\tau}_{\text{small}} + \vec{\tau}_{\text{large}}$$
$$|\vec{\tau}_{\text{net}}| = -1.5T \cdot r + T \cdot 2r$$
$$|\vec{\tau}_{\text{net}}| = +0.5Tr$$

b)
As calculated above, the net torque on the pulley is positive. This corresponds to a net positive angular acceleration. Since the pulley is initially rotating clockwise, it will be turning clockwise with a decreasing speed until it reaches a turning point and starts turning counterclockwise.

c)

If we want to maintain its angular velocity, the net torque must be zero. In order to achieve that, the tension in the left rope must decrease by a quarter of the value it was. Mathematically, we would then obtain:

$$|\vec{\tau}_{\text{net}}| = -1.5T \cdot r + (0.750T) \cdot 2r = 0$$

d)
If we want to maintain its angular velocity, the net torque must be zero. In order to achieve that, the tension in the right rope must increase by one third compared to the value it was before. Mathematically, we would then obtain:

$$|\vec{\tau}_{\text{net}}| = -1.5T \cdot (4/3) \cdot r + T \cdot 2r = 0$$

Question7.
The tension in the rope is found using Newton's 2nd law applied to 2.0-kg pail:

$$mg - T = ma$$
$$T = m(g - a)$$

The equation for tension should be use for the net torque on the disk:

$$\tau_{\text{disk}} = I\alpha$$
$$+T \cdot r = I\left(\frac{a}{r}\right)$$
$$m(g - a) \cdot r = I\left(\frac{a}{r}\right)$$
$$a = \frac{mgr^2}{I + mr^2} = \frac{(2.0\text{ kg}) \cdot \left(9.81\frac{\text{N}}{\text{kg}}\right) \cdot (0.20\text{ m})^2}{0.800\text{ kgm}^2 + (2.0\text{ kg}) \cdot (0.20\text{ m})^2} = 0.892\text{ m/s}^2$$
$$\alpha = \frac{a}{r} = 4.46\text{ rad/s}^2$$

The rotational velocity after 2.0 s will be:
$$\omega = 0 + \alpha t = (4.46\text{ rad/s}^2)(2.0\ s) = 8.92\text{ rad/s}$$

Question 8.
Diagrams a. and b. both provide the largest torque about the pivot. They differ only in the direction of rotation given to the arm.

Question 9.

a.

They all possess the same total mass, but distributed differently is each case.

b.

Arrangements A and C possess the same moment of inertia, and it's the largest among the combinations shown. Shown below are the moment of inertia of the five combinations:

$$I_A = MR^2$$
$$I_B = M\left(\frac{R}{2}\right)^2 = \frac{MR^2}{4}$$
$$I_C = MR^2$$
$$I_D = \left(\frac{M}{2}\right)\left(\frac{R}{2}\right)^2 + \left(\frac{M}{2}\right)R^2 = \frac{3MR^2}{4}$$
$$I_E = \frac{1}{2}MR^2$$

c.

Since $I_A = I_C > I_D > I_E > I_B$, the largest torque needed will be applied to combinations A and C.

d.

The rotational kinetic energy is proportional to the moment of inertia and the angular velocity squared:

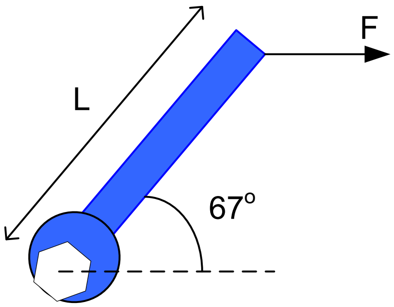
$$E_R = \frac{1}{2}I\omega^2$$

And since the angular velocity is the same for all five combinations, the ranking in terms of rotational kinetic energy will follow the same order as the moment of inertia.

$$E_{R_A} = E_{R_C} > E_{R_D} > E_{R_E} > E_{R_B}$$

PROBLEMS

1.



$\tau = 95 N \cdot m$

$L = 45 cm = 0.45 m$

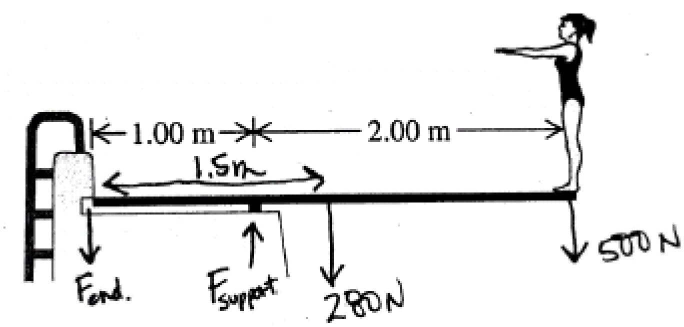
$\vec{\tau} = \vec{r} \times \vec{F}$ with $r = L$

magnitude of torque is $\tau = rF \sin\theta = 95$

θ is the angle between $\vec{r}$ and $\vec{F}$

$$F = \frac{\tau}{r \sin\theta} = \frac{\tau}{L \sin\theta} = \frac{95 N \cdot m}{0.45 m \cdot \sin 67^\circ} = \boxed{229.3 N}$$

2.



balance torques at the right end of the board and balance forces.
Clockwise torque is negative.

a)

$\vec{\tau}_{net} = 0$

$\vec{\tau}_{girl} + \vec{\tau}_{board} + \vec{\tau}_{\text{support}} + \vec{\tau}_{end} = 0$

$-500 N \cdot 3 m - 280 N \cdot 1.5 m + F_{\text{support}} \cdot 1 m + 0 = 0$

$$F_{\text{support}} = \frac{500 \cdot 3 + 280 \cdot 1.5}{1} = \boxed{1920 N}$$

b)

$\Sigma F_y = 0$

$\vec{F}_{end} + \vec{F}_{\text{support}} + \vec{F}_{board} + \vec{F}_{girl} = 0$

$\vec{F}_{end} = 1920 - 280 - 500 = \boxed{-1140 N\hat{j}}$

3.

$I = 2.7 kg \cdot m^2$

$700 rpm = 700(2\pi)/60 = 73.3 rad/s$

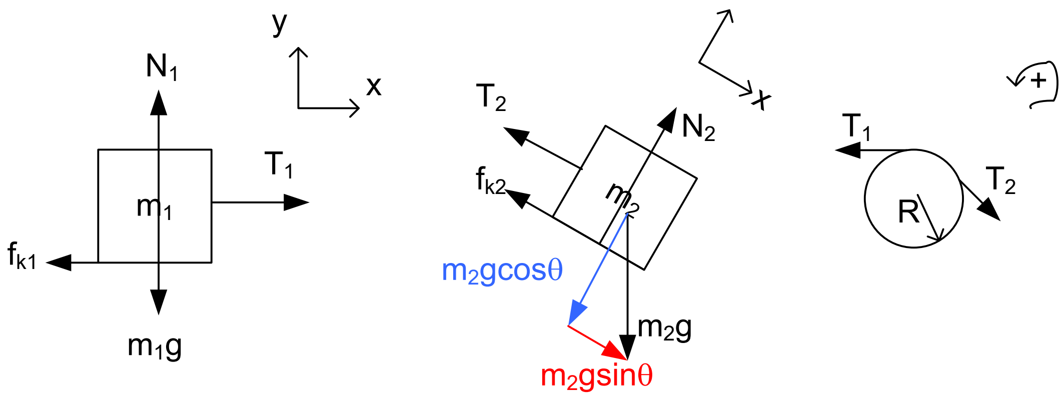
use $\tau = I\alpha$ but what is α?

$$\omega^2 = \omega_o^2 + 2\alpha(\Delta\theta)$$

$$\alpha = \frac{\omega^2 - \omega_o^2}{2(\theta - \theta_o)} = \frac{(73.3 rad/s)^2 - 0}{2(2\pi(25) - 0)} = 17.11 rad/s^2$$

$$\tau = 2.7 kg \cdot m^2 \cdot 17.11 rad/s^2 = \boxed{46.2 N \cdot m}$$

4.



consider m_1:

$\Sigma F_x = T_1 - f_{k1} = m_1 a$

$\Sigma F_y = N_1 - m_1 g = 0 \rightarrow N_1 = m_1 g$

$T_1 - \mu m_1 g = m_1 a \rightarrow T_1 = m_1 a + m_1 \mu g$

consider m_2:

$\Sigma F_x = -T_2 - f_{k2} + m_2 g \sin 30 = m_2 a$

$\Sigma F_y = N_2 - m_2 g \cos 30 = 0 \rightarrow N_2 = m_2 g \cos 30$

$-T_2 - \mu m_2 g \cos 30 + m_2 g \sin 30 = m_2 a \rightarrow T_2 = -m_2 a + m_2 g \sin 30 - m_2 \mu g \cos 30$

consider pulley:

$\Sigma \vec{\tau} = I\alpha$ (α is clockwise, so negative)

$$\tau_1 - \tau_2 = I\alpha = -\left(\frac{1}{2}MR^2\right)\frac{a_{\tan}}{R}$$

$$T_1R - T_2R = -\frac{1}{2}MRa$$

$$T_1 - T_2 = -\frac{1}{2}Ma$$

substitute expression for T_1 and T_2

$$(m_1a + m_1\mu g) - (m_2 g\sin 30 - \mu m_2 g\cos 30 - m_2 a) = -\frac{1}{2}Ma$$

$$m_1\mu g - m_2 g sin30 + \mu m_2 g\cos 30 = \left(-\frac{1}{2}M - m_1 - m_2\right)a$$

$$a = \frac{m_2 g\sin 30 - \mu m_1 g - \mu m_2 g\cos 30}{0.5M + m_1 + m_2} = \frac{6kg\cdot 9.81m/s^2\cdot\sin 30 - 0.360\cdot 9.81m/s^2\cdot(2kg + 6kg\cos 30)}{(2+6+0.5\cdot 10)kg} =$$

$$\boxed{a = 0.309m/s^2}$$

then find T_1 and T_2

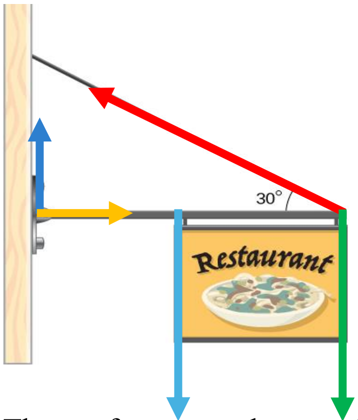
$$T_1 = m_1(a + \mu g)$$

$$T_1 = 2kg\cdot(0.309m/s^2 + 0.360\cdot 9.81m/s^2) = \boxed{7.67N = T_1}$$

$$T_2 = m_2(g\sin 30 - \mu g\cos 30 - a)$$

$$T_2 = 6kg\cdot(9.81m/s^2\sin 30 = 0.360\cdot 9.81m/s^2\cos 30 - 0.309m/s^2) = \boxed{9.21N = T_2}$$

6.



The net force must be zero for the translational static equilibrium condition to be met.

Along the y-axis:

$$\sum F_y = 0$$

$$\boldsymbol{F}_{\textbf{wall,vertical}} - w_{\text{strut}} - \boldsymbol{w}_{\textbf{sign}} + \boldsymbol{F}_{\textbf{cable,vertical}} = 0 \text{ Eq. (1)}$$

Along the x-axis:

$$\sum F_x = 0$$

$$\boldsymbol{F}_{\textbf{wall,horizontal}} - \boldsymbol{F}_{\textbf{cable,horizontal}} = 0 \text{ Eq. (2)}$$

The net torque also has to be zero about the pivot, taken to be at the hinge on the wall. We can write the condition for reaching rotational static equilibrium:

$$\sum \tau = 0$$

$$\tau_{\text{wall,vertical}} + \tau_{\text{wall,horizontal}} + \tau_{\text{strut}} + \tau_{\text{sign}} + \tau_{\text{cable}} = 0 \text{ Eq. (3)}$$

$$0 + 0 - 400 \cdot \left(\frac{L_{\text{strut}}}{2}\right) - 200 \cdot L_{\text{strut}} + \boldsymbol{F}_{\textbf{cable,vertical}} \cdot L_{\text{strut}} = 0$$

$$\boldsymbol{F}_{\textbf{cable,vertical}} = 400 \text{ N}$$

Inserting the result obtained above in Eq. (1), we get:

$$\boldsymbol{F}_{\textbf{wall,vertical}} = 200 \text{ N}$$

The tension in the cable can be found using trigonometry:

$$F_{\text{cable}} = \frac{F_{\text{cable,vertical}}}{\sin 30.0°} = 800 \text{ N}$$

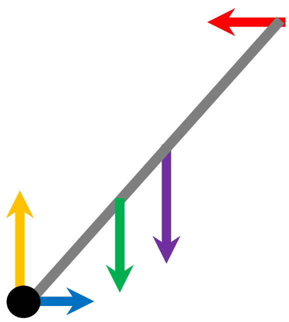
Now, using the above into Eq. (2), we get:

$$\boldsymbol{F}_{\textbf{wall,horizontal}} = 800 \cdot \cos 30.0° = 693 \text{ N}$$

Therefore, the magnitude of the force at the hinge is:

$$F_{\text{wall}} = \sqrt{F_{\text{wall,horizontal}}^2 + F_{\text{wall,vertical}}^2} = \sqrt{693^2 + 200^2} = 721 \text{ N}$$

7.



First, let's find the angle the ladder makes with the ground using basic trigonometry:

$$\theta = \cos^{-1}\frac{2}{6} = 70.5°$$

Conditions for getting the ladder to be in translational, static equilibrium:

$\vec{F}_{\text{net}} = 0$

Along the x-axis:

$F_{\text{net,x}} = 0$

$\boldsymbol{f}_{\textbf{base}} = \boldsymbol{N}_{\textbf{gutter}}$ Eq. (1)

Along the y-axis:

$F_{\text{net,y}} = 0$

$\boldsymbol{N}_{\textbf{ground}} = \boldsymbol{w}_{\textbf{ladder}} + \boldsymbol{w}_{\textbf{person}}$ Eq. (2)

The normal force exerted by the ground on the ladder is thus:

$\boldsymbol{N}_{\textbf{ground}} = 785\text{ N}$

Conditions for getting the ladder to be in rotational, static equilibrium:

$\vec{\tau}_{\text{net}} = 0$

$\tau_{\text{base}} + \tau_{\text{ground}} + \tau_{\text{ladder}} + \tau_{\text{person}} + \tau_{\text{gutter}} = 0$

0 + 0 - (2.00)(10.0·9.81)(sin160.5°) - (3.00)(70.0·9.81)(sin160.5°) + (6.00)(Ngutter)(sin109.5°) = 0 Eq. (3)

This could also be evaluated using the perpendicular force components:

$0 + 0 - (2.00)(w_{\text{ladder},\perp}) - (3.00)(w_{\text{person},\perp}) + (6.00)(N_{\text{gutter},\perp}) = 0$ so:

$-(2.00)(10.0 \cdot 9.81 \cdot \cos 70.5°) - (3.00)(70.0 \cdot 9.81 \cdot \cos 70.5°) + (6.00)(N_{\text{gutter}} \sin 70.5°) = 0$ Eq. (3)

Using either Eq. (3) and Eq. (1), we have that

$\boldsymbol{f}_{\textbf{base}} = \boldsymbol{N}_{\textbf{gutter}} = 133\text{ N}$

8.

$$x_{\text{cm}} == \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i}$$

$$x_{\text{cm}} = \frac{(1.00 \cdot 1.00) + (1.00 \cdot 2.00) + (2.00 \cdot 0.00)}{(1.00 + 1.00 + 2.00)} = 0.750 \text{ m}$$

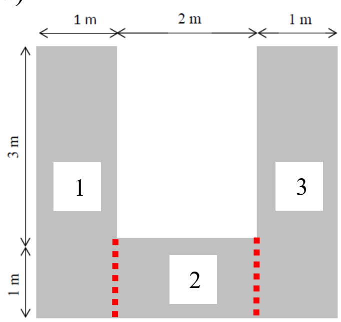
$$y_{\text{cm}} = \frac{\sum_{i=1}^{3} m_i y_i}{\sum_{i=1}^{3} m_i} = \frac{(1.00 \cdot 0.00) + (1.00 \cdot 0.00) + (2.00 \cdot 2.00)}{(1.00 + 1.00 + 2.00)} = 1.00 \text{ m}$$

The center of mass is located at:

$$r_{\text{cm}} = (0.750\,\hat{\imath} + 1.00\,\hat{\jmath}) \text{ m}$$

9.

a)



The center of mass about the bottom left center can be found if we first choose a system of coordinates. We'll be using the conventional x- and y- axis so that the origin coincides with the bottom left corner. We also must divide the shape into shapes for which the center of mass is easily found.

$$x_{\text{cm}} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{\sum_{i=1}^{3} \sigma_i A_i x_i}{\sum_{i=1}^{3} \sigma_i A_i} = \frac{\sum_{i=1}^{3} A_i x_i}{\sum_{i=1}^{3} A_i}$$

$$x_{\text{cm}} = \frac{(4 \cdot 1) \cdot (0.500) + (2 \cdot 1) \cdot (2.00) + (4 \cdot 1) \cdot (3.50)}{(4 \cdot 1) + (2 \cdot 1) + (4 \cdot 1)} = 2.00 \text{ m}$$

$$y_{\text{cm}} = \frac{\sum_{i=1}^{3} A_i y_i}{\sum_{i=1}^{3} A_i} = \frac{(4 \cdot 1) \cdot (2.00) + (2 \cdot 1) \cdot (0.500) + (4 \cdot 1) \cdot (2.00)}{(4 \cdot 1) + (2 \cdot 1) + (4 \cdot 1)} = 1.70 \text{ m}$$

The center of mass is located at:

$$r_{\text{cm}} = (2.00\,\hat{\imath} + 1.70\,\hat{\jmath}) \text{ m}$$

b) and c) on next page

9 b) c)

Gravity on an extended objected is exerted vertically downward at the centre of mass of the object. The shape will hang such that the net torque and the net force are zero. If only gravity and tension are acting then,

- according to Newton's second law, the net force must be zero, the forces will be equal in magnitude and in opposite directions.
- the net torque must be zero and therefore gravity must be along the line joining the hole and the centre of mass: in $r \cdot F \cdot \sin\theta$, we need $\sin\theta=0$. (The force at the hole always exerts zero torque since the hole is the pivot point. This makes $r=0$.)

The shape will therefore hang so that the line joining the hole and the centre of mass is vertical.

<table>
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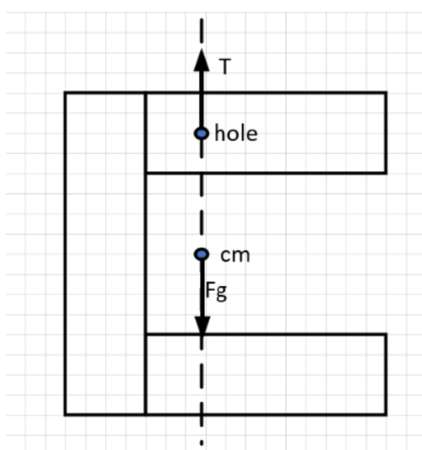
<td>b) If we want the 2 parallel arms to be horizontal, as in b, we should drill the hole along the line that goes through the centre of mass of the shape and that is parallel to the bottom part of the shape, because we want the bottom part of the shape to hang vertically.



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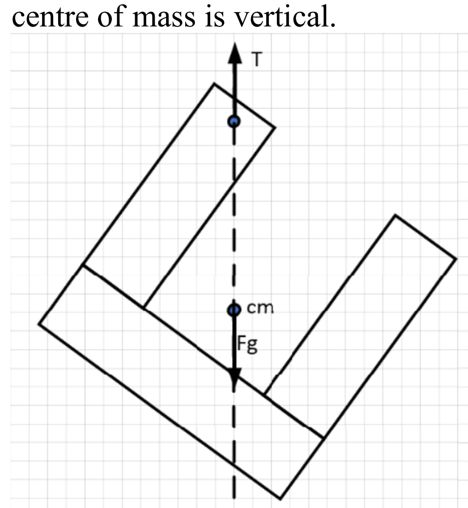
<td>c) If we hang the shape from the hole at the top left-hand branch as in c), the shape will hang so that the line joining the hole and the centre of mass is vertical.



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